

Title: 3D Mirror Symmetry and HOMFLY-PT Homology

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Series: Mathematical Physics

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Abstract: A recent construction of HOMFLY-PT knot homology by Oblomkov-Rozansky has its physical origin in $\mathcal{B}$ -twisted 3D $N=4$ gauge theory, with adjoint and fundamental matter. Mathematically, the construction uses certain categories of matrix factorization. We apply 3D Mirror Symmetry to identify an A-twisted mirror of this construction. In the case of algebraic knots, we find that knot homology on the A side gets expressed as cohomology of affine Springer fibers (related but not identical to work of Gorsky-Oblomkov-Rasmussen-Shende). More generally, we propose a Fukaya-Seidel category mirror to the Oblomkov-Rozansky matrix factorization. Joint work with N Garner, J Hilburn, A Oblomkov, and L Rozansky.

$$N = (2, 2)^u$$

2d B-model to $T^2 \text{Flag}_n$

$SL(n, \mathbb{C}) / \text{Borel}$

phys th depends on

$N-1$ (FI parameters)

or Kähler params

$$\vec{t} = (t_1, \dots, t_N) \quad (t_i \sim t_i + c)$$

$$N = (2, 2)^w$$

2d B-model to $T^2 \text{Flag}_n$

$SL(n, \mathbb{C}) / \text{Borel}$

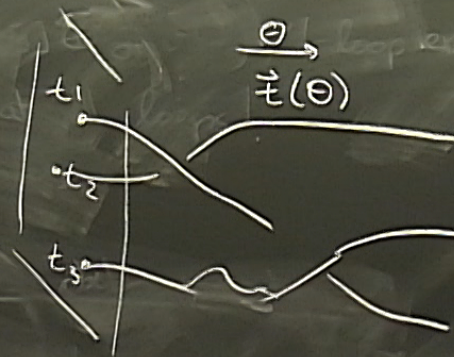
phys th depends on

$N-1$ (FI parameters)

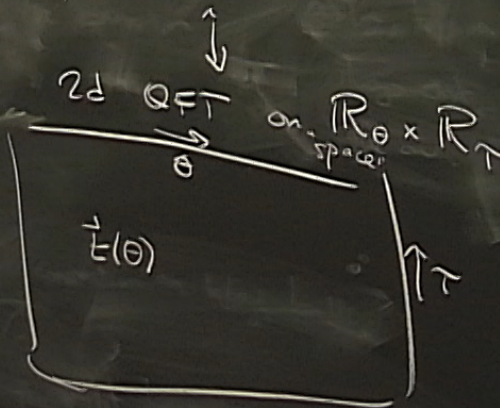
or Kähler params

$$\vec{t} = (t_1, \dots, t_n) \quad (t_i \sim t_i + c)$$

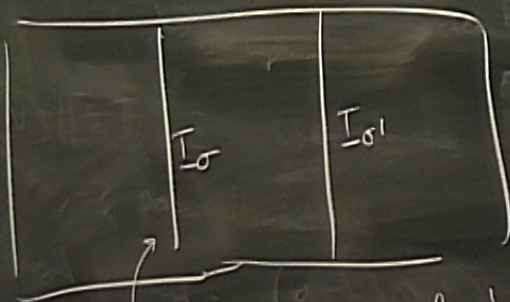
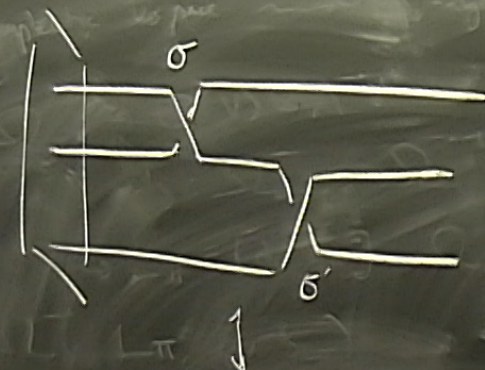
think: positions of n strands of braid



θ loop extent (Gm/int quantization)



Alg. geom:



$I_\sigma \in \text{Int cat of interfaces}$
 $= \text{Coh}(T^* \text{Flag}_n \times T^* \text{Flag}_n)$

Find: representation of Br_n
in Int

Note Int is a $\otimes$ category
(convolution)

Brn

$$S^1_0 \times \mathbb{R}_T$$

close up the braid



$$\text{Hilb}(\text{cylinder}) = \text{Hilb}(\text{circle})$$

dg rec space
invariant of the braid
closure

of B_n

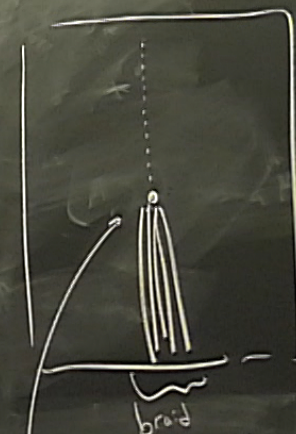
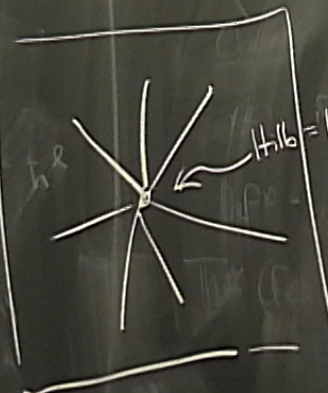
$S_0 \times \mathbb{R}_+$
close up the braid



$$\text{Hilb}(\text{cylinder}) = \text{Hilb}(\text{circle})$$

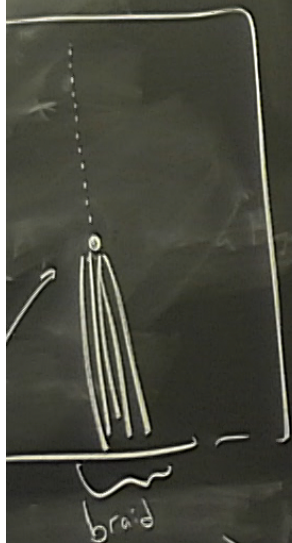
dg rec space
invariant of the braid
closure

State.op.



$$\text{Hom}_{\text{Int}}(\text{oooo}, \mathbb{1})$$

(mod Serre)



$000'0, 1)$
 $\uparrow$
 0_Δ
 (seel)

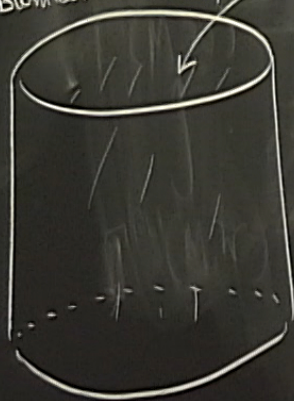
→ Jones

Want HOMFLY

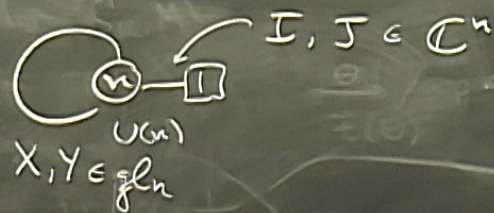
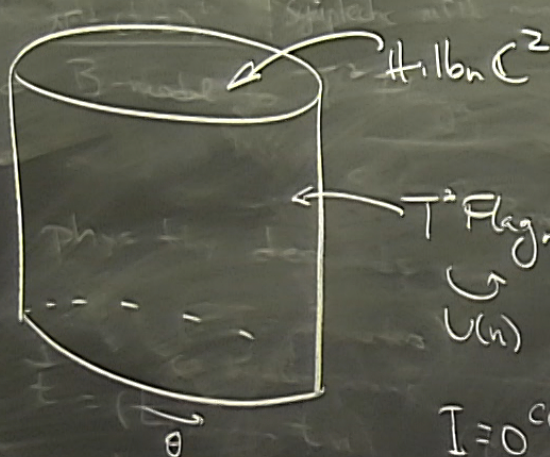
Comes from adding a bullet

Oblomkov-Rozansky:

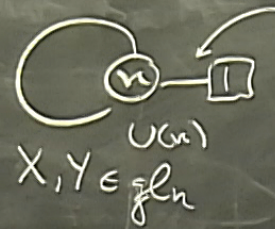
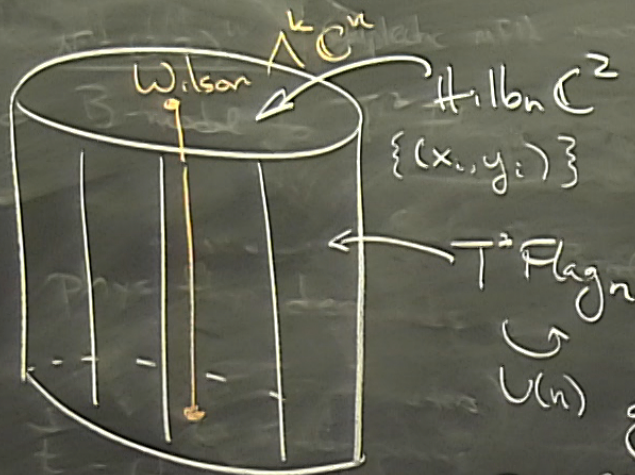
$3\frac{1}{2}N=4$ B-twist



$\textcircled{n} - \square$
 (σ-model to $\text{Hilb}_n(\mathbb{C}^2)$)



$U(n)$ gauged in
 $I=0$ coupling to bulk
 $X = \mu_c : TFlag : gl_n^{(c)}$



$I, J \in \mathbb{C}^n$

Expect



Conj (OR)
Hilb ()

braided bdy FI's

$I=0$ coupling to bulk

$X = \mu_C$

$T^*Flag \cdot gl_n^{(u)}$

a row of
HOMFLY-PT homology
(of braid closure)

$x_i = \text{evals}(\mu_C)$
 $y_i = 0$

Computation: done in
Int = cat of bdy interfaces
= Coh($T^*Flag_n \times T^*Flag_n$)
Hilbn $\mathbb{C}^2$

Hilbn $\mathbb{C}^2$
 $\{(x_i, y_i)\}$

$I, J \in \mathbb{C}^n$
 $U(n)$
 $X, Y \in \mathfrak{gl}_n$

Expect

$T^* \text{Flag}_n$

$U(n)$ gauged in

$I=0$ coupling to bulk

$X = \mu_C$

$T^* \text{Flag}_n \cdot \mathfrak{gl}_n^{(u)}$

$x_i = \text{evals}(\mu_C)$

$y_i = 0$

$M5$
 $M2$

Computation: done in

Int = cat of bdy interfaces

$= \text{Coh}(T^* \text{Flag}_n \times T^* \text{Flag}_n)$

$T^*(\mathfrak{gl}_n \times \mathbb{C}^n) // \text{GL}_n$

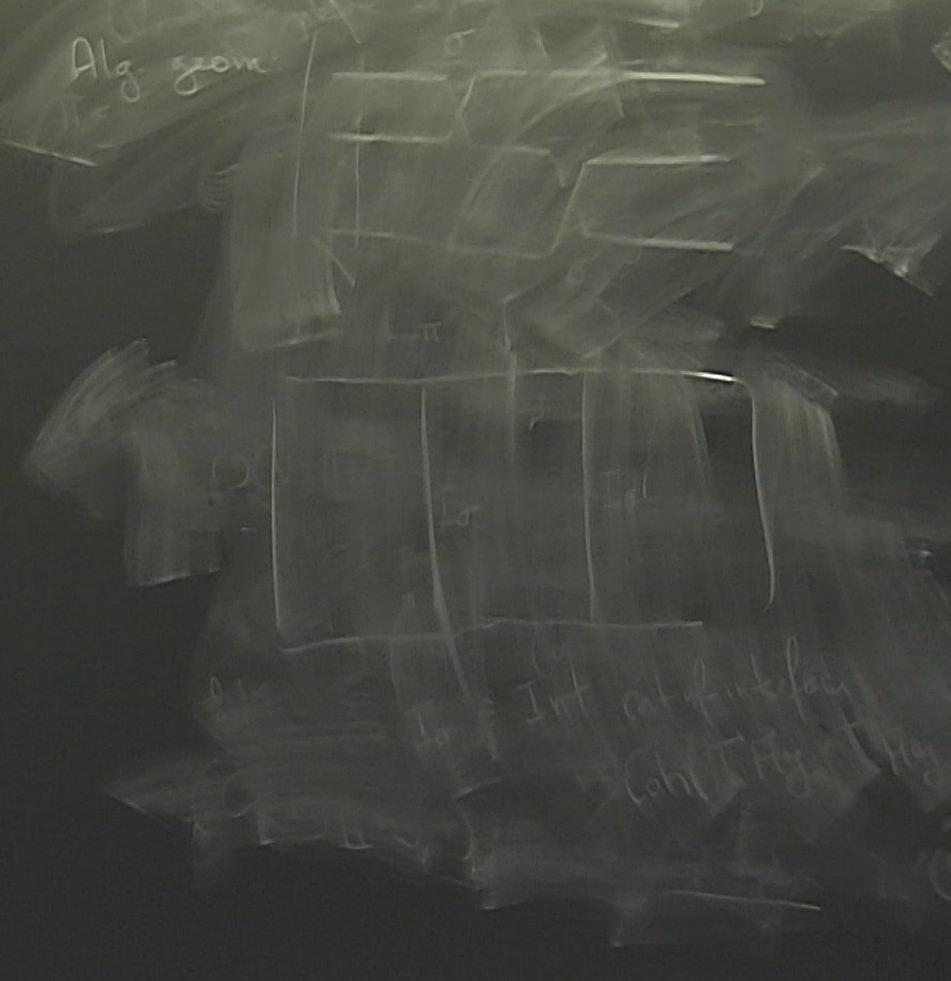
orig (OR)

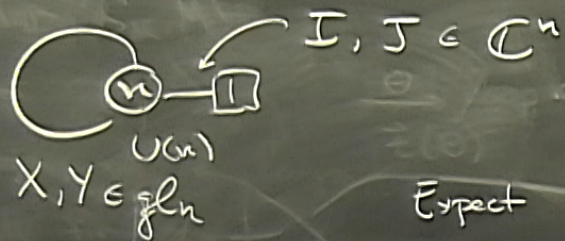
θ braid bdy FI's

Hilb () = a row of HOMFLY-PT homology (of braid closure)

$I, J \in \mathbb{C}^n$
 $\frac{\partial}{\partial z}$ (Goursat quadrilateral)
 Expect

 Computation: done in
 $\text{Int} = \text{cat of bdy interfaces}$
 $= \text{Coh}(T\text{Flag}_n \times T\text{Flag}_n)$
 $\frac{T^*(\text{gl}_n \times \mathbb{C}^n)}{\text{GL}_n}$
 $= \text{MF}(\dots)$

Alg. geom.




Expect



Flag $_n$

$U(n)$ gauged in

$I=0$ coupling to bulk

$X = \mu_C$
 $TFlag \cdot gl_n^{(u)}$

ow of
 /-PT homology
 closure

$x_i = \text{evals}(\mu_C)$

$y_i = 0$

Computation: done in
 $Int = \text{cat of bdy interfaces}$
 $= Coh(TFlag_n \times TFlag_n)$
 $(gl_n \times U^*)$

GL_n
 $= MF(\dots)$

Alg geom

3d mirror

3d Bulk

$$\bigcirc \textcircled{n} - \square$$

Now $M_c \simeq \text{Hilb}_n \mathbb{C}^2$

$= \text{Spec } A$

$$A = H^*(\text{Maps}(\mathbb{D}_{\mathbb{D}^2}^U, \mathbb{C}^n / G_n))$$

mon ops of fundamental weight for $U(n)$

$$\{(x_i, y_i)\}$$

Find representation of Br

A-twist

$\wedge^k \mathbb{C}^n$ Wilson line $\rightsquigarrow$ vortex line

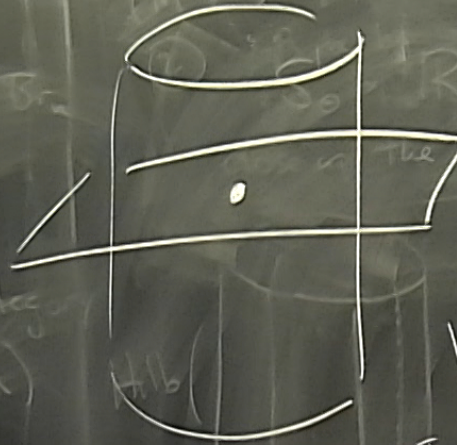
mon ops of fundamental weight for $U(n)$

$$\{(x_i, y_i)\}$$

$\mathbb{C}^2$

$$(\mathbb{D} \cup \mathbb{D}^*, \mathfrak{gl}_n \times \mathbb{C}^n / \mathfrak{gl}_n)$$

x line for



X, Y, I, J
behave holomorph

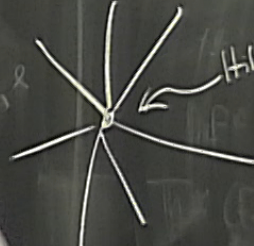
Vortex line V_k

I, J regular

$$X = \begin{pmatrix} k & z \\ 1 & z \end{pmatrix}$$

$$Y = \begin{pmatrix} k & \\ 1 & z \end{pmatrix} \begin{matrix} n-k \\ n-k \end{matrix}$$

State-op.

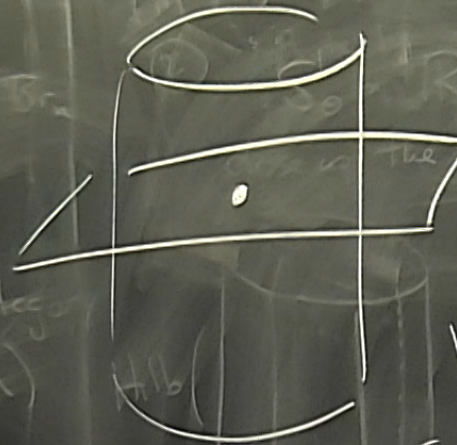


mon ops of fundamental weight for $U(n)$

$$\{(x_i, y_i)\}$$

$$(\mathbb{D} \cup \mathbb{D}^*, \mathfrak{gl}_n \times \mathbb{C}^n / \mathfrak{gl}_n)$$

x line for



X, Y, I, J
behave holomorphically

Vortex line V_k

Gauge gp near origin
is $GL_n[[z]]$

broken to P_k

$$P_k = \begin{pmatrix} & z \\ & \end{pmatrix}$$

$$X = \begin{pmatrix} k & z \\ & n-k \end{pmatrix}$$

$$Y = \begin{pmatrix} k & \\ \frac{1}{z} & n-k \end{pmatrix}$$

Stack op.

X, Y, I, J
behave holomorphically

Vortex line V_k

I, J regular

$$X = \begin{pmatrix} k & z \\ n-k & \end{pmatrix}$$

$$Y = \begin{pmatrix} k & \\ \frac{1}{z} & n-k \end{pmatrix}$$

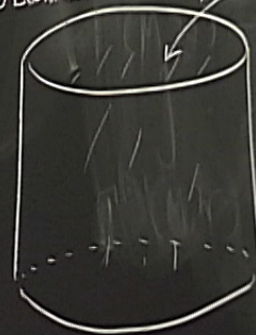
Physically: coupling to
 $T = G(k, n) \hookrightarrow U(n)$
 $N=2$ QM on line

→ Jones

Want HOMFLY

Comes from adding a bullet

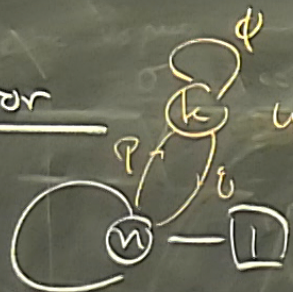
Oblomkov-Rozansky: $3d N=4$



$(\sigma\text{-model})$

3d mirror

3d Bulk



$$w_{1,2} = p\phi e + \chi \Big|_{z=0}$$

$$\frac{\partial \phi^L}{\partial p}$$

mon ops of fundamental weight for $U(n)$

$$\{(x_i, y_i)\}$$

$$\text{Now } M_C \approx \text{Hilb}_n \mathbb{C}^2$$

$$= \text{Spec } A$$

$$A = H^*(\text{Maps}(\mathbb{D}_{D^*}^U \mathbb{D}, gl_n \times \mathbb{C}^n / Gl_n))$$

A-twist

$\wedge^k \mathbb{C}^n$ Wilson line $\rightsquigarrow$ vortex line

Gauge gp near
is $GL_n(\mathbb{C})$

$$P_k = \begin{pmatrix} \end{pmatrix}$$

X, Y, I, J
behave holomorphically

Vortex line V_k

I, J regular

$$X = \begin{pmatrix} k & z \\ + & n-k \end{pmatrix}$$

$$Y = \begin{pmatrix} k & + \\ \frac{1}{z} & n-k \end{pmatrix}$$

Physically: coupling to
 $T^*G(k, n) \hookrightarrow U(n)$
 $N=2$ QM on line

Boundary condition: pure Dirichlet

$$J = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$X = \begin{pmatrix} t_1 & 0 \\ 0 & t_n \end{pmatrix}$$

→ Jones
 Want HOMFLY
 from adding a link

Link-Rozansky

X, Y, I, J
behave holomorphically

Vortex line V_k

I, J regular

$$X = \begin{pmatrix} k & z \\ + & n-k \end{pmatrix}$$

$$Y = \begin{pmatrix} k & + \\ \frac{1}{z} & n-k \end{pmatrix}$$

Physically: coupling to
 $T^*G(k, n) \hookrightarrow U(n)$
 $N=2$ QM on line

Bdy condⁿ pure Dirichlet

$$J = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$X = \begin{pmatrix} t_1 & 0 \\ 0 & t_n \end{pmatrix}$$

Lag in $gl_n \times \mathbb{C}^n$

→ Jones
Want HOMFLY
Comes from adding
Oblomkov-Rozensky

X, Y, I, J
behave holomorphically

Vortex line V_k

J regular

$$= \begin{pmatrix} k & z \\ n-k & \end{pmatrix}$$

$$Y = \begin{pmatrix} k & \\ \frac{1}{z} & n-k \end{pmatrix}$$

Physically: coupling to
 $T = G(k, n) \hookrightarrow U(n)$
 $N=2$ QM on line

Bdy condⁿ pure Dirichlet

$$J = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

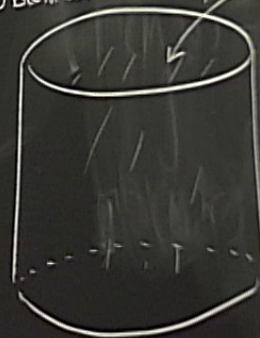
$$X = \begin{pmatrix} t_1 & 0 \\ 0 & t_n \end{pmatrix}$$

Lag in $T \otimes \ln \times \mathbb{C}^n$

→ Jones
Want HOMFLY

Comes from adding a link

Oblomkov-Rozensky



X, Y, I, J
behave holomorphically

Vortex line V_k

I, J regular

$$X = \begin{pmatrix} k & z \\ + & n-k \end{pmatrix}$$

$$Y = \begin{pmatrix} k & + \\ \frac{1}{z} & n-k \end{pmatrix}$$

Physically: coupling to
 $T^*G(k, n) \hookrightarrow U(n)$
 $N=2$ QM on line

Bdy condⁿ pure Dirichlet

$$I = +$$

$$I = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$Y = +$$

$$X = \begin{pmatrix} t_1 & 0 \\ 0 & t_n \end{pmatrix}$$

Note: $\text{Stab}_{G(n)}(\dots) = 1$

} Log in $T^*G(n) \times \mathbb{C}^n$

→ Jones
Want HOMFLY
Comes from
Oblomkov

Easy compⁿ of $\text{Hilb}(\mathbb{C}^n)$

Want for algebraic knots
 Can be expressed as braid w/
 positive twist.

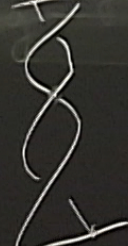
Given $\gamma(z) \in \mathfrak{gl}_n[z]$
 st evals of $\gamma(e^{i\theta})$ sweep out
 a braid closure.

$$\{\det(\gamma(z) - wI) = 0\} \subset \mathbb{C}_{zw}^2$$

Eg $n=2$

$$\gamma(z) = \begin{pmatrix} & 1 \\ z^3 & \end{pmatrix} \rightsquigarrow \text{trefoil}$$

evals: $\frac{z^{3/2}}{-z^{-3/2}}$

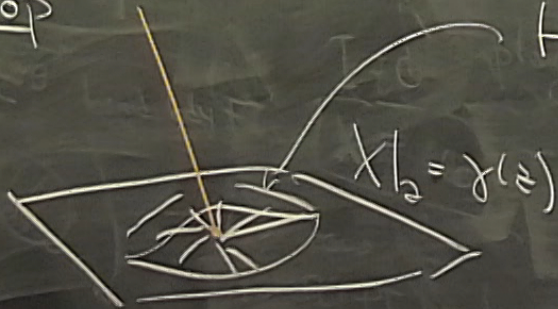


$\gamma(z)$ define a wrapped bdy condn

$$X|_{|z|=1} = \gamma(e^{i\theta})$$

$$\vec{I} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

State-op



$$I, J \in \mathbb{C}^n$$

For $k=0$ $\pi: GL_n(\mathbb{C}) \times g_n(\mathbb{C}) \times \mathbb{C}^n$

$$= H^*(GL_n(\mathbb{C}) \times \pi^{-1}(\gamma(z), \vec{I})) \times \mathbb{C}^n(z)$$

Hilb/space = $H^*(Maps(D, T(GL_n \times \mathbb{C}^n)/GL_n))$

satisfying the bdy condn and w/ singularity V_k at $z=0$

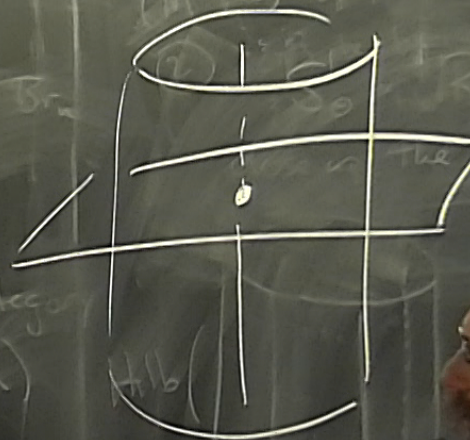
$$= \bigoplus_{d \in \mathbb{Z}} H^*(\dots |_{deg d})$$

$$d = deg(\det g(z))$$

Thm

Garner
-Kirichen

$$[GL_n \backslash \pi^{-1}(\gamma(z) \times \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix})]_{\deg d} = \text{Hilb}_d(\det(\gamma(z) - w) = 0)$$



Gauge gp near origin
is $GL_n(\mathbb{C})$
↓ broken to P_k
 $P_k = \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$

dw

ABJ

Thm

Garner - Kirichen = $\text{Hilb}_d \left(\left(\det(\gamma(z) - wI) = 0 \right) \right) \quad (k=0)$

$\left[GL_n(\mathbb{C}) \times \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right] \deg d$

Conj

$\text{Int}_A = \text{FS} \left(GL_n \times gl_n \times \frac{\mathbb{C}^n}{J} \right)$

$(\text{red } w) \quad g \quad Y \quad J$

$X = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ same as Var

$I = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$W_\alpha = \text{Tr} \left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} - g \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} g^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} Y \right)$

$+ J(g^{-1}) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$+ \alpha \text{Tr } Y^2$

$(k=0)$

$\mathbb{C}^n$

J

$r \left(\begin{pmatrix} 1 \\ \vdots \\ n \end{pmatrix} - g \begin{pmatrix} 1 \\ \vdots \\ n \end{pmatrix} g^{-1} \begin{pmatrix} 1 \\ \vdots \\ n \end{pmatrix} y \right)$

$+ J(g^{-1}) \begin{pmatrix} 1 \\ \vdots \\ n \end{pmatrix}$

$+ \alpha \sqrt{r} y^2$

$dW=0 : \left\{ g \in \text{Weyln}, J=0, \right\}$

$\simeq \text{Weyln}$

Physically: coupling

$T^* G(k,n)$

$N=2$ QM

Bdy condⁿ: Pure

$J=+$

$Y=+$

Note: St

ABJ

Thm

$$\left[GL_n(\mathbb{C}) \times \left(\begin{smallmatrix} 1 \\ \vdots \\ 1 \end{smallmatrix} \right) \right] \deg d$$

$$dW = 0$$

Garner
-Kirichen

$$= H^k(H, b_d \left(\det(y(z) - w) = 0 \right)) = a^0 \text{ row of HOMFLY-PT homology}$$

g-degree can d

Conj

$$Int_A = FS \left(GL_n \times gl_n \times \mathbb{C}^n, \begin{smallmatrix} (red w) g \\ y \\ J \end{smallmatrix} \right)$$

$$X = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \text{ same as } Y$$

$$I = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$W_\alpha = Tr \left(\left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} - g \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} g^{-1} \right) Y \right. \\ \left. + J(g^{-1}) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right. \\ \left. + \alpha Tr Y^2 \right)$$