Title: General Relativity for Cosmology - Lecture 23

Speakers: Achim Kempf

Collection: General Relativity for Cosmology (Kempf)

Date: November 29, 2019 - 4:00 PM

URL: http://pirsa.org/19110008

GR for Cosmology, Achim Kempf — Lecture 23

More on singularity theorems

- Assume a set of symmetries of matter and spacetime has been chosen.
- Assume an exact solution or at least its asymptotic properties at early times have been found.
- Assume, we choose a timelike congruence e.g. of geodesics.

properties at early times have been found.

- Assume, we choose a timelike congruence e.g. of geodesics.

$\Rightarrow$ We can now explicitly calculate the twist, shear and expansion along the congruence:

The Hubble functions:

The Hubble functions:

In particular, we can see how the expansion or contraction of the universe behaves dynamically e.g. when the condition of perfect isotropy is relaxed:

- Now we have different expansions in different directions, nonlinearly influencing another.

- **Recall:**

  The expansion in one direction can be enhanced by shear, as long as shear shrinks other directions.

- **Definition:**

  We define a rate of expansion tensor that includes or contraction of the universe behaves dynamically e.g. when the condition of perfect isotropy is relaxed:

- Now we have different expansions in different directions, nonlinearly influencing another.

- Recall:

The expansion in one direction can be enhanced by shear, as long as shear shrinks other directions.

- Definition:

We define a rate of expansion tensor that includes shear:

$$\Theta_{\mu\nu} := \overset{\text{shear}}{\sigma_{\mu\nu}} + \frac{1}{3}\,\theta\, h_{\mu\nu}$$

projector $\perp$ to the timelike u-field

symmetric part of $B_{\mu\nu}$

expansion scalar.

- $\Theta_{\mu\nu}$ is fully spacelike and symmetric $\Rightarrow$ $\Theta_{\mu\nu}$ can be diagonalized in suitable ON frame $\{e_0, e_1, e_2, e_3\}$:

$$\Theta_{\mu\nu} = \begin{pmatrix} 0 & & & 0 \\ & \theta_1 & & \\ & & \theta_2 & \\ 0 & & & \theta_3 \end{pmatrix}$$

3 space-like directions.

with the traditional expansion being the trace (because $\sigma_{\mu\nu}$ is traceless):

$$\theta = \theta_1 + \theta_2 + \theta_3$$

why $\frac{1}{3}$? Recall that $\overbrace{Tr(h_{\mu\nu})}^{\Rightarrow \text{ is not quite projector}} = 3$

- **Definition:** $H_i := \frac{1}{3}\theta_i$ Local Hubble expansion function in direction e_i

- $\theta_{\mu\nu}$ is fully spacelike and symmetric $\Rightarrow$ $\theta_{\mu\nu}$ can be diagonalized in suitable ON frame $\{e_0, e_1, e_2, e_3\}$:

$$\theta_{\mu\nu} = \underbrace{\begin{pmatrix} 0 & & & \\ & \theta_1 & & 0 \\ & 0 & \theta_2 & \\ & & & \theta_3 \end{pmatrix}}_{3 \text{ space-like directions.}}$$

with the traditional expansion being the trace (because $\sigma_{\mu\nu}$ is traceless):

$$\theta = \theta_1 + \theta_2 + \theta_3$$

why $\frac{1}{3}$? Recall that $\underbrace{Tr(h_{\mu\nu}) = 3}_{\Rightarrow \text{ is not quite projector}}$

- **Definition:**

$H_i := \frac{1}{3}\theta_i$ Local Hubble expansion function in direction e_i.

$H := \frac{1}{3}\theta$ Overall local Hubble expansion function.

□ Definition :

We use H_i, H to define local directional and general scale factors ℓ_i, ℓ :

The ℓ_i, ℓ are defined as the solutions to :

$$\frac{\dot{\ell}_i}{\ell_i} = H_i$$

$$\frac{\dot{\ell}}{\ell} = H$$

Here, the time derivative is defined as:

recall: u is timelike.

$$\dot{\ell} = u(\ell) = u^\mu \frac{\partial}{\partial x^\mu} \ell$$

Full set of cases not yet known.

But:

Explicit examples are known where e.g.:

- All $\ell_i \to 0$ as in FL cosmologies
- $\ell_1, \ell_2 \to 0$, $\ell_3 \to \infty$ "cigar singularity"
- $\ell_1, \ell_2 \to 0$, $\ell_3 \to \text{const}$ "barrel singularity"
- $\ell_1, \ell_2 \to \text{const}$, $\ell_3 \to 0$ "pancake singularity"

- **Note:** For homogeneous, isotropic FL models, H is the regular Hubble parameter and ℓ is

Singularity theorems for black holes

In 1915, Schwarzschild found this black hole solution to the Einstein equation:

Mass of black hole

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + d\varphi^2 \sin^2\theta\right)$$

Singularity : $r = 0$

Horizon : $r = 2M$

Here, $x = (t, r, \varphi, \theta)$ are called the Schwarzschild coordinates.

Schwarzschild coordinates long misled intuition:

- The singularity at $r = 2M$ is not real: it disappears in

In 1915, Schwarzschild found this black hole solution to the Einstein equation:

Mass of black hole

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + d\varphi^2 \sin^2\theta\right)$$

Singularity: $r = 0$

Horizon: $r = 2M$

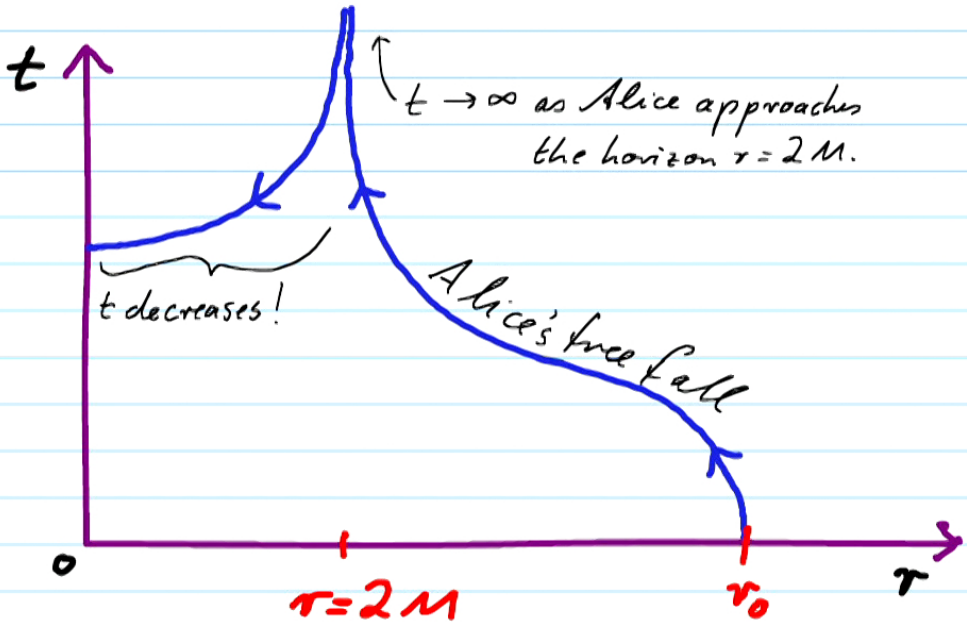
Here, $x = (t, r, \varphi, \theta)$ are called the Schwarzschild coordinates.

Schwarzschild coordinates long misled intuition:

- The singularity at $r = 2M$ is not real: it disappears in other coordinate systems. The curvature is smooth across $r = 2M$.

- Due to the sign changes across $r = 2M$, for $r < 2M$ dt is spacelike and dr is timelike for $r < 2M$.

- Consider, for example, a traveler, Alice, who is freely falling from $r = r_0$ to $r = 0$:



$$r(\alpha) = \frac{r_0}{2}\left(1 + \cos(\alpha)\right)$$

$$t(\alpha) = \left(\frac{r_0}{2} + 2M\right) w\alpha + \frac{r_0}{2} w \sin(\alpha) + 2M \log\left|\frac{w + \tan(\alpha/2)}{w - \tan(\alpha/2)}\right|$$

$$\tau(\alpha) = \frac{r_0}{2}\left(\frac{r_0}{2M}\right)^{1/2}\left(\alpha + \sin(\alpha)\right)$$

Here: $0 < \alpha < \pi$ and $w := \left(\frac{r_0}{2M} - 1\right)^{1/2}$

Simplification:

For now, we drop the φ and θ coordinates.

First design of a new cds (T, R) - Alice's choice (for $r_0 = 2M$):

- Require $g_{\mu\nu}(T, R)$ to be regular across $r = 2M$.
- Require $g_{\mu\nu}(0,0) = \eta_{\mu\nu}$ at $r = 2M$. If there's really no singularity at $r = 2M$ this must be possible.
- Extend this cds so that $g_{\mu\nu}(T, R) = f(T, R)\, \eta_{\mu\nu}$

⇒ Alice's choice are the Kruskal-Szekeres coordinates (T,R):

$$T(t,r) := 4M\left|\frac{r}{2M}-1\right|^{1/2} e^{\frac{r-2M}{4M}}\left(\sinh\left(\frac{t}{4M}\right)\theta(r-2M)+\cosh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

$$R(t,r) := 4M\left|\frac{r}{2M}-1\right|^{1/2} e^{\frac{r-2M}{4M}}\left(\cosh\left(\frac{t}{4M}\right)\theta(r-2M)+\sinh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

This map is, in principle, invertible, to obtain $t(T,R)$, $r(T,R)$.

The Schwarzschild metric now takes this form:

$$ds^2 = \frac{2M}{r(T,R)}\, e^{1-\frac{r(T,R)}{2M}}\left(dT^2 - dR^2\right)$$ Obeys all conditions!

$$T(t,r) := 4M\left|\frac{r}{2M}-1\right|^{1/2} e^{\frac{r-2M}{4M}}\left(\sinh\left(\frac{t}{4M}\right)\theta(r-2M)+\cosh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

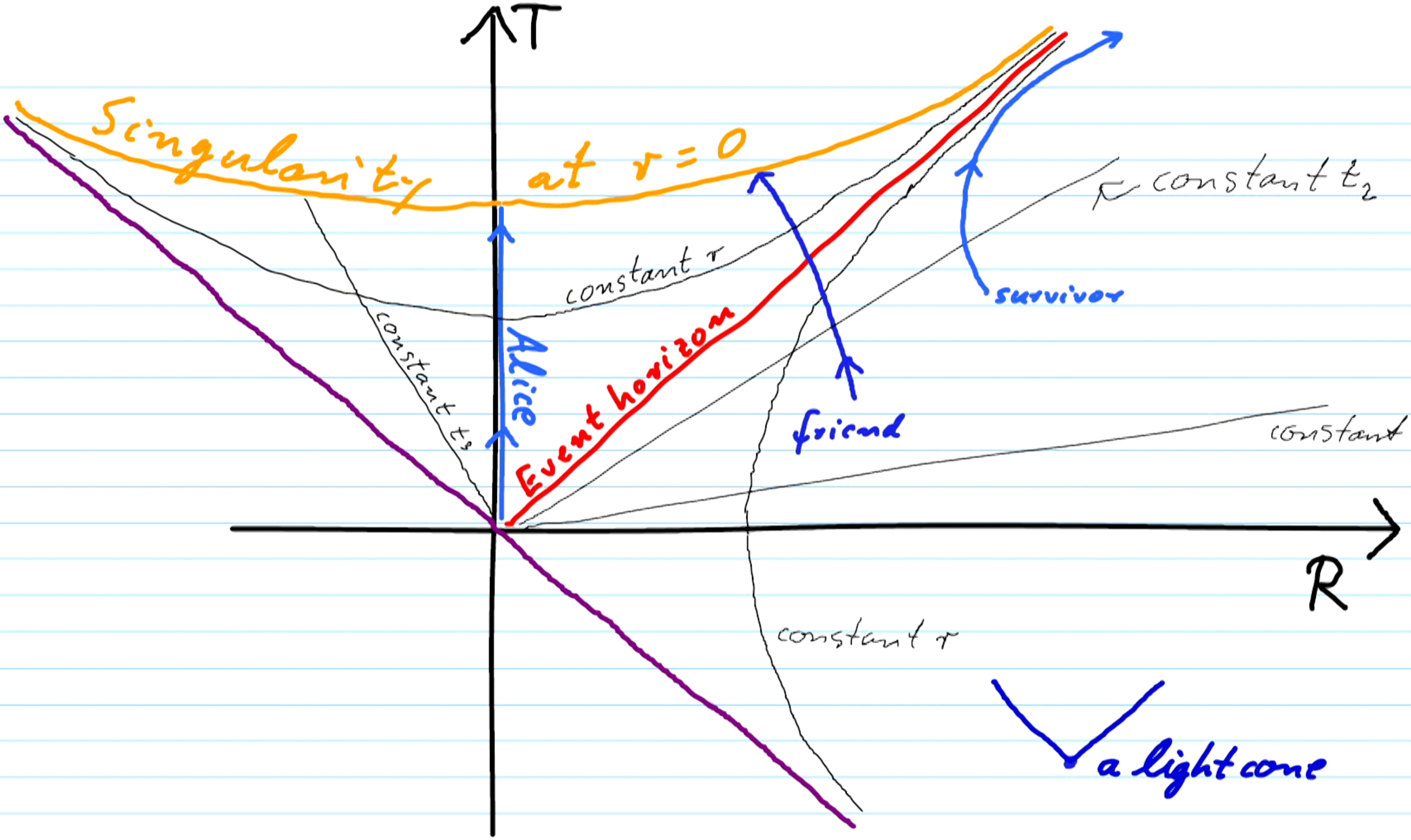
$$R(t,r) := 4M\left|\frac{r}{2M}-1\right|^{1/2} e^{\frac{r-2M}{4M}}\left(\cosh\left(\frac{t}{4M}\right)\theta(r-2M)+\sinh\left(\frac{t}{4M}\right)\theta(2M-r)\right)$$

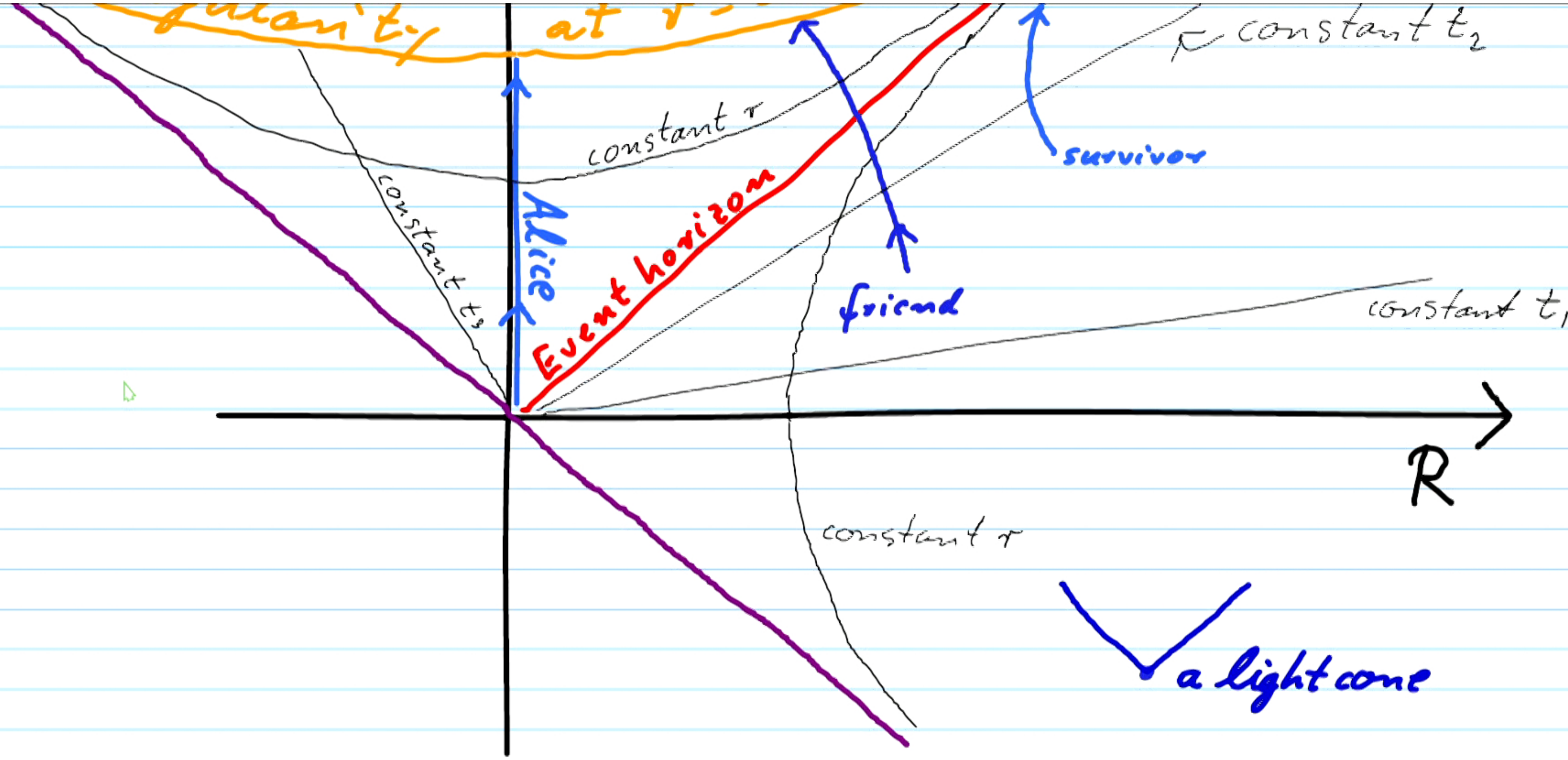
This map is, in principle, invertible, to obtain $t(T,R)$, $r(T,R)$.

The Schwarzschild metric now takes this form:

$$ds^2 = \underbrace{\frac{2M}{r(T,R)}\, e^{1-\frac{r(T,R)}{2M}}}_{\text{Conformal prefactor} = 1 \text{ as } r = 2M} \underbrace{\left(dT^2 - dR^2\right)}_{= \eta_{\mu\nu}}$$

Obeys all conditions!


T
Singularity at r = 0
constant t2
constant r
survivor
constant t0
Alice
Event horizon
friend
constant t1
R
constant r
a light cone



- Alice was at rest at the event horizon.
- The singularity is at $T(R) = \left(R^2 + \frac{16M^2}{e}\right)^{1/2}$ and is spacelike.

Alice's light cone coordinates:

$$u := T - R, \quad v := T + R$$

Metric: $$ds^2 = \underbrace{\frac{2M}{r(u,v)}\, e^{1-\frac{r(u,v)}{2M}}}_{\substack{\text{conformal factor}\\ \text{(which is 1 at horizon)}}} \underbrace{du\, dv}_{\substack{\text{light cone}\\ \text{Minkowski}}}$$

$\Rightarrow$ The action $S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\sqrt{g}\, d^2x$ becomes:

$$= \frac{1}{2}\int_{T>-R} \left(\partial_T\phi(T,R)\right)^2 - \left(\partial_R\phi(T,R)\right)^2 dT\, dR$$

$$= 2\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\partial_u\phi(u,v)\right)\left(\partial_v\phi(u,v)\right) dv\, du$$

$\leftarrow$ b/c region $T > -R$ means $T + R > 0$, i.e. $v > 0$.

$$u := T - R, \quad v := T + R$$

Metric: $$ds^2 = \underbrace{\frac{2M}{r(u,v)}\, e^{1-\frac{r(u,v)}{2M}}}_{\substack{\text{conformal factor} \\ \text{(which is 1 at horizon)}}} \underbrace{du\, dv}_{\substack{\text{lightcone} \\ \text{Minkowski}}}$$

$\Rightarrow$ The action $S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\, \phi_{,\alpha}\, \phi_{,\beta} \sqrt{g}\, d^2x$ becomes:

$$= \frac{1}{2}\int_{T>-R} \left(\partial_T \phi(T,R)\right)^2 - \left(\partial_R \phi(T,R)\right)^2 dT\, dR$$

$$= 2\int_{-\infty}^{\infty}\int_{0}^{\infty} \left(\partial_u \phi(u,v)\right)\left(\partial_v \phi(u,v)\right) dv\, du$$

$\leftarrow$ b/c region $T > -R$ means $T + R > 0$, i.e. $v > 0$.

$\Rightarrow$ Eqn of motion: $\partial_u \partial_v \phi(u,v) = 0$

Bob's choice of coordinate system

Bob is far from the black hole.

He wants a cds in which:

- $g_{\mu\nu}(x) \to \eta_{\mu\nu}$ as $r \to \infty$.
- $g_{\mu\nu}(x) = f(x)\,\eta_{\mu\nu}$ everywhere.

This is so that in his cds too

- photons travel at 45°
- equations of motion of matter fields will be simple (useful in QFT!)

He wants a cds in which:

- $g_{\mu\nu}(x) \to \eta_{\mu\nu}$ as $r \to \infty$.
- $g_{\mu\nu}(x) = f(x)\,\eta_{\mu\nu}$ everywhere.

This is so that in his cds too

- photons travel at 45°
- equations of motion of matter fields will be simple (useful in QFT!)

⇝ Bob's choice is the Tortoise coordinate system.

Tortoise cds (t^*, r^*):

- In terms of the Schwarzschild cds:

$$t^* := t$$

must require $r > 2M$!

$$r^* := r - 2M + 2M \log\left(\frac{r}{2M} - 1\right)$$

$\Rightarrow$ Important: This is in principle invertible, to obtain

$$r = r(r^*)$$

but only for $r > 2M$!

$\Rightarrow$ The tortoise cds only cover the BH's outside!

Metric: $ds^2 = \left(1 - \frac{2M}{r(r^*)}\right)\left(dt^{*2} - dr^{*2}\right)$

In terms of the Schwarzschild cds:

$$t^* := t$$

must require $r > 2M$!

$$r^* := r - 2M + 2M \log\left(\frac{r}{2M} - 1\right)$$

⇒ Important: This is in principle invertible, to obtain

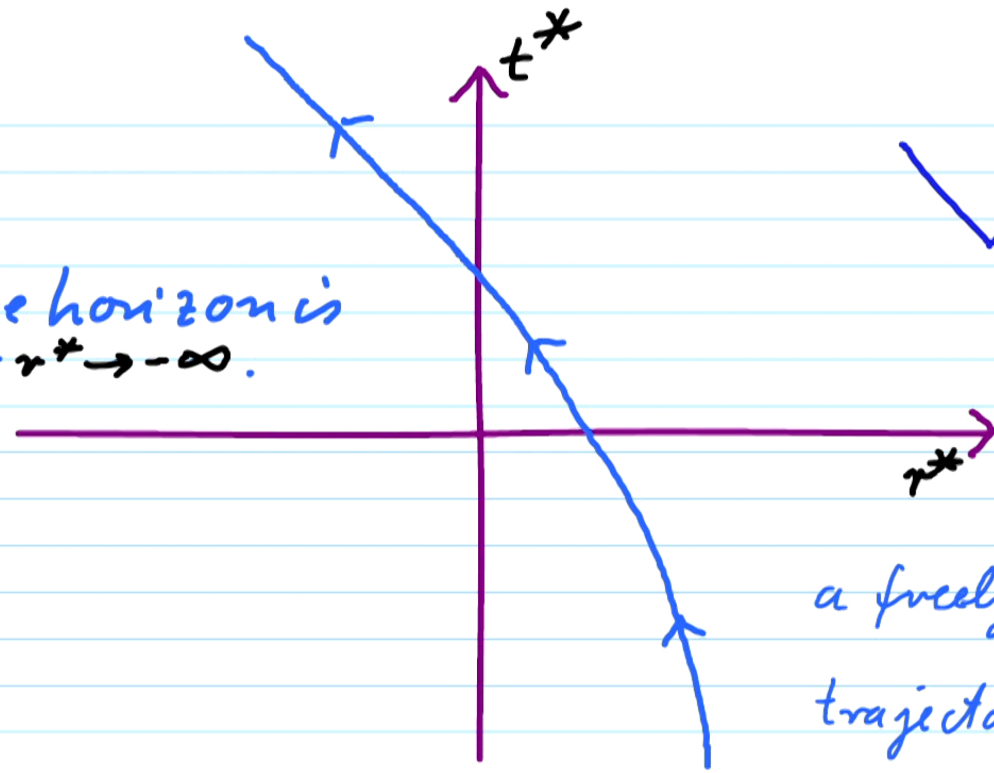
$$r = r(r^*)$$

but only for $r > 2M$!

⇒ The tortoise cds only cover the BH's outside!

Metric:
$$ds^2 = \underbrace{\left(1 - \frac{2M}{r(r^*)}\right)}\left(dt^{*2} - dr^{*2}\right)$$

Conformal factor $\to 1$ as $r \to \infty$, as planned but $\to 0$ at horizon.



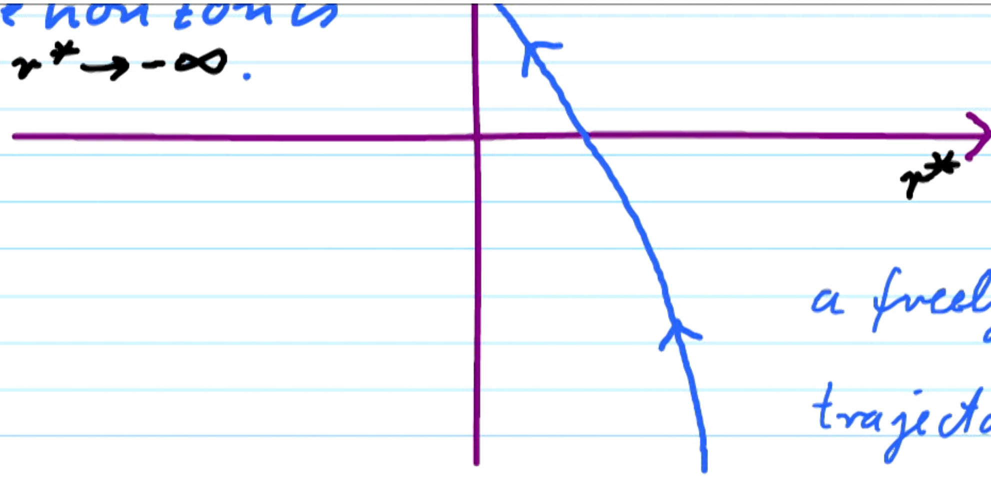
$\leftarrow$ The horizon is at $r^* \to -\infty$.

$\vee$ a light cone

a freely infalling observer's trajectory is bent towards 45° as he speeds up.

Bob's light cone coordinates: $\bar{u} := t^* - r^*$, $\quad \bar{v} := t^* + r^*$

The horizon is at $r^* \to -\infty$.



a freely infalling observer's trajectory is bent towards 45° as he speeds up.

Bob's light cone coordinates: $\bar{u} := t^* - r^*, \quad \bar{v} := t^* + r^*$

The metric is then: $$ds^2 = \underbrace{\left(1 - \frac{2M}{r(\bar{u},\bar{v})}\right)}_{\to 1 \text{ as } r \to \infty \text{ and } \to 0 \text{ as } r \to 2M} d\bar{u}\, d\bar{v}$$

$\Rightarrow$ The action:

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta}\sqrt{g}\,d^2x \quad \text{becomes:}$$

$$= \frac{1}{2}\int_{\mathbb{R}^2} \left(\partial_{t^*}\phi(t^*,r^*)\right)^2 - \left(\partial_{r^*}\phi(t^*,r^*)\right)^2 \, dt^* dr^*$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\partial_{\bar{u}}\phi(\bar{u},\bar{v})\right)\left(\partial_{\bar{v}}\phi(\bar{u},\bar{v})\right) d\bar{v}\, d\bar{u}$$

$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\, \phi_{,\alpha}\, \phi_{,\beta} \sqrt{g}\, d^2x$ becomes:

$$= \frac{1}{2}\int_{\mathbb{R}^2} \left(\partial_{t^*}\phi(t^*,r^*)\right)^2 - \left(\partial_{r^*}\phi(t^*,r^*)\right)^2 dt^* dr^*$$

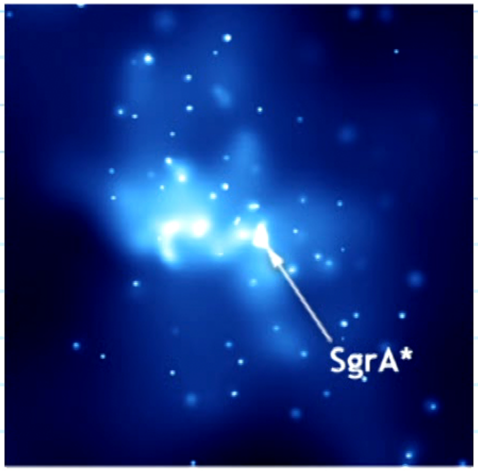
$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\partial_{\bar{u}}\phi(\bar{u},\bar{v})\right)\left(\partial_{\bar{v}}\phi(\bar{u},\bar{v})\right) d\bar{v}\, d\bar{u}$$

$\Rightarrow$ Eqn of motion: $\partial_{\bar{u}}\partial_{\bar{v}}\phi(\bar{u},\bar{v}) = 0$

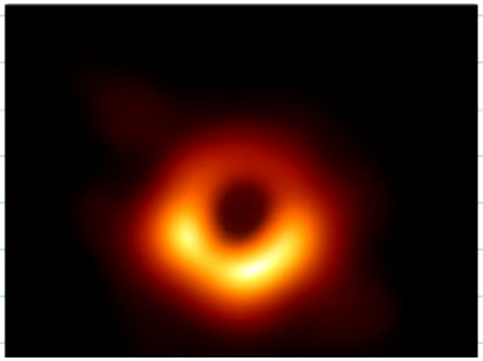
Do real black holes possess a singularity?

Sagittarius A*

- 4 Mio stellar masses
- Diameter 44 Mio km
- 26000 light years away at centre of Milky Way.

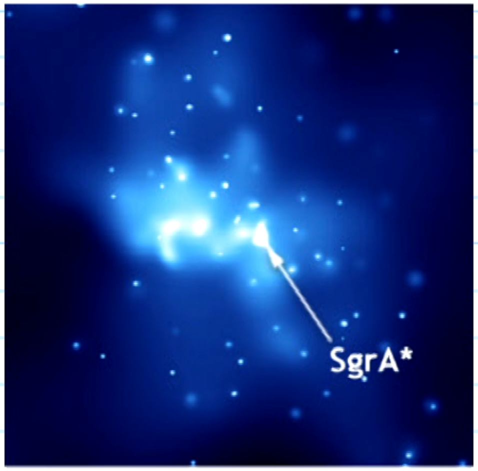


→ Observation of M87 by the Event Horizon Telescope (in mm band) with enough

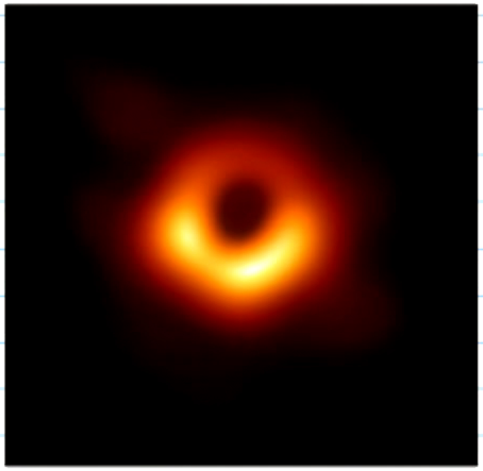


Sagittarius A*

- 4 Mio stellar masses
- Diameter 44 Mio km
- 26000 light years away
  at centre of Milky Way.



→ Observation of M87 by the
Event Horizon Telescope
(in mm band) with enough
resolution to see the event horizon:



How to model properties of real black holes roughly?

Singularity theorems suitable for black holes involve the concept and assumption of a trapped surface:

Def:

- Let Σ be a spacelike hypersurface. (Note: Σ is 3-dimensional)
- Let $T \subset \Sigma$ be a compact, 2-dimensional smooth spacelike submanifold of Σ. Consider the ingoing and the outgoing future-directed null geodesics that are orthogonal to T.
- If all these geodesics possess negative expansion, $\theta < 0$, then T is called a trapped surface.

How to model properties of real black holes roughly?

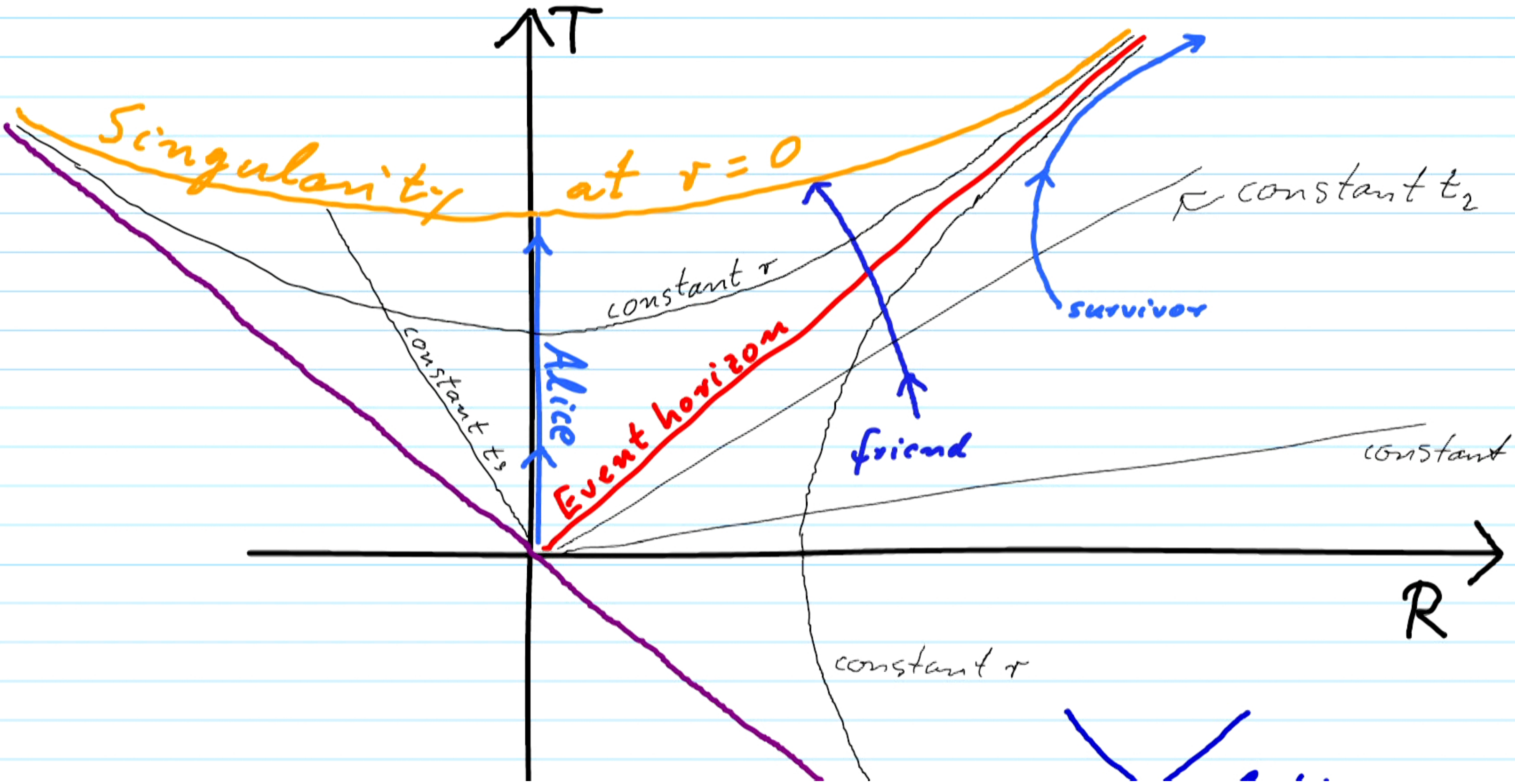
Singularity theorems suitable for black holes involve the concept and assumption of a trapped surface:

Def:

- Let Σ be a spacelike hypersurface. (Note: Σ is 3-dimensional)
- Let $T \subset \Sigma$ be a compact, 2-dimensional smooth spacelike submanifold of Σ. Consider the ingoing and the outgoing future-directed null geodesics that are orthogonal to T.
- If all these geodesics possess negative expansion, $\theta < 0$, then T is called a trapped surface.

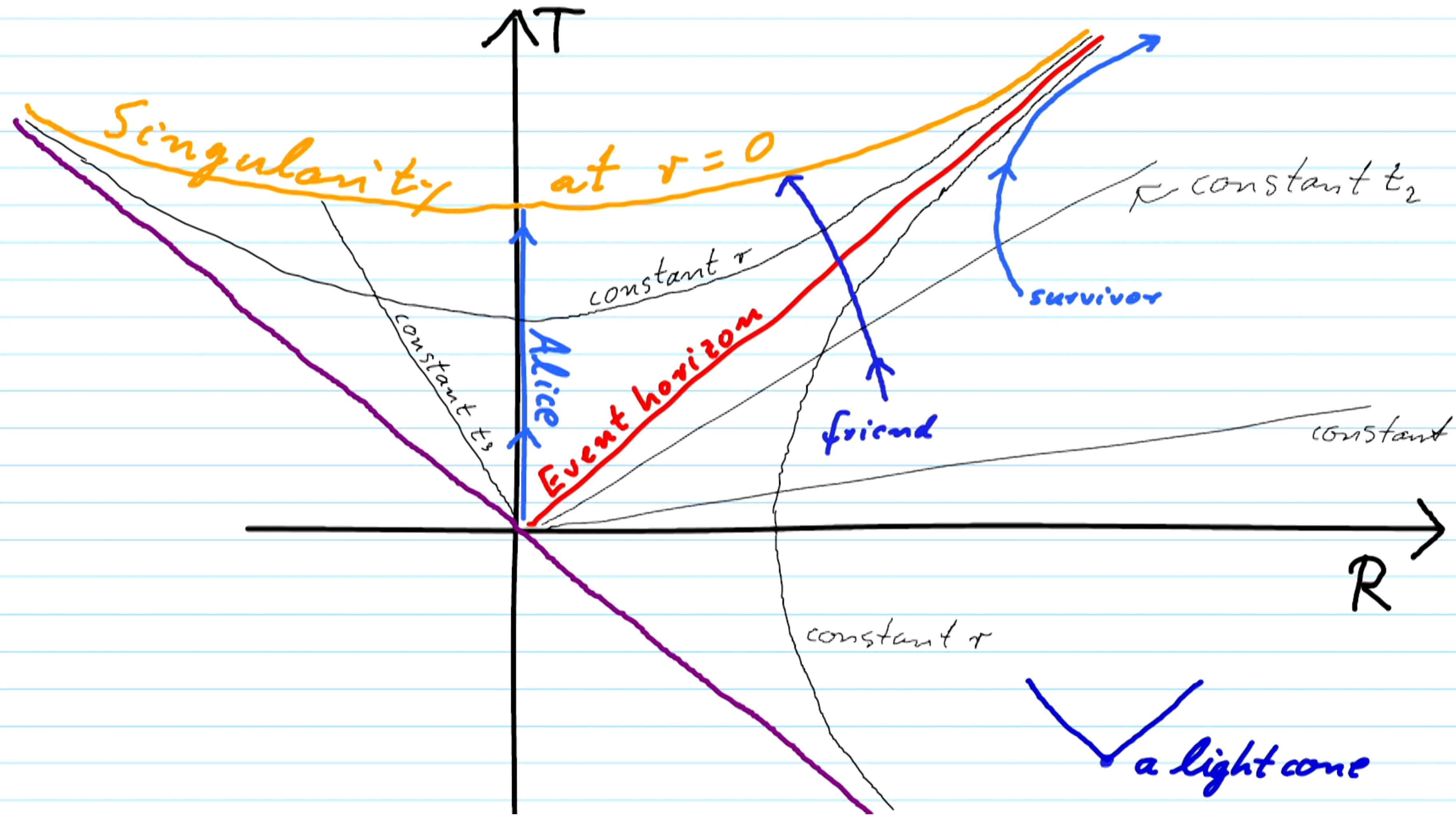
Examples of trapped surfaces:

All spheres $r = \text{const.}$ inside a Schwarzschild black hole.



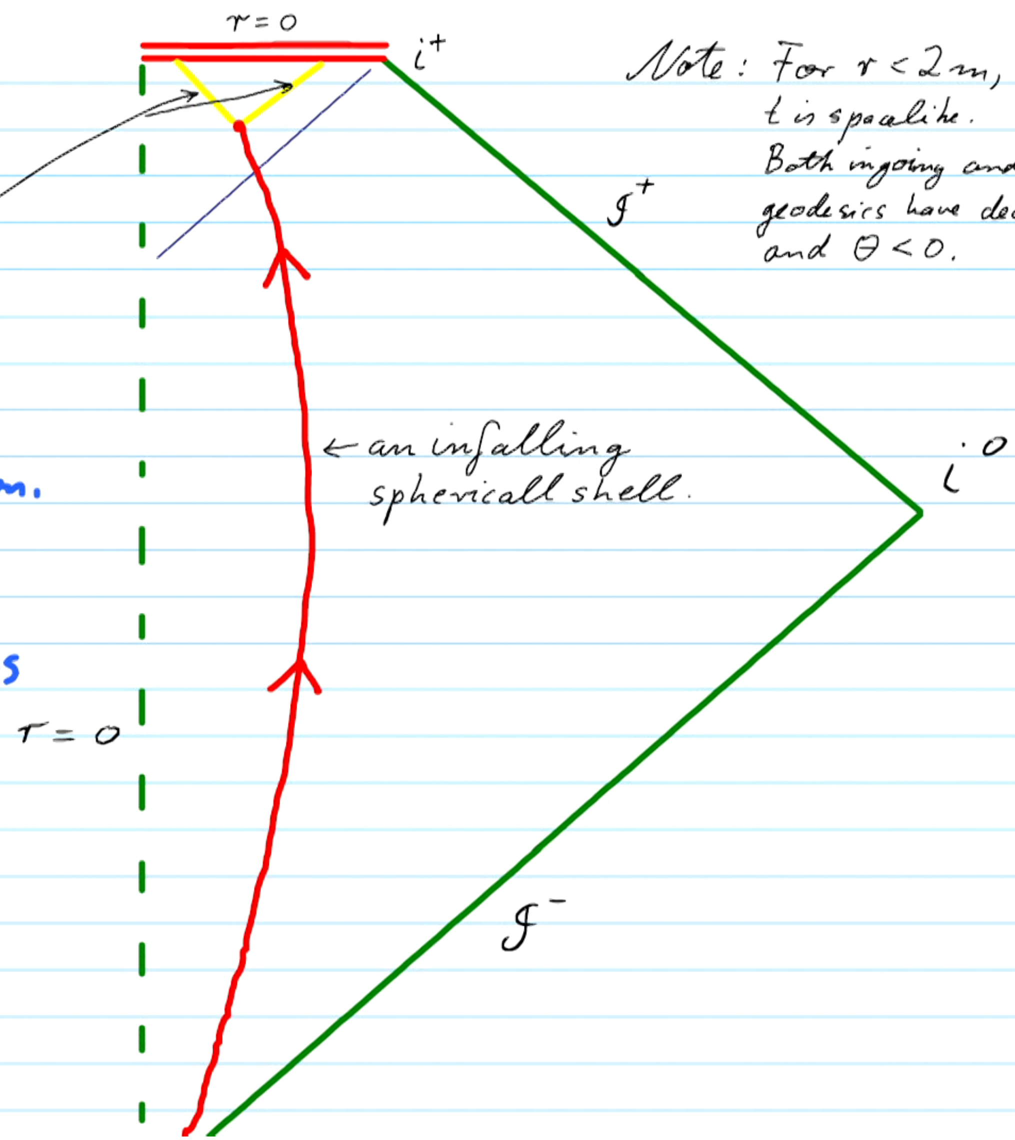
Examples of trapped surfaces:

All spheres $r = \text{const.}$ inside a Schwarzschild black hole.



Generally:

The in- and outgoing null geodesics both have negative expansion. Can't see it here b/c the neighboring geodesics are neighbors in the suppressed angular directions.



Note: For $r<2m$, r is timelike and t is spacelike.
Both ingoing and outgoing null geodesics have decreasing r, and $\theta<0$.

Def: Let Σ be a spacelike hypersurface.

Then, the (3-dim. spacelike) union, $\mathfrak{T}$, of all trapped surfaces $T \subset \Sigma$ is called the **trapped region** of Σ.

Def: The boundary $\partial \mathfrak{T} \subset \Sigma$ is called the **apparent horizon** of the spacelike hypersurface Σ.

Note: $\partial \mathfrak{T}$ is 2-dimensional and spacelike.

Def: If we foliate spacetime into spacelike hypersurfaces

$$\Sigma_\alpha, \alpha \in I \subset \mathbb{R}$$

each with its apparent horizon, $\mathfrak{T}_\alpha$, then their union

$$\mathcal{A} := \bigcup \mathfrak{T}_\alpha$$

Then, the (3-dim. spacelike) union, $\mathcal{T}$, of all trapped surfaces $T \subset \Sigma$ is called the trapped region of Σ.

Def: The boundary $\partial\mathcal{T} \subset \Sigma$ is called the apparent horizon of the spacelike hypersurface Σ.

Note: $\partial\mathcal{T}$ is 2-dimensional and spacelike.

Def: If we foliate spacetime into spacelike hypersurfaces

$$\Sigma_\alpha, \alpha \in I \subset \mathbb{R}$$

each with its apparent horizon, $\mathcal{T}_\alpha$, then their union

$$\mathcal{A} := \bigcup_\alpha \mathcal{T}_\alpha$$

is called the Trapping horizon of the spacetime.

Remarks:

- To check for the existence of an event horizon
  $\mathcal{J}^-$ (worldline to i^+)
  in principle requires knowledge of the full future.

- But one can check for the existence of an apparent horizon in any spacelike hypersurface by calculating the expansions only at that time!

- The notion of apparent horizons is dependent on the the choice of foliation of spacetime into spacelike hypersurfaces.

Remarks:

- To check for the existence of an event horizon

  $\mathcal{J}^-$ (worldline to i^+)

  in principle requires knowledge of the full future.

- But one can check for the existence of an apparent horizon in any spacelike hypersurface by calculating the expansions only at that time!

- The notion of apparent horizons is dependent on the the choice of foliation of spacetime into spacelike hypersurfaces.

- For static Schwarzschild black holes the event and apparent horizons coincide.

- Singularity theorems for black holes make assumptions of apparent horizons.

Comment:

Hawking radiation is usually thought to emanate from the apparent horizon.