Title: General Relativity for Cosmology - Lecture 19

Speakers: Achim Kempf

Collection: General Relativity for Cosmology (Kempf)

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# GR for Cosmology, Achim Kempf — Lecture 19

Note Title

## Classification of solutions of GR

(and along the way we will introduce some generally useful methods of group theory)

**Recall:**

- The task is to solve the equations of motion of matter, jointly with the Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- In practice, this problem must be simplified, i.e., the number of to-be-determined functions must be reduced.

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**Question:** How much can we weaken the symmetry assumptions of Friedmann-Lemaître and still get exact solutions?

**Strategy:**

- Classify cosmological models $(M, g)$, $T_{\mu\nu}$ by the amount and type of symmetry assumed.
- For each amount and type of symmetry assumed, try to find exact solutions or at least (asymptotic) properties of exact solutions.

**Remark:** Among the high symmetry models, some come arbitrarily close to F.L. at finite times!

See, e.g., text by Wainwright & Ellis.

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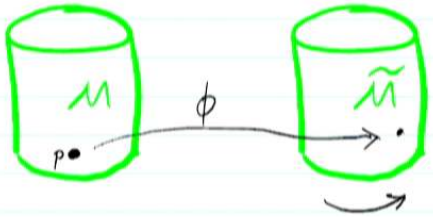
# Recall: Symmetries & Killing vector fields

- Two spacetimes $(\mathcal{M}, g)$, $(\tilde{\mathcal{M}}, \tilde{g})$ are isometric (and therefore of exactly identical shape) if there is a diffeomorphism $\phi: \mathcal{M} \to \tilde{\mathcal{M}}$ so that the image of the metric $g$ in $\tilde{\mathcal{M}}$ is $\tilde{g}$: $Tg = \tilde{g}$.
- A space-time has a symmetry, if we find such a $\phi$ for $\tilde{\mathcal{M}} = \mathcal{M}$.
- Example:




$\phi$ performs a rotation of $\mathcal{M}$ about a symmetry axis, to obtain $\tilde{\mathcal{M}} = \mathcal{M}$ with $Tg = \tilde{g}$.

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**Note:** The set of all symmetries of a manifold $(M, g)$ forms a "group":

**Definition:** A "group" $G$ is a set, with an operation, say "$\circ$",

$$\circ : G \times G \to G$$

and a "neutral element", say "$e$", $e \in G$, such that

$$(a \circ b) \circ c = a \circ (b \circ c) \qquad \forall a, b, c \in G$$

$$a \circ e = e \circ a = a \qquad \forall a \in G$$

$$\exists a^{-1}: \; a^{-1} \circ a = a \circ a^{-1} = e \qquad \forall a \in G$$

($\exists$ "there exists", $\forall$ "for all")

**Definition:** A group $G$ is called a **Lie group** if $G$ is also a finite-dimensional smooth manifold.

**Example:** The sets of rotations in $\mathbb{R}^3$ forms a 3-dimensional Lie group, $SO(3)$.

The angles $\alpha, \beta, \gamma$ are coordinates for elements $g \in SO(3)$.

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**Remarks:**
- The symmetries of a manifold $(M, g)$ can be discrete, such as reflections.
- But often, the symmetry group of a manifold $(M, g)$ is actually a Lie group.

**Note:**
- Each $h \in G$ yields an isometric diffeomorphism, by assumption.
  $h : M \to M$, namely $h : p \to h(p) \quad \forall p \in M$
- Consider the set $\mathcal{O}_p \subset M$ defined by: $\mathcal{O}_p := \{q \in M \mid \exists\, h \in G : h(p) = q\}$

**Definition:** The set $\mathcal{O}_p$ is called the **Orbit** of $p$ under the action of the group $G$.

**Note:** If $G$ is a Lie group then each orbit $\mathcal{O}_p$ is $p$ or a submanifold of $(M, g)$.

**Question:** What are the infinitesimal isometric diffeomorphisms?
And what type of mathematical structure do the infinitesimal symmetries form?

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**Recall:** The Lie derivative,

$$\mathcal{L}_\xi Q^{a\ldots b}{}_{c\ldots d} = Q^{a\ldots b}{}_{c\ldots d;k}\xi^k - Q^{k\ldots b}{}_{c\ldots d}\xi^a{}_{;k} - \ldots - Q^{a\ldots k}{}_{c\ldots d}\xi^b{}_{;k} + Q^{a\ldots b}{}_{k\ldots d}\xi^k{}_{;c} + \ldots + Q^{a\ldots b}{}_{c\ldots k}\xi^k{}_{;d}$$

yields the rate of change of a tensor $Q$ along the flow of diffeomorphisms $\phi$ generated by a vector field $\xi$.

⇝ Here, can use $\mathcal{L}_\xi$ to differentiate along symmetry group orbits.

Thus, if $\mathcal{L}_\xi g_{\mu\nu} = 0$

then $\xi$ generates isometries $\phi: \mathcal{M} \to \mathcal{M}$, $g \to \tilde{g} = g$.

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$\Rightarrow$ A vector field $\xi$ generates a symmetry of spacetime if it is a Killing vector field:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \qquad (X)$$

**Q:** Maximum number, $d$, of Killing vector fields in $n$ dims.?

**A:** $\boxed{d = n(n+1)/2}$ To see this, note that there are 2 ways to obey Eq. (X):

a) $\xi_{\mu;\nu} = 0 \;\forall \nu$, i.e. $\nabla\xi = 0$
Can have maximally $n$ such indep. vectors

b) $\nabla\xi \neq 0$, but then $K_{\mu\nu} := \xi_{\mu;\nu}$ is antisymmetric
Can have at most $n(n-1)/2$ indep. such cases.

$\Rightarrow d = n + n(n-1)/2 = n(n+1)/2$

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# From a symmetry Lie group to a "symmetry Lie algebra":

## General idea:

Normally the points of a manifold cannot be multiplied!

- A Lie group is a smooth manifold with extra structure: the multiplication.
- Notice: Product of group elements close to $1 \in G$ yields a group element close to $1$.
- Consider the tangent space $T_1(G)$ to the point $1 \in G$ of the Lie group manifold $G$.
- $T_1(G)$ is a vector space and it has extra structure, inherited from the group's multiplication.
- Define the Lie algebra of a group $G$ to be $T_1(G)$, equipped with the inherited "multiplication".

The identity element of the group, $p = 1$, is also a point of the group's manifold.
$T_1(G)$ is the tangent space to this point.

Crucial fact: From knowledge of only the Lie algebra, i.e., only $T_1(G)$ and

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## Crucial fact:

From knowledge of only the Lie algebra, i.e., only $T_1(G)$ and its "multiplication", the group $G$ can be constructed!

(though not always uniquely)

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## Summary of the big picture:

1. The symmetries of any $(M, g)$ form a group: they can be concatenated associatively, and all possess an inverse. (Recall: there can be discrete symmetries too.) Some symmetries are differentiable, parametrized by the flow $\Rightarrow$ the symmetries form a Lie group.

2. Each Killing vector field is the infinitesimal generator of a flow of isometric diffeomorphisms, i.e., of a symmetry.

3. We see here that the Killing vector fields indeed form a Lie algebra.

4. Recall that every Lie algebra generates a Lie group.

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# Surfaces of homogeneity and the isotropy subgroup:

- Definition:

  Let $r$ be the dimension of the Lie algebra, i.e., also the dimension of the Lie group of symmetries.

- Recall this definition:

  - Consider the set of points $\mathcal{O}(p)$ that a point $p$ can flow to along the Killing vector fields.

  - $\mathcal{O}(p)$ is called the **orbit** of $p \in M$ under the action of the symmetry group. We denote the dimension of the orbit by $s$.

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- **Clearly:**

  The dimension of an orbit cannot be larger than the dimension of the symmetry group, i.e.

  $$s \leq r,$$

  but $s < r$ easily happens:

- **Example:**
  - Consider $M := \mathbb{R}^2$ and $p = (0,0)$.
  - Then $r = r_{max} = n(n+1)/2 = \underline{3}$ is dim. of sym. group. ($n = 2$)
  - $\Rightarrow$ The three-dimensional Lie algebra of Killing vector fields is spanned by three Killing vector fields:

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$\Rightarrow$ The three-dimensional Lie algebra of Killing vector fields is spanned by three Killing vector fields:

Concretely:

$$K^{(1)} := \frac{\partial}{\partial x} \quad , \quad K^{(2)} := \frac{\partial}{\partial y}$$

$$K^{(3)} := y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

(Group elements generated by them are $e^{a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}}$ and they act as $e^{a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}} f(x,y) = f(x+a, y+b)$ by Taylor expansion. $\downarrow$)

Orbit of $p=(0,0)$:

$$\mathcal{O}(p) = \mathbb{R}^2 \text{ because generators } \frac{\partial}{\partial x} \text{ and } \frac{\partial}{\partial y}$$

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**Definition:**

We say that those Killing vector fields which do not generate a homogeneity surface (generalized translations), i.e., which generate a trivial group orbit for a point are generating the isotropy subgroup (generalized rotations) (of the full symmetry group generated by all Killing vectors).

**Dimension, $d$, of the isotropy subgroup?**

Clearly: $d = r - s$

($d$: isotropy; $r$: full; $s$: homogeneous.)

## Classification of cosmological models

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# Classification of cosmological models

- The classification is with respect to:
  - Dimension of isotropy subgroup $d$:

(# of conserved 'angular momenta') →

$d = 0, 1, 2, 3, 4, 5, 6$ ← e.g. full Lorentz symmetry

- $d = 0$: anisotropic case
- $d = 1$: at each pt one rotational symmetry axis
- $d = 3$: e.g. spatially isotropic case

  - Dimension of homogeneity surfaces $s$:

(# of conserved 'momenta') →

$s = 0, 1, 2, 3, 4$

- $s = 0$: inhomogeneous
- $s = 1$: homogeneous on 3dim orbits
- $s = 4$: homogeneous

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- A large body of literature exists on most cases of $(d, s)$:
  - Many exact solutions are known!
  - Many asymptotic behaviors are known!
  - Comprehensive text:
    Wainwright & Ellis, Dyn. systems in cosmology, Cambridge Univ. Press (1997)

- Examples:

| homogeneity $s$ | isotropy $d$ | |
|---|---|---|
| 4 | 3 | Einstein's static model |
| 4 | 1 | Gödel's model |
| 4 | 0 | Ozsvath-Kerr models |
| 3 | 3 | Friedmann-Lemaître models |
| 3 | 1 | spatially hom. & locally one rot. sym. axis |
| 3 | 0 | Bianchi models |
| ⋮ | ⋮ | |

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## Powerful alternative classification approach:

**Idea:** Classify the possible $T_{\mu\nu}$, then use Einstein equation to obtain classification of curvature.

**Proposition:**

For every physical energy momentum tensor $T_{\mu\nu}$ there exists a unique timelike vector field $u$ so that $T_{\mu\nu}$ takes this standard form:

$$T_{ab} = \underset{\text{scalar}}{\mu}\, u_a u_b + \underset{\text{vector}}{q_a}\, u_b + q_b\, u_a + \underset{\text{scalar}}{p}\,(g_{ab} + u_a u_b) + \underset{\text{tensor}}{\pi_{ab}}$$

where $q$ and $\pi$ are a vector field and a tensor field obeying:

$$q_a u^a = 0\,, \quad \pi_{ab} u^b = 0\,, \quad \pi_a{}^a = 0\,, \quad \pi_{ab} = \pi_{ba}$$

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## Segré classification:

- A systematic classification of $T_{\mu\nu}$ can be performed, by the analysis of its eigenvalues / eigenvectors. Nontrivial because:
- $T_{\mu\nu}$ is symmetric.
  But, the inner product in the vector space is $g_{\mu\nu} \Rightarrow T_{\mu\nu}$ is generally not hermitean!
- $T^{\mu}{}_{\nu}$ is in a space with the inner product $g^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$, but $T^{\mu}{}_{\nu}$ is generally not symmetric!

## Use Jordan normal form:

$\Rightarrow$ Segré classification yields 4 main

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Use Jordan normal form:

$\Rightarrow$ Segré classification yields 4 main types of energy momentum tensors $T_{\mu\nu}$.

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of possible curvatures.

**Indeed:** In 3+1 dimensions the Einstein equation also reads:

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

where $T = T^{\mu}{}_{\mu}$

**Exercise:** Prove this and notice the dimension-dependence

$\Rightarrow$ The 10 degrees of freedom of $T_{\mu\nu}$ (as a symmetric 4×4 matrix) determine the 10 degrees of freedom of $R_{\mu\nu}$.

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$\rightarrow$ The [illegible] of possible $T$ [illegible]

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$\Rightarrow$ The Segré classification of possible $T_{\mu\nu}$ yields, via the Einstein equation also a classification of possible Ricci tensors $R_{\mu\nu}$.

**Q:** Does this yield also a classification of the possible Riemann tensors $R^{\nu}{}_{\sigma\rho}$ ?

**A:** No! The Ricci tensor contains only 10 of the 20 degrees of freedom of the Riemann tensor! (In 3+1 dim)

**Prop.:** The information in $R^{\nu}{}_{\sigma\rho}$ is shared among the Ricci tensor $R_{\mu\nu}$ and the so-called Weyl tensor, $C^{\nu}{}_{\sigma\rho}$.

$\Rightarrow$ It remains to classify the possible Weyl tensors!

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$\Rightarrow C^{\alpha}{}_{\mu\nu\beta}$ contains all that curvature information which is not determined via the Einstein equation by $T_{\mu\nu}$.

$\Rightarrow$ $C^{\alpha}{}_{\mu\nu\beta}$ describes all that curvature which can exist even where there is no matter! (e.g.: gravity waves)

(also e.g. sun's gravity away from the sun in empty space)

**Proposition**

- Assume $(M, g)$ is a 3+1 dimensional Lorentzian manifold.
- Choose any smooth positive scala function $\phi$ on $M$.
- Define $(M, \tilde{g})$ with the new metric $\tilde{g}$ obtained through the "conformal transformation":

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) := \phi(x)\, g_{\mu\nu}(x)$$

Intuition:
Weyl curvature distorts ($\circ \to \diamond$)
but only Ricci curvature
shrinks or expands overall: ($\circ \to \circ$)

**Then:** $\tilde{C}^{\alpha}{}_{\mu\nu\beta}(x) = C^{\alpha}{}_{\mu\nu\beta}(x) \quad \forall x \in M$

(Exercise: what would be a proof strategy?)

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Then: $\tilde{C}^{\mu}{}_{\nu\alpha\beta}(x) = C^{\mu}{}_{\nu\alpha\beta}(x) \quad \forall x \in M$ (Exercise: what would be a proof strategy?)

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## Historical remark

- Consider the equivalence class of spacetimes $(M, \tilde{g})$ that are conformally equivalent to Minkowski space:

$$g_{\mu\nu}(x) = \phi^2(x)\,\eta_{\mu\nu}$$

- Einstein and Fokker initially considered a theory in which the metric possesses only this conformal degree of freedom $\phi$ (to play rôle of Newton's gravitational potential).

Newton gravity does come out correctly as a limiting case!

- Then, $S = \int_M R\sqrt{g}\,d^4x + \int_M \mathcal{L}_{matter}\sqrt{g}\,d^4x$ and $\frac{\delta S}{\delta \phi} = 0$ yield:

$$R = 8\pi G\, T^{\mu}{}_{\mu}$$

↳ In electromagnetism $T^{(EM)\mu}{}_{\mu} = 0$ i.e. EM fields would not gravitate.

No gravity waves here because $C^{ab}{}_{cd} = C^{ab\,(Minkowski)}{}_{cd} = 0$

- Equivalence principle ok.
- Light bending & Mercury perihelion shift wrong.

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- Thus $C^{\mu\nu}{}_{\rho\varepsilon}$ can locally be viewed as a symmetric map from the antisymmetric part $A_p(M)^2$ of $T_p(M)^2$ (so called bi-vectors) into itself:

$$C : A_p(M)^2 \to A_p(M)^2$$

- But, the inner product in $A_p(M)^2$ is not positive definite!

$\Rightarrow$ $C$ is generally not hermitean.

Therefore, use Jordan normal form again:

**Result:** 6 main Petrov classes for Weyl curvature: according to eigenvalue/eigenvector decomposition.

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according to eigenvalue / eigenvector decomposition.

Type O: Weyl curvature vanishes

Type D: "Static" Weyl curvature, e.g., in vicinity of a star.

Type N: Transverse gravitational waves, the type LIGO aims to detect. Like light, their strength decays $\sim \frac{1}{r}$ from the source.

Type I: Longitudinal gravitational waves

These waves cause a shear effect

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