Title: PSI 2019/2020 - Quantum Field Theory II (David) - Lecture 2

Speakers: Francois David

Collection: PSI 2019/2020 - Quantum Field Theory II (David)

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Why Euclidean time is interesting ?

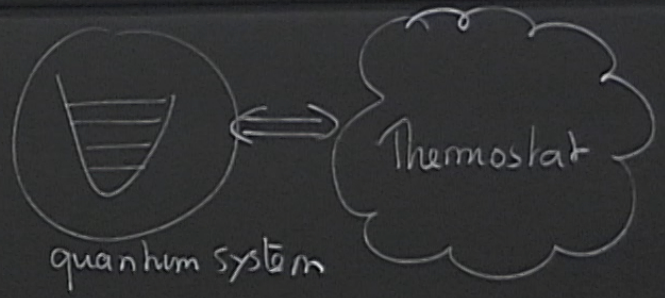
real time $t \longrightarrow t = -i\tau$ imaginary time

- path integrals $\longleftrightarrow$ classical statistical physics / probability — random walk, Brownian Process
- quantum statistical physics

$$\hat{U}(t) = \exp\left(\frac{t}{i\hbar}\hat{H}\right) \longrightarrow \hat{U}_E(\tau) = \exp\left(-\frac{\tau}{\hbar}\hat{H}\right)$$

$$\hat{\rho}(\tau) = \exp\left(-\frac{\tau}{\hbar}\hat{H}\right) \Big/ \mathrm{Tr}\left(\exp\left(-\frac{\tau}{\hbar}\hat{H}\right)\right) \qquad \hat{\rho} \geq 0 \quad \mathrm{Tr}\,\hat{\rho} = 1$$

energy e.s. $|i\rangle$, E_i



$$\hat{\rho} = \sum_i p_i |i\rangle\langle i|$$

$$p_i = \frac{\exp(-\frac{1}{\hbar} E_i)}{\sum_i \exp(-\frac{1}{\hbar} E_i)}$$

Boltzmann Weight

$$\exp\left(-\frac{1}{k_B T} E_i\right)$$

temperature T

access

$\hat{\rho} = 1 \rightarrow$ density matrix $\rightarrow$ mixed quantum state

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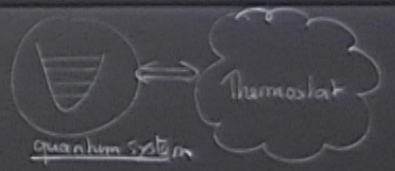
$$p_i = \frac{\exp(-\frac{\tau}{\hbar}E_i)}{\sum_i \exp(-\frac{\tau}{\hbar}E_i)}$$

Boltzmann Weight $\exp\left(-\frac{1}{k_B T}E_i\right)$ — temperature T

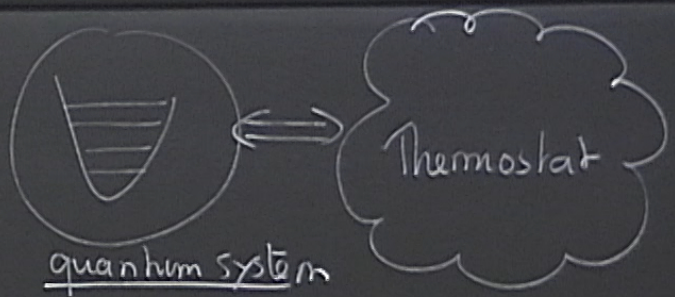
Gibbs State: Gibbs distribution at thermal equilibrium

$$\frac{\tau}{\hbar} = \frac{1}{k_B T} \qquad k_B T = \frac{\hbar}{\tau}$$

thermal fluctuations ↖ quantum fluctuations



energy e.s. $|i\rangle$, E_i



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Gibbs State

Gibbs distribution at thermal equilibrium

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thermal fluctuations

$\mathrm{Tr}\,\hat{\rho} = 1$ → density matrix → mixed quantum state

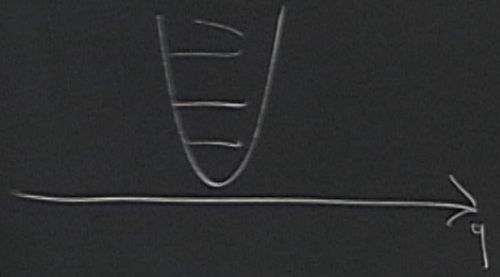
$$\sum_i \exp\left(-\frac{1}{k_B T} E_i\right) = Z(T)$$ partition function

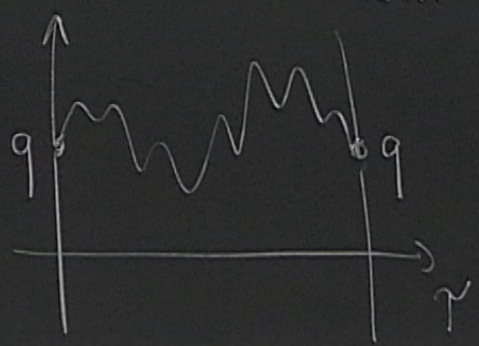
$$Z(T) = \mathrm{Tr}\left[\exp\left(-\frac{\tau}{\hbar}\hat{H}\right)\right] = \int_{-\infty}^{+\infty} dq \, \langle q | U_E(\tau) | q \rangle = \int_{-\infty}^{+\infty} dq \int \mathcal{D}[$$

$q(0) = q$

$q(\tau) =$

position eigenstates $|q\rangle$ $\quad \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{\phi})$





$$S_E[q] = \int_0^\tau d\sigma \, \frac{m}{2}\left[\dot{q}^2(\sigma) + V(q(\sigma))\right]$$

$$\int_{-\infty}^{+\infty} dq \; |q\rangle = \int \mathcal{D}[q] \; e^{-\frac{1}{\hbar} S_E[q]} = \int \mathcal{D}[q] \; e^{-\frac{1}{\hbar} S_E[q]}$$

$q(0) = q$

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periodic (in Euclidean time) trayectories

$q(0) = q(\tau)$

$$= \int_0^{\tau} d\sigma \; \frac{m}{2}\left[\dot{q}^2(\sigma) + V(q(\sigma))\right]$$

$\exp\left(-\frac{\tau}{\hbar}\hat{H}\right)$

$\hat{\rho} \geq 0 \quad \mathrm{Tr}\,\hat{\rho} = 1 \longrightarrow$ density matrix $\longrightarrow$ mixed quantum state

$\frac{\tau}{\hbar} = \frac{1}{k_B T}$ $\quad k_B T = \frac{\hbar}{\tau}$ ← quantum fluctuations

thermal fluctuations

function = path integral for periodic Euclidean time period

$$\tau = \frac{\hbar}{k_B T}$$

General quite deep

$$\langle q | U_E(\tau) | q \rangle = \int_{-\infty}^{+\infty} dq \int \mathcal{D}[q]\, e^{-\frac{1}{\hbar} S_E[q]} = \int \mathcal{D}[q]\, e^{-\frac{1}{\hbar} S_E[q]}$$

$q(0) = q$
$q(\tau) = q$

periodic (in Euclidean time) Trajectories
$q(0) = q(\tau)$

quantum system at finite temperature $\Longleftrightarrow$ periodic Euclidean Time

$V(\hat{Q})$

$$S_E[q] = \int_0^{\tau} d\sigma \left[\frac{m}{2}\dot{q}^2(\sigma) + V(q(\sigma))\right]$$

- quantum statistical physics

Brownian process

Gibbs distribution at thermal equilibrium

temperature T

$\hat{U}(t) = \exp\left(\frac{t}{i\hbar}\hat{H}\right) \rightarrow \hat{U}_E(\tau) = \exp\left(-\frac{\tau}{\hbar}\hat{H}\right)$

$\hat{\rho}(\tau) = \exp\left(-\frac{\tau}{\hbar}\hat{H}\right) / Tr\left(\exp\left(-\frac{\tau}{\hbar}\hat{H}\right)\right)$ $\quad \hat{\rho} \geq 0$ $\quad Tr\,\hat{\rho} = 1$ $\rightarrow$ density matrix $\rightarrow$ mixed quantum state

$\frac{\tau}{\hbar} = \frac{1}{k_B T}$ $\quad k_B T = \frac{\hbar}{\tau}$ thermal fluctuations ↖ quantum fluctuations

$\sum_i \exp\left(-\frac{1}{k_B T} E_i\right) = Z(\tau)$ partition function = path integral for periodic Euclidean time period $\quad \boxed{\tau = \frac{\hbar}{k_B T}}$ General quite deep

$Z(\tau) = Tr\left[\exp\left(-\frac{\tau}{\hbar}\hat{H}\right)\right] = \int_{-\infty}^{+\infty} dq \langle q | U_E(\tau) | q \rangle = \int_{-\infty}^{+\infty} dq \int \mathcal{D}[q]\, e^{-\frac{1}{\hbar} S_E[q]} = \int \mathcal{D}[q]\, e^{-\frac{1}{\hbar} S_E[q]}$

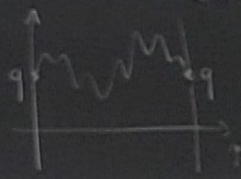
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periodic (on Euclidean time) Trajectories $q(0) = q(\tau)$

quantum system at finite temperature $\Leftrightarrow$ periodic Euclidean Time

position eigenstates $|q\rangle$ $\quad \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$





$S_E[q] = \int_0^{\tau} d\sigma \frac{m}{2}\left[\dot{q}^2(\sigma) + V(q(\sigma))\right]$

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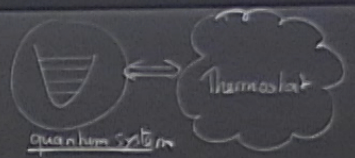
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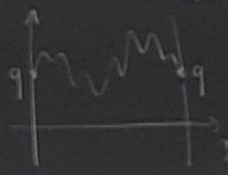
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$q(0) = q$, $q(\tau) = q$ — periodic (in Euclidean time) trajectories $q(0) = q(\tau)$

position eigenstates $|q\rangle$ $\quad \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{\Phi})$





$$S_E[q] = \int_0^\tau d\sigma \left[\frac{m}{2}\dot{q}^2(\sigma) + V(q(\sigma))\right]$$

Complex time → tunneling phenomenon, decay rates

quantum system at finite temperature (Equilibrium) ⟺ periodic Euclidean Time

↓

non equilibrium stat mechanics, real & imaginary time, 2-time Schwinger-Keldish formalism

$\sum_i \exp(-\frac{\tau}{\hbar} E_i)$ temperature T at thermal equilibrium

$\frac{\tau}{\hbar} = \frac{1}{k_B T}$ $k_B T = \frac{\hbar}{\tau}$ ← quantum fluctuations

$\hat{\rho} \geq 0$ $\mathrm{Tr}\hat{\rho} = 1$ → density matrix → mixed quantum state

thermal fluctuations

= path integral for periodic Euclidean time period

$\tau = \frac{\hbar}{k_B T}$ General quite deep

$\tau)|q\rangle = \int_{-\infty}^{+\infty} dq \int \mathcal{D}[q]\, e^{-\frac{1}{\hbar} S_E[q]} = \int \mathcal{D}[q]\, e^{-\frac{1}{\hbar} S_E[q]}$

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periodic (in Euclidean time) trajectories
$q(0) = q(\tau)$

quantum system at finite temperature ⟺ periodic Euclidean Time
(Equilibrium)

$= \int_0^{\tau} d\sigma \frac{m}{2}\left[\dot{q}^2(\sigma) + V(q(\sigma))\right]$

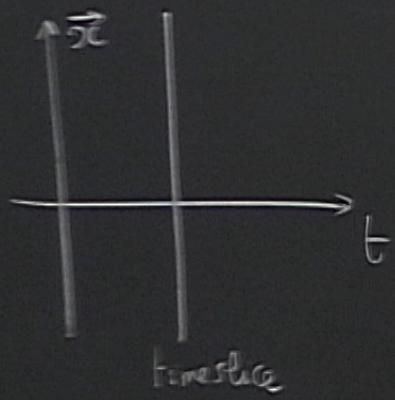
complex time → tunneling phenomenon
decay rates

non equilibrium stat mechanics
real & imaginary time
2-time Schwinger-Keldish formalism

Q.F.T

Klein-Gordon or Scalar free quantum field
↳ non-interacting

classical field



spacetime is Minkowski $\mathbb{M}^{1,d-1}$

Lorentz invariance

$x = (t, \vec{x})$

$c = 1$
speed of light

metric $ds^2 = -dt^2 + d\vec{x}^2$ West coast

Signature $(-,+\cdots)$
time ↑ ↳ space

$ds^2 = dt^2 - d\vec{x}^2$ East coast

$\phi(x)$ real field - scalar

$\|$ 1 component spin = 0

$\phi(t, \vec{x})$

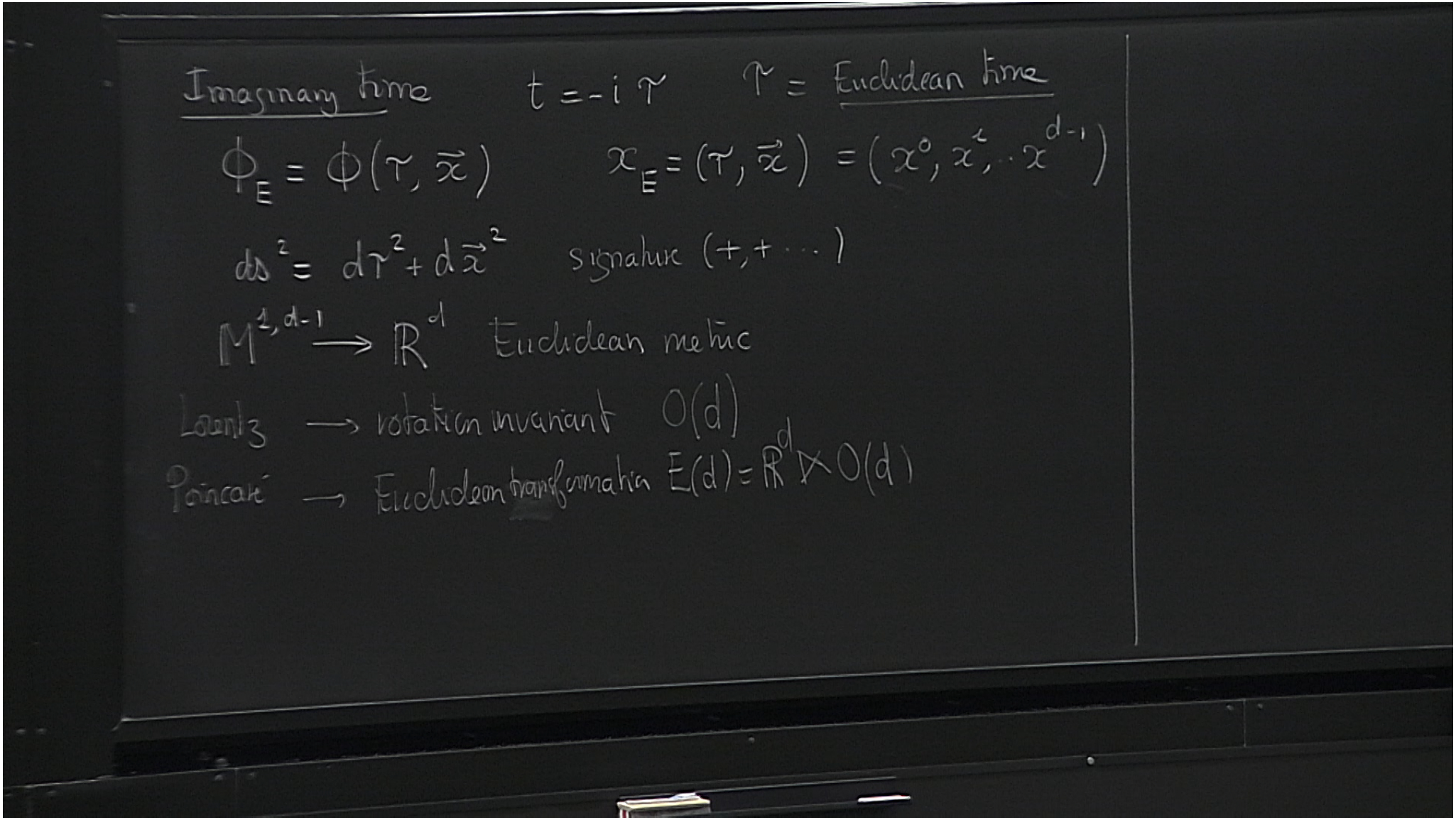
configuration $\phi_t = \{ \phi(t,x), x \in \mathbb{R}^{d-1} \}$

Action at real time

$$S[\phi] = \int dt \, \mathcal{L}[\phi_t, \dot{\phi}_t] = \int dt \int d^{d-1}\vec{x} \left[\frac{1}{2}(\partial_t \phi)^2 - \left(\frac{1}{2}(\partial_{\vec{x}} \phi)^2 + \frac{m^2}{2}\phi^2 \right) \right]$$

↑ Kinetic ↑ potential energy

$$-\partial_t^2 \phi + \partial_{\vec{x}}^2 \phi - m^2 \phi = 0$$ KG equ → plane waves


Imaginary time $t = -i\tau$ τ = Euclidean time
$\phi_E = \phi(\tau, \vec{x})$ $x_E = (\tau, \vec{x}) = (x^0, x^1, \cdots x^{d-1})$
$ds^2 = d\tau^2 + d\vec{x}^2$ signature $(+,+\cdots)$
$M^{1,d-1} \longrightarrow \mathbb{R}^d$ Euclidean metric
Lorentz $\longrightarrow$ rotation invariant $O(d)$
Poincaré $\longrightarrow$ Euclidean transformation $E(d) = \mathbb{R}^d \rtimes O(d)$

2-time Schwinger-Keldish formalism

$\phi(x)$ real field - scalar

$\|$ 1 component spin = 0

$\phi(t, \vec{x})$

configuration $\phi_t = \{ \phi(t, x), x \in \mathbb{R}^{d-1} \}$

Action at real time

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↑ Kinetic ↑ potential energy

$-\partial_t^2 \phi + \partial_{\vec{x}}^2 \phi - m^2 \phi = 0$ KG equ. → plane waves Elliptic equation

$$S[\phi] = S_E[\phi] = \int dx^0 \int d^{d-1}\vec{x} \left[\underbrace{\frac{1}{2}(\partial_{x_0}\phi)^2 + \frac{1}{2}(\partial_{\vec{x}}\phi)^2}_{\frac{1}{2}(\partial_x \phi)^2} + \frac{m^2}{2}\phi^2 \right]$$

Euclidean Action

Euclidean Klein gordon Parabolic Equation

$$-\Delta\phi + m^2\phi = 0$$

Δ ordinary Laplacian in $\mathbb{R}^d$ $\quad \frac{\partial^2}{\partial x_0^2} + \cdots \frac{\partial^2}{\partial x_{d-1}^2}$

signature (-,+,+,+)

time ↑ ↳ space

$ds^2 = dt^2 - d\vec{x}^2$ East coast

$-\partial_t^2\phi + \partial_{\vec{x}}^2\phi - m^2\phi =$

Path Integral Quantization = functional integration

$$Z = \int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$$

$$= \int \mathcal{D}[\phi_t]$$

$$= \int \prod_t d\phi_t$$

$$= \int \prod_t \prod_{\vec{x}} d\phi(t,\vec{x})$$

all possible field configuration in spacetime

At real time ⚠

$$\int \mathcal{D}[\phi_E] \exp\left(-\frac{1}{\hbar} S_E[\phi_E]\right) = Z_E$$

at euclidean time

$$\int \prod_\tau \prod_{\vec{x}} d\phi_E(\tau,\vec{x})$$

$|\phi_I\rangle$ inital state $t=0$ $\phi(0,\vec{x}) = \phi_I(\vec{x})$

classical field inital configuration

$\neq$ IN and OUT states with particle with definite momenta

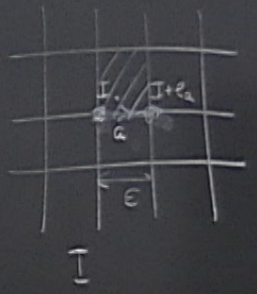
no particles yet

$$S[\phi] = S_E[\phi] = \int dx^0 \int d^{d-1}\vec{x} \left[\frac{1}{2}(\partial_{x_0}\phi)^2 + \frac{1}{2}(\partial_{\vec{x}}\phi)^2 + \frac{m^2}{2}\phi^2 \right]$$

Euclidean Action

$$\frac{1}{2}(\partial_x \phi)^2$$

Euclidean Klein gordon Parabolic Equation $-\Delta\phi + m^2\phi = 0$ Δ ordinary Laplacian in $\mathbb{R}^d = \frac{\partial^2}{\partial x_0^2} + \cdots \frac{\partial^2}{\partial x_{d-1}^2}$



discretize the action ordinary derivatives $\frac{\partial}{\partial \vec{x}}$ → Finite difference derivatives

$a = 0, \ldots d-1$ $\frac{\partial}{\partial x^a}\phi(x) \Rightarrow$ compact d $\partial_a \phi(I) = \frac{\phi(I+e_a) - \phi(I)}{\epsilon}$ $d^d x \to \sum_{I \in \mathbb{Z}^d} \epsilon^d$ volume element

$\vec{x} = \epsilon I$ $I \in \mathbb{Z}^d$ $e^a = (0 \ldots 1 \ldots 0)$

$x = \epsilon I \quad I \in \mathbb{Z}$

discretized Euclidean action

$$S_E[\phi] = \sum_{I \in \mathbb{Z}^d} \epsilon^d \left[\frac{1}{2} \sum_{a=0}^{d-1} \left(\frac{\phi(I+e_a) - \phi(I)}{\epsilon} \right)^2 + \frac{1}{2} m^2 \phi(I)^2 \right]$$

ϵ^{d-2} factor

ϵ^d factor

same thing for real discretized time

$S[\phi]$

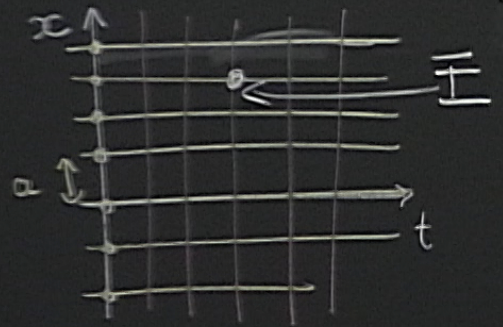
functional integral with space-time discretization ← useful for renormalization

$\lim \epsilon = a \to 0$ continuous limit for space time ⚠

∞ KG equ → plane waves Elliptic equation

∞ number of degree of freedom ← discretize space-time

Ultra-violet regularization



1+1 dim case

measure $\mathcal{D}[\phi] = \prod_{\vec{I}\in\mathbb{Z}^d} d\phi(\vec{I}) \left[\frac{2\pi\hbar}{\epsilon^{d-2}}\right]^{-1/2}$

discretized s-time

discretized euclidean space-time $\mathcal{D}_E[\phi] = \prod_{\vec{I}\in\mathbb{Z}^d} d\phi_E(\vec{I}) \left(\frac{2\pi\hbar}{\epsilon^{d-2}}\right)^{-1/2}$

Space $\mathbb{R} \to \mathbb{Z}$ integers time discretization ← Q.T. already

$x = ia \quad i\in\mathbb{Z} \qquad t = j\epsilon \qquad \epsilon = a$

Special cut-off Time cut-off

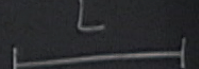
$\phi(t,\vec{x}) = \phi(j,\vec{i}), \ (j,\vec{i}) = \vec{I}\in\mathbb{Z}^d \quad (j,\vec{i}) = \vec{I}\in\mathbb{Z}^d$

[continuous ∞ # d.o.f
↳ discrete ∞ # d.o.f

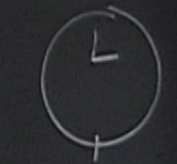
Exercise

Large distance / time Infra Red Regularization

space is infinite $\Longrightarrow$ might lead to problems / questions / issues



<u>space finite</u> $\rightarrow$ QFT in a spacial box $\leftarrow$ Casimir Effect

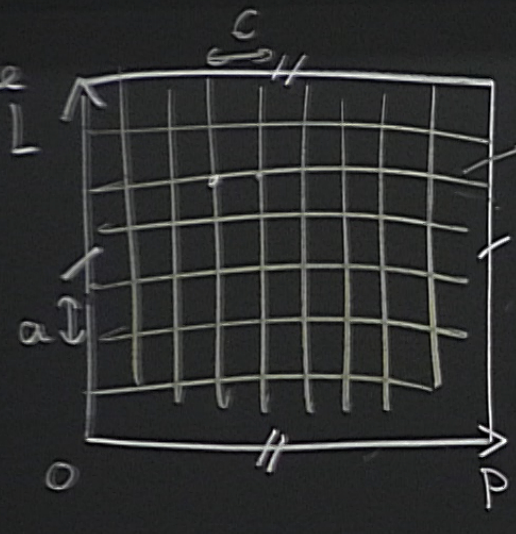


$\phi_t(\vec{x} + L\cdot\vec{m}) = \phi_t(\vec{x})$ on a cylinder space $\mathbb{R}^{d-1} \rightarrow \mathbb{T}^{d-1} = T^{\otimes d-1}$ dimension L

$\vec{m} \in \mathbb{Z}^{d-1}$ periodic bc in space

time finite : Euclidean time periodic with period P $\Longleftrightarrow$ Finite temperature T

$k_B T = \frac{\hbar}{P}$



Spacetime → d-dimensional torus

Infra-red regularization

continuum limit $\epsilon \to 0$

large distance limit

$L = P \to \infty$

$\int \mathcal{D}[\phi]$ → finite dimensional integral over functions

$$\frac{P}{\epsilon}\left(\frac{L}{a}\right)^{d-1}$$

QFT at finite temperature

$\epsilon \to 0$

$\epsilon = a = 0 \qquad L = \infty$

$L = P \to \infty$

P still finite

dimensional

d over functions