Title: Equivariant localization and Atiyah-Segal completion for Hochschild and cyclic homology

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Abstract: There is a close relationship between derived loop spaces, a geometric object, and Hochschild homology, a categorical invariant, made possible by derived algebraic geometry, thus allowing for both intuitive insights and new computational tools. In the case of a quotient stack, we discuss a "Jordan decomposition" of loops which is made precise by an equivariant localization result. We also discuss an Atiyah-Segal completion theorem which relates completed periodic cyclic homology to Betti cohomology.

Plan

Review PL, categorification

② Hochschild homology, loop spaces

③ Equiv. loc.

④ Atiyah-Segal compl.

Note: $R = \mathbb{C}$.

(1)

Problem: Classify $\mathrm{Irr}_{\text{finite}}(\mathcal{H}^{\text{aff}})$.

- $\mathcal{H}^{\text{aff}}$ is deformation over $k[q, q^{-1}]$ of $\mathbb{C}[\tilde{W}^{\text{aff}}]$
- $\mathcal{H}^{\text{aff}}_q$ Iwahori-Hecke algebra. e.g. controls the rep thy of reps of ${}^{L}G(\mathbb{Q}_p)$ w/ Iwahori-fixed vector.

<u>Thm</u> (Ginzburg, Kazhdan, Lusztig)

$$\mathrm{Irr}_{f.d.}(\mathcal{H}^{\mathrm{aff}}) \longrightarrow \text{set of "tamely ramified" Langlands parameters} = \left\{ \begin{array}{ll} s \in G & \text{semisimple} \\ n \in \mathcal{N} & \text{nilpotent} \\ q \in \mathbb{C}^\times & \\ \multicolumn{2}{l}{\text{such that}} \\ sns^{-1} = qn & \end{array} \right\}.$$

Ideas in proof

Main players:

Springer resolution.

$\mu: T^*G/B = \tilde{\mathcal{N}} \longrightarrow \mathcal{N}$

$\{(x, \mathcal{F}) \in \mathcal{N} \times G/B \mid x \cdot \mathcal{F} \subseteq \mathcal{F}\}$

$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ Steinberg

Construct iso: $\mathcal{H}^{aff} \xrightarrow{\simeq} K_0(Coh(Z/G \times \mathbb{G}_m))$

$\mathcal{H}^{aff} \supseteq Z(\mathcal{H}^{aff})$, $K_0(Coh(Z/G \times \mathbb{G}_m)) \supseteq K_0(Coh(BG \times B\mathbb{G}_m))$

$Z(\mathcal{H}^{aff}) \xrightarrow{\simeq} K_0(Coh(BG \times B\mathbb{G}_m)) = k[T]^W[q, q^{-1}]$

To specify a central character is to choose closed point of
$G//G \times \mathbb{G}_m$.

<u>Equivariant localization</u>

$$\mathcal{H}^{aff}_{\hat{z}} = K_0(\mathrm{Coh}(Z/G\times\mathbb{G}_m))_{\hat{z}} \overset{E.L.}{\simeq} K_0(\mathrm{Coh}(Z^z/G^z\times\mathbb{G}_m))_{\hat{z}} \overset{A.S.}{\simeq} H^{BM}_\bullet(Z^z/G^z\times\mathbb{G}_m, \mathbb{C})$$

$$\simeq \mathrm{REnd}_{N^z/G^z\times\mathbb{G}_m}(\mu^z_* \mathbb{C}_{\tilde{N}^z/G^z\times\mathbb{G}_m})$$

uses yoga of 6 functors

<u>Decomposition theorem</u>. $\mu_z \mathbb{C}_{\tilde{\mathcal{N}}^z} = \bigoplus L_x \otimes IC_x[r_i]$

$x = (\mathcal{O}, \mathcal{L})$

$$\mathrm{End}(\mu_z \mathbb{C}_{\tilde{\mathcal{N}}^z}) / \text{radical} = \bigoplus \mathrm{End}(L_x)$$

L_x irreducibles of $\mathcal{H}^{aff}_z$

IC_x "irred. char sheaves"

<u>Categorification</u>

$$\mathcal{H}^{aff}_z\text{-mod} \simeq \langle \mathcal{S}_z = \mu_z \mathbb{C}_{\tilde{\mathcal{N}}^z} \rangle \hookrightarrow \mathrm{Perv}(\mathcal{N}^z).$$

Categorical Deligne Langlands + Equiv. loc + AS-compl.

(joint w/ Davids Ben-Zvi, Helm, Nadler)

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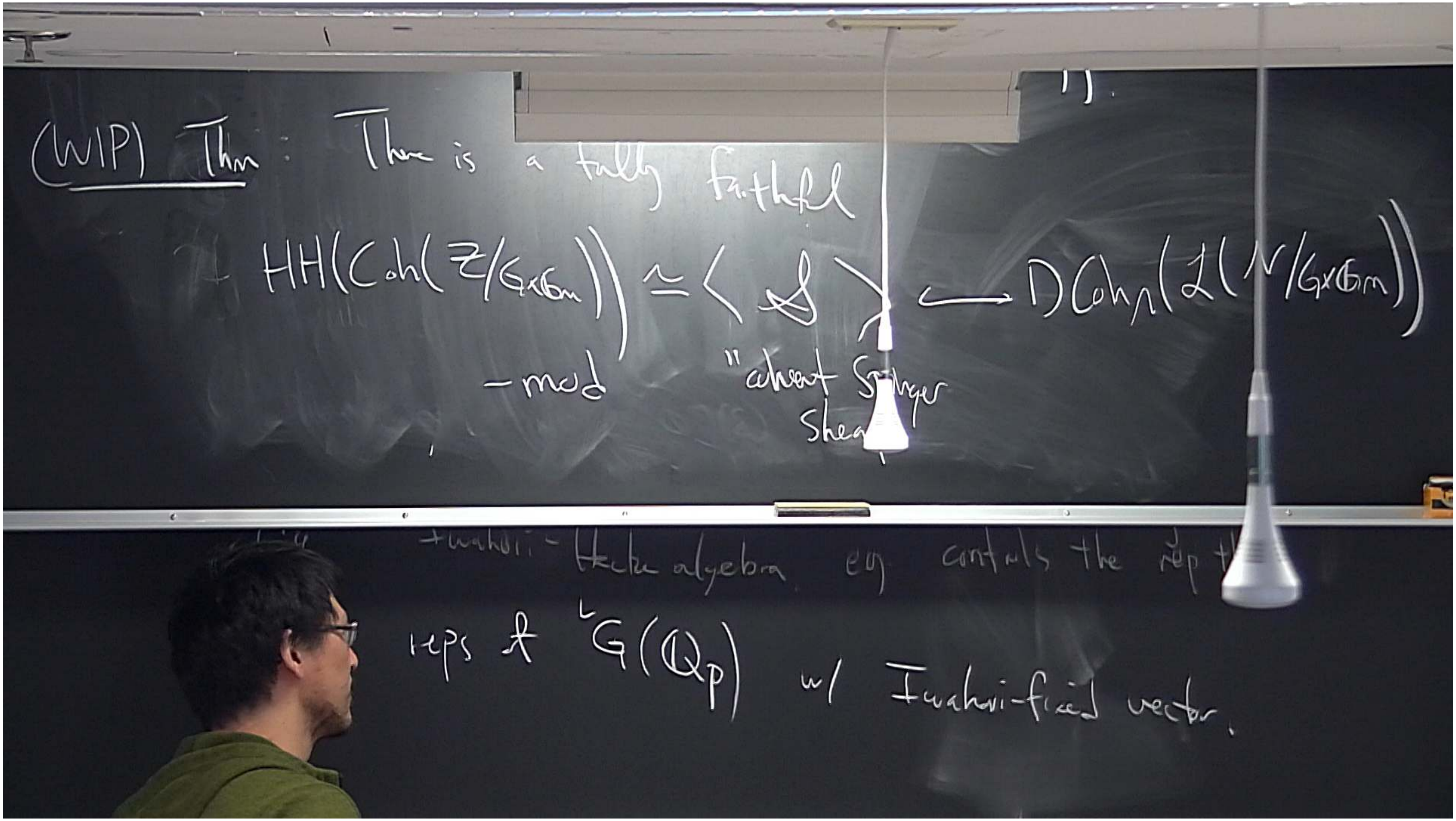
(WIP) Thm: There is a fully faithful
$HH(Coh(Z/G\times\mathbb{G}_m)) \simeq \langle S \rangle \hookrightarrow DCoh_\Lambda(\mathcal{L}(N/G\times\mathbb{G}_m))$
-mod
"coherent Springer sheaf"
Iwahori-Hecke algebra e.g. controls the rep th
reps of ${}^LG(\mathbb{Q}_p)$ w/ Iwahori-fixed vector.

where $\mathcal{L}(N/G\times\mathbb{G}_m)(\mathbb{C}) = \left\{ \begin{array}{l} n\in N(\mathbb{C}) \\ g\in G(\mathbb{C}) \\ q\in\mathbb{C}^\times \end{array} \;\middle|\; gng^{-1} = qn \right\}.$

moduli stack of tamely ramified Lang. param.

② Hochschild hom $\quad \mathcal{C}$ small dg category.

① vertical trace (Hoch. hom)

$HH(\mathcal{C}) = \mathcal{C} \otimes_{\mathcal{C} \otimes \mathcal{C}^{op}} \mathcal{C}$ $\quad$ when $\mathcal{A}$ is dg category,

$\mathcal{A}\text{-mod} = \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Ch}_k) = \hat{\mathcal{A}}$ $\quad$ "ind-completion"

$\mathcal{C}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$

② horizontal trace. (trace) $(\mathcal{C}, \otimes)$ monoidal categ.

$$\mathcal{C} \xrightarrow{tr} \mathrm{Tr}(\mathcal{C}) = \mathcal{C} \otimes_{\mathcal{C} \otimes \mathcal{C}^{op}} \mathcal{C}$$

$(\mathcal{A}, \otimes)$ monoidal category, $\mathcal{A}$-module category $\mathcal{M}$

$\mathcal{C} \otimes \mathcal{M} \to \mathcal{M}$.

Relationship: · If $\mathcal{C}$ monoidal, b/c HH monoidal, then $HH(\mathcal{C})$ is an algebra.

Example If $\mathcal{C} = A\text{-mod}$ for dg alg A, $HH(\mathcal{C}) = HH(A)$.

If S is generating set for $\mathcal{C}$,

$$HH(\mathcal{C}) = \bigoplus_{X_0 \in S} \mathrm{Hom}(X_0, X_0) \leftleftarrows \bigoplus_{X_0, X_1 \in S} \mathrm{Hom}(X_0, X_1) \otimes \mathrm{Hom}(X_1, X_0) \;\; \cdots$$

$$\begin{array}{cc} fg & gf \\ \uparrow & \uparrow \\ f \otimes g & \end{array}$$

Abstract characterization of HH: dimension of dualizable object in $\Pr^L_k$

Def $\mathcal{C} \in \Pr^L_k$ dualizable with dual $\mathcal{C}^\vee$

$$\dim(\mathcal{C}) : \mathrm{Ch}_k \xrightarrow{coev} \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{ev} \mathrm{Ch}_k$$

$$HH(\mathcal{C})(k)$$

Abstract characterization of HH: dimension of dualizable object in Pr^L_k

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$$HH(\mathcal{C}) = \dim(\mathcal{C})(k)$$

(2) If $\mathcal{C}$ geometric, e.g. $\mathcal{C} = QCoh(X)$ X global quotient stack in char 0.

$\mathcal{C}^\vee = QCoh(X)$ $\mathcal{C} \otimes \mathcal{C}^\vee = QCoh(X \times X)$

$$QCoh(*) \xrightarrow{\Delta_* p^*} QCoh(X \times X) \xrightarrow{p_* \Delta^*} QCoh(*)$$

$$\begin{tikzcd}
 & & LX \arrow[dl] \arrow[dr] & & \\
 & X \arrow[dl, "p"'] \arrow[dr, "\Delta"] & & X \arrow[dl, "\Delta"'] \arrow[dr, "p"] & \\
* & & X \times X & & *
\end{tikzcd}$$

Thm (Ben-Zvi, Francis, Nadler)

$$HH(\mathrm{Perf}(X)) = \Gamma(\mathcal{L}X, \mathcal{O}_{\mathcal{L}X}) = \mathcal{O}(\mathcal{L}X).$$

$$HH(\mathrm{Coh}(X)) = \omega(\mathcal{L}X).$$

<u>Def^n</u> The <u>derived loop space</u> of stack X is

$$\mathcal{L}X = \mathrm{Map}_{DSt}(S^1, X) = \mathrm{Map}(* \amalg_{S^0} *, X) = X \times_{X \times X} X$$

Ex ① $X = \mathrm{Spec}(A)$ scheme, affine

Then $\mathcal{L}X = X \times^{h}_{X \times X} X = \mathrm{Spec}\left(A \overset{L}{\otimes}_{A \otimes A} A\right) = \mathrm{Spec}\left(\underbrace{C_\bullet(A,A)}_{\text{cyclic bar complex, shuffle product.}}\right)$

$\overset{HKR}{\simeq} \mathrm{Spec}_X \, \mathrm{Sym}_X \, \Omega^1_X[1]$.

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②
$$\begin{tikzcd}
\mathcal{L}(X/G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \times G/G \arrow[d, "(a,p)"] \\
X/G \arrow[r, "\Delta"] & X \times X/G
\end{tikzcd}$$

$\pi_0(\mathcal{L}(X/G)) = I(X/G)$

$\mathbb{C}$-points: $\left\{ \begin{matrix} g \in G(\mathbb{C}) \\ x \in X(\mathbb{C}) \end{matrix} \;\middle|\; g \cdot x = x \right\}$.

Geometric

③ Equivariant localization

Def^n:

Ref:

$$HN(\mathcal{C}) = HH(\mathcal{C})^{S^1}$$

$$HC(\mathcal{C}) = HH(\mathcal{C})_{S^1}$$

$$HP(\mathcal{C}) = HH(\mathcal{C})^{S^1} \otimes_{k[[u]]} k((u))$$

Defn Note $\mathcal{L}(X/G)$ lives over $\mathcal{L}(BG) = G/G$, z semisimple.

G reductive.

z-formal loops

$$\begin{array}{ccc} \hat{\mathcal{L}}_z(X/G) & \longrightarrow & \mathcal{L}(X/G) \\ \downarrow & & \downarrow \\ \{G\cdot z\}/G & \longrightarrow & G/G \end{array}$$

z-unipotent loops

$Y \hookrightarrow X$ any G^z-closed subscheme

Retrn maps $l_z : \mathcal{L}(Y/G^z) \longrightarrow \mathcal{L}(X/G^z) \longrightarrow \mathcal{L}(X/G)$

$\hat{l}_z : \hat{\mathcal{L}}_z(Y/G^z) \longrightarrow \hat{\mathcal{L}}(X/G)$

$l^u_z : \mathcal{L}^u_z(Y/G^z) \longrightarrow \mathcal{L}^u(X/G)$

$\pi_0(X_0^Z) = \pi_0(X^Z)$, $\mathcal{L}(X_0^{\wedge}) = X$.

<u>Thm</u> The following have $(L\alpha)$. G reductive.

- X smooth, $X_0^Z = \pi_0(X^Z)$
- X, Y, W smooth, $(X \times_W Y)_0^Z = \pi_0(X^Z) \times_{\pi_0(W^Z)} \pi_0(Y^Z)$

Cor If (X, G) has (L_G) then:

$$HH(\mathrm{Perf}(X/G))^{\wedge}_{z} \simeq HH(\mathrm{Perf}(X^z/G^z))^{\wedge}_{z}$$

same is true for HN.

<u>Cor</u> If (X, G) has (L_G) then

$$HH(\mathrm{Perf}(X/G))^{\wedge}_{z} \cong HH(\mathrm{Perf}(X^z/G^z))^{\wedge}_{z}$$

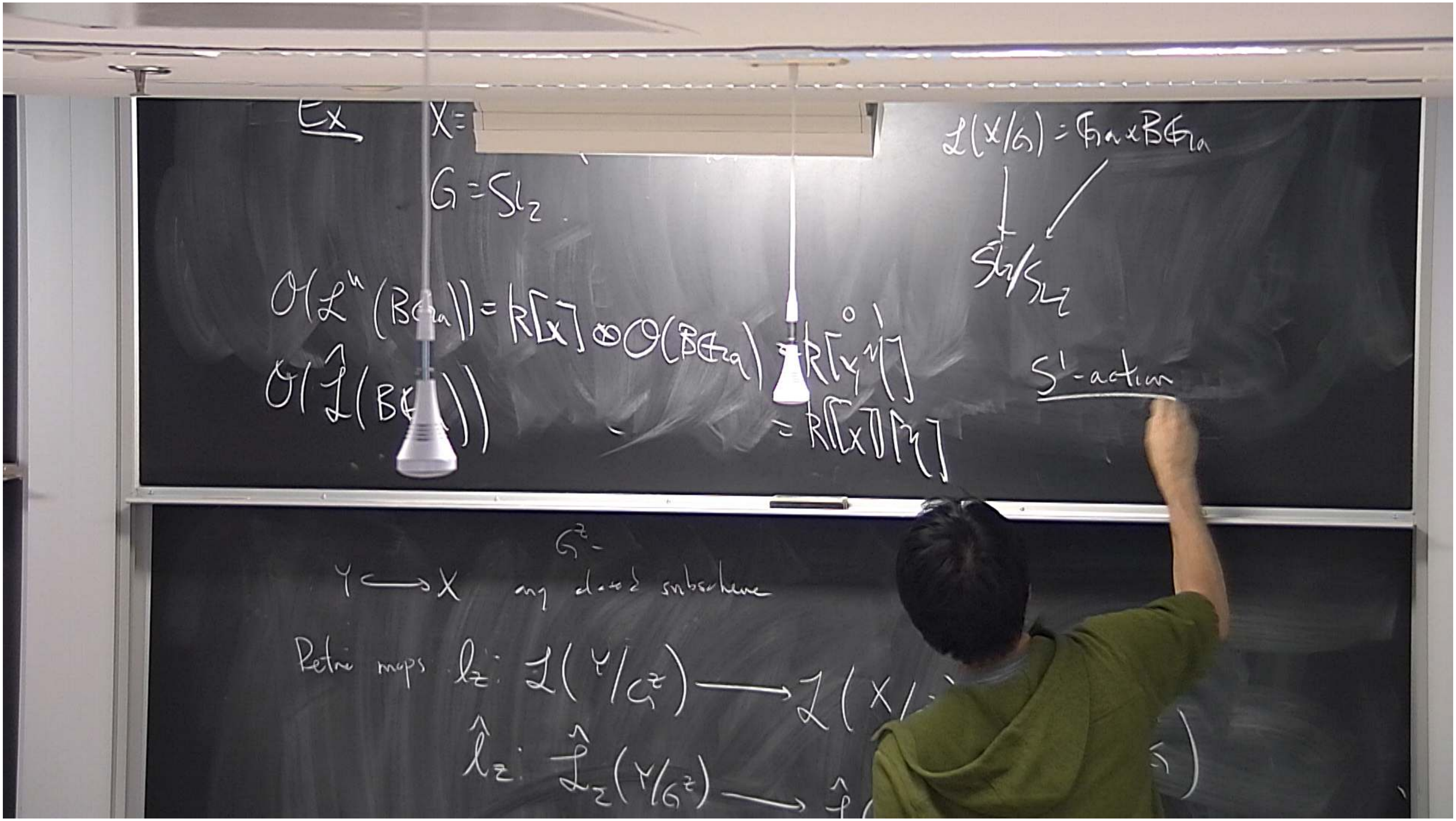
same is true for HN.

<u>Cor</u> If X is smooth, the same is true for HC, HP.

④ Atiyah-Segal completion

Thm If X stack, then

$$\mathcal{O}(\hat{\mathcal{L}}X)^{\mathrm{Tate}} \simeq C^{\bullet}_{dR}(X;\mathbb{C}) \,\hat{\otimes}\, k((u))$$


Ex
G = Sl_2
O(L^u(BGa)) = k[x] ⊗ O(BGa)
S^1-action
any closed subscheme
Retro maps

$\mathcal{L}(D(E_a))$ $\quad R[[x]][\hbar] \quad \eta \longmapsto x$

<u>Thm</u> X smooth, quasi-projective, $G \curvearrowright X$ reductive,

$$HP(\mathrm{Perf}(X/G))_{\hat{z}} \simeq C^{\bullet}_{dR}(X^{z}/G^{z}, \mathbb{C}) \,\hat{\otimes}\, \mathbb{C}((u))$$

$$\underline{HH(\mathrm{Coh}(Z/G \times \mathbb{G}_m))}\text{-mod} \hookrightarrow D\mathrm{Coh}(\sim)$$