

Title: Field Theory With a Vector Global Symmetry

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# Introduction

- Motivated by recent advances with fractons, but not sure this talk is relevant to fractons.
- Study global symmetries, whose conserved charge is a vector.
- One example is the momentum  $P^i = \int_{space} T_0^i$ . Its conserved Noether current  $T^{\mu\nu}$  is symmetric.
- Oneform symmetry is common in relativistic field theories.
- We'll present a hierarchical set of three symmetries, starting with the most special one and generalizing it.
- For each symmetry (for simplicity  $U(1)$ ) we will present:
  - the conserved currents
  - a class of field theories with the symmetry
  - a concrete example based on a scalar field theory

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## Relativistic oneform global symmetry

A lot of earlier work, here we follow [Gaiotto, Kapustin, NS, Willett]

$$\partial_\mu J^{[\mu\nu]} = 0$$

Charges

$$Q(\mathcal{C}) = \int_{\mathcal{C}} J^{[\mu\nu]} n_{[\mu\nu]}$$

$\mathcal{C}$  a codimension 2 manifold in spacetime orthogonal to  $n_{[\mu\nu]}$ .

$Q$  is topological – it does not change under small deformations of  $\mathcal{C}$ . Specifically, it does not change unless  $\mathcal{C}$  crosses another operator.

# Relativistic oneform global symmetry

In nonrelativistic terms

$$\begin{aligned}\partial_0 J_0^j - \partial_i J^{[ij]} &= 0 \\ G = \partial_i J_0^i &= 0\end{aligned}$$

Charges at fixed time

$$Q(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j$$

$\mathcal{C}$  a codimension 1 manifold in space orthogonal to  $n_j$ .

$Q$  is topological – it does not change under small changes of  $\mathcal{C}$ .

# Relativistic oneform global symmetry

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 $p, l, s$  are plaquettes, links, and sites

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In  $A_0 = 0$  the Hamiltonian is

$$\mathcal{H} = \sum_p (U_p + U_p^+) + \sum_l E_l^2$$

The variables are  $U(1)$  elements  $U_l$  and their conjugate momenta  $E_l$ .  $U_p = \prod_{l \subset p} U_l$  (oriented product)

Impose Gauss law  $G = \sum_{l \supset s} E_l = 0$  (oriented sum)

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$Q(\mathcal{C}) = \sum_{l \subset \mathcal{C}} E_l$  (oriented sum over the links pierced by  $\mathcal{C}$ ) is the electric flux through  $\mathcal{C}$ . It is conserved and topological.

## Nonrelativistic oneform global symmetry

As before,

$$\partial_0 J_0^j - \partial_i J^{[ij]} = 0 ,$$

but now we do not impose  $G = \partial_i J_0^i = 0$ .

The conservation leads to  $\partial_0 G = \partial_i \partial_0 J_0^i = 0$ ,

i.e.  $G$  is conserved at every point, but it is not zero.

As before, conserved charges at fixed time

$$Q(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j ,$$

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but now  $Q$  is not topological.

A cruder charge is  $Q^j = \int_{space} J_0^j$ .

Correspondingly, point operators can transform under  $Q(\mathcal{C})$ ,  $Q^j$ .

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## Nonrelativistic oneform global symmetry

$$\partial_0 J_0^j - \partial_i J^{[ij]} = 0$$

Therefore

$$\partial_0 G = \partial_i \partial_0 J_0^i = 0$$

$$Q(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j$$

If there is an  $\mathcal{M}$ , such that  $\mathcal{C} = \partial\mathcal{M}$

$$Q(\mathcal{C}) = \int_{\mathcal{M}} \partial_i J_0^i = \int_{\mathcal{M}} G$$

The conservation of  $Q(\mathcal{C})$  implies the conservation of  $G$ .

But the conservation of  $Q(\mathcal{C})$  contains more information, because  $\mathcal{C}$  might not be a boundary.

## Nonrelativistic oneform global symmetry

Lattice example:

$U(1)$  lattice gauge theory in  $A_0 = 0$ , but without imposing Gauss law  $G = \sum_{l \supset s} E_l = 0$  [Kitaev]

$$\mathcal{H} = \sum_p (U_p + U_p^+) + \sum_l E_l^2 + \sum_s \left( \sum_{l \supset s} E_l \right)^2$$

The last term imposes Gauss law energetically.

Interpret as  $U(1)$  gauge theory with charged matter at the lattice scale.

$G = \sum_{l \supset s} E_l$ , is conserved, but it is nonzero.

$Q(\mathcal{C}) = \sum_{l \subset \mathcal{C}} E_l$  are conserved, but they are not topological.

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Related discussion in [Hermele, Fisher, Balents; Williamson, Bi, Cheng].

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# Nonrelativistic oneform global symmetry

## A class of continuum examples

$$\partial_0 J_0^j - \partial_j J^{[ij]} = 0$$

Couple the  $U(1)$  gauge theory to charged matter fields, such that we still have

$$\begin{aligned} J_0^j &= F_0^j \\ J^{[ij]} &= F^{[ij]} \end{aligned}$$

For that, the matter theory should have an operator  $\mathcal{O}$  satisfying  $\partial_0 \mathcal{O} = 0$ .

The  $U(1)$  gauge theory couples to  $(J_0^{matter} = \mathcal{O}, J^{i matter} = 0)$

$$\mathcal{L}_1 = \mathcal{L}_0 + (F_0^i)^2 - (F^{ij})^2 + A_0 \mathcal{O}$$

with  $\mathcal{L}_0$  the matter Lagrangian. This is  $U(1)$  gauge invariant and has the nonrelativistic oneform symmetry with  $G = \partial_j J_0^i = \mathcal{O}$ .

# Nonrelativistic oneform global symmetry

## A class of continuum examples

$$\partial_0 \mathcal{O} = 0$$

Such a matter theory has infinitely many conserved charges.

$C(x^i)\mathcal{O}$  is conserved for every  $C(x^i)$ .

The charged matter is not mobile.

A concrete example:

A complex scalar field  $\Phi$  with

$$\mathcal{L}_0 = i\bar{\Phi}\partial_0\Phi - \partial_i(\bar{\Phi}\Phi)\partial^i(\bar{\Phi}\Phi) - |\Phi|^4 + \dots$$

No  $\partial_i\bar{\Phi}\partial^i\Phi$  term. Highly nonstandard. Similar to fractons.

Invariance under  $\Phi \rightarrow e^{iC(x^i)}\Phi$

The charge density  $\mathcal{O} = |\Phi|^2$  at a point is conserved.

## A more general vector symmetry

Previous case  $\partial_0 J_0^j - \partial_i J^{ij} = 0$

with antisymmetric  $J^{ij}$ .

Generalize to  $J^{ij}$  with no restriction on the symmetry of  $ij$ .

Now  $\partial_0 G = \partial_i \partial_0 J_0^i = \partial_i \partial_j J^{ij} \neq 0$ .

Therefore, the charge operators on codimension one manifold  $\mathcal{C}$   
 $Q(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j$  are no longer conserved, but the cruder charge

$$Q^j = \int_{space} J_0^j$$

is conserved.

# A more general vector symmetry

## A class of continuum examples

$$\partial_0 J_0^j - \partial_i J^{ij} = 0$$

For  $J^{(ij)} = 0$ , we coupled charged matter with  $\partial_0 \mathcal{O} = 0$  to a  $U(1)$  gauge field.

Now, we take a matter theory with

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0 \quad (\text{with } \mathcal{O}^{(ij)})$$

Couple the  $U(1)$  gauge field to  $(J_0^{matter} = \mathcal{O}, J^{i matter} = \partial_j \mathcal{O}^{ij})$

$$\mathcal{L}_1 = \mathcal{L}_0 + (F_0^i)^2 - (F^{ij})^2 + A_0 \mathcal{O} - A_i \partial_j \mathcal{O}^{ij}$$

The conserved current of the global symmetry are

$$J_0^j = F_0^j$$
$$J^{ij} = F^{[ij]} - \mathcal{O}^{ij}$$



# A more general vector symmetry

## A class of continuum examples

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0$$

Such a matter theory (before coupling to the gauge field) has the conserved charges

$$Q = \int_{space} J_0^{matter} = \int_{space} \mathcal{O}$$
$$Q^j = \int_{space} x^j J_0^{matter} = \int_{space} x^j \mathcal{O}$$

They can be interpreted as:

- a global  $U(1)$  symmetry (which we gauge)
- a vector symmetry, dipole symmetry, with charge  $Q^j$

After the gauging we are left only with the vector symmetry.

# A more general vector symmetry

## A class of continuum examples

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0$$

This defining equation is local and therefore, the discussion makes sense on every manifold.

This is not true for the charge,  $Q^j = \int_{space} x^j \mathcal{O}$ , which makes sense only on  $\mathbb{R}^D$ .

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After coupling to the  $U(1)$  gauge field the conserved current of the vector symmetry is

$$\begin{aligned} J_0^j &= F_0^j \\ J^{[ij]} &= F^{[ij]} - \mathcal{O}^{ij} \end{aligned}$$

No explicit  $x^j$  dependence.

## A more general vector symmetry

### A concrete continuum example [Pretko]

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0$$

A complex scalar field  $\Phi$  with

$$\begin{aligned} \mathcal{L}_0 = & i\bar{\Phi}\partial_0\Phi - \partial_i(\bar{\Phi}\Phi)\partial^i(\bar{\Phi}\Phi) - |\Phi|^4 \\ & + i(\bar{\Phi}^2\partial_i\Phi\partial^i\Phi - \Phi^2\partial_i\bar{\Phi}\partial^i\bar{\Phi}) + \dots \end{aligned}$$

Again, no  $\partial_i\bar{\Phi}\partial^i\Phi$  term.

The new term  $i(\bar{\Phi}^2\partial_i\Phi\partial^i\Phi - \Phi^2\partial_i\bar{\Phi}\partial^i\bar{\Phi})$  breaks the symmetry  $\Phi \rightarrow e^{iC(x^i)}\Phi$  to  $\Phi \rightarrow e^{i\alpha + ic_i x^i}\Phi$ .

Here

$$\begin{aligned} \mathcal{O} &= |\Phi|^2 \\ \mathcal{O}^{ij} &= |\Phi|^4 \delta^{ij} + \dots \end{aligned}$$

Higher order terms can lead to a traceless symmetric tensor in  $\mathcal{O}^{ij}$ .

# Gauging

Start with a theory with  $\partial_0 J_0^j - \partial_i J^{ij} = 0$ .

We gauge by introducing sources  $B_{0j}, \mathcal{A}_{ij} = \frac{1}{2}(B_{[ij]} + S_{(ij)})$ .

(No need to introduce  $S_{(ij)}$  when  $J^{(ij)} = 0$ .)

Add to the Lagrangian the minimal coupling terms

$$B_{0j}J_0^j - \mathcal{A}_{ij}J^{ij}$$

Invariance under the gauge symmetry

$$B_{0i} \rightarrow B_{0i} + \partial_0 c_i$$

$$\mathcal{A}_{ij} \rightarrow \mathcal{A}_{ij} + \partial_i c_j$$

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- No  $c_0$
- $\mathcal{A}_{ij}$  has no symmetry in  $ij$  – peculiar gauge field
- $c_i$  is not a  $U(1)$  gauge field

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# Gauging

$$B_{0i} \rightarrow B_{0i} + \partial_0 c_i$$

$$\mathcal{A}_{ij} \rightarrow \mathcal{A}_{ij} + \partial_i c_j$$

We can also add kinetic terms for these fields using the gauge invariant field strengths

$$\epsilon_{ij} = \partial_0 \mathcal{A}_{ij} - \partial_i B_{0j}$$

$$\beta_{ijk} = \partial_i \mathcal{A}_{jk} - \partial_j \mathcal{A}_{ik}$$

The standard  $H_{\mu\nu\rho}$  are linear combinations of them

$$H_{0ij} = \epsilon_{ij} - \epsilon_{ji}$$

$$H_{ijk} = \beta_{ijk} - \beta_{ikj} + \beta_{jki}$$



# Gauging

Let us implement it explicitly in the  $U(1)$  theory coupled to a matter system. We started with a matter theory with

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0$$

Above we gauged an ordinary  $U(1)$  with

$(J_0^{matter} = \mathcal{O}, J^{i matter} = \partial_j \mathcal{O}^{ij})$ , i.e.

$$\mathcal{L}_1 = \mathcal{L}_0 + (F_0^i)^2 - (F^{ij})^2 + A_0 \mathcal{O} - A_i \partial_j \mathcal{O}^{ij}$$

The remaining global symmetry

$$\begin{aligned} J_0^j &= F_0^j \\ J^{[ij]} &= F^{[ij]} - \mathcal{O}^{ij} \end{aligned}$$

Now, gauge this remaining vector global symmetry.

# Gauging

$$J_0^j = F_0^j$$
$$J^{ij} = F^{[ij]} - \mathcal{O}^{(ij)}$$

Add new gauge fields  $B_{0j}$ ,  $\mathcal{A}_{ij} = \frac{1}{2}(B_{[ij]} + S_{(ij)})$  with minimal coupling

$$\mathcal{L}_2 = \mathcal{L}_0 + (F_0^i + B_0^i)^2 - (F^{ij} + B^{ij})^2 + A_0 \mathcal{O} + A_{ij} \mathcal{O}^{ij} + \dots$$

where  $A_{ij} = \frac{1}{2}(S_{ij} + \partial_i A_j + \partial_j A_i)$ .

The  $U(1)$  gauge symmetry acts as

$$A_0 \rightarrow A_0 + \partial_0 \alpha$$

$$A_i \rightarrow A_i + \partial_i \alpha$$

$$A_{ij} \rightarrow A_{ij} + \partial_i \partial_j \alpha$$

In addition, we also have the gauge symmetry...

# Gauging

Additional gauge symmetry

$$\begin{aligned}A_i &\rightarrow A_i + c_i \\B_{0i} &\rightarrow B_{0i} + \partial_0 c_i \\B_{ij} &\rightarrow B_{ij} + \partial_i c_j - \partial_j c_i\end{aligned}$$

Note,  $A_{ij} = \frac{1}{2}(S_{ij} + \partial_i A_j + \partial_j A_i)$  is invariant under  $c_i$

Can add kinetic terms using

$$\begin{aligned}H_{0ij} &= \partial_0 B_{ij} - \partial_i B_{0j} + \partial_j B_{0i} \\H_{ijk} &= \partial_i B_{jk} - \partial_j B_{ik} + \partial_k B_{ij} \\E_{ij} &= \partial_0 A_{ij} - \partial_i \partial_j A_0 \\B_{ijk} &= \partial_i A_{jk} - \partial_j A_{ik}\end{aligned}$$

$A_i$  is “Higgsed” and is lifted (becomes massive) with  $B_{i0}$  and  $B_{ij}$ .

We are left with  $A_0$  and  $A_{ij}$ .

# Gauging

We can do the gauging in one step. (Similar to discussions in [Rasmussen, You, Xu; Pretko; Slagle, Prem, Pretko;...].)

We started with a matter theory with

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0$$

We couple these operators to sources

$$\mathcal{L}_0 + A_0 \mathcal{O} + A_{ij} \mathcal{O}^{ij} + \dots$$

Hence, there is a  $U(1)$  gauge symmetry

$$A_0 \rightarrow A_0 + \partial_0 \alpha$$

$$A_{ij} \rightarrow A_{ij} + \partial_i \partial_j \alpha$$

And we can add kinetic terms using

$$\mathcal{E}_{ij} = \partial_0 A_{ij} - \partial_i \partial_j A_0$$

$$\mathcal{B}_{ijk} = \partial_i A_{jk} - \partial_j A_{ik}$$

## Summary

- A hierarchy of global symmetries, whose charges  $Q^i$  carry a spatial vector index.
- For every one of these we presented a large class of theories exhibiting them.
  - All these examples are based on a  $U(1)$  gauge theory coupled to a special matter theory
  - We showed concrete examples of these matter theories
- We can gauge these new symmetries. The needed gauge field is an antisymmetric tensor  $B_{[\mu\nu]}$  and in the general symmetry we also need a symmetric tensor  $S_{(ij)}$ .