Title: A Universal Operator Growth Hypothesis

Speakers: Thomas Scaffidi

Series: Condensed Matter

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Abstract: Thanks to the Lanczos algorithm, the Hamiltonian dynamics of any operator can be written as a hopping problem on a semi-infinite one-dimensional chain. Our hypothesis states that the hopping strength grows linearly down the chain, with a universal growth rate α that is an intrinsic property of the system. This leads to an exponential motion of the operator down the chain, capturing the irreversible process of simple operators inevitably evolving into complex ones. This exponential growth exists for generic quantum systems, even away from large-N or semiclassical limits. In fact, α gives an upper bound for the exponential growth rate of a large class of operator complexity measures, including out-of-time-order correlations. As a result, we conjecture a new bound on Lyapunov exponents $\lambda_L \leq 2 \alpha$, which generalizes the known universal low-temperature bound $\lambda_L \leq 2 \pi T$. We illustrate the hypothesis in paradigmatic examples such as non-integrable spin chains, the q-SYK model, and chaotic coupled top models, and show that some of them saturate the conjectured bound.

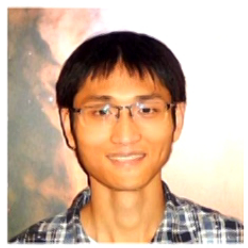
A Universal Operator Growth Hypothesis

Thomas Scaffidi
Perimeter Institute
October 2019

arXiv:1812.08657, soon to appear in Phys Rev X

Collaborators:

Daniel Parker

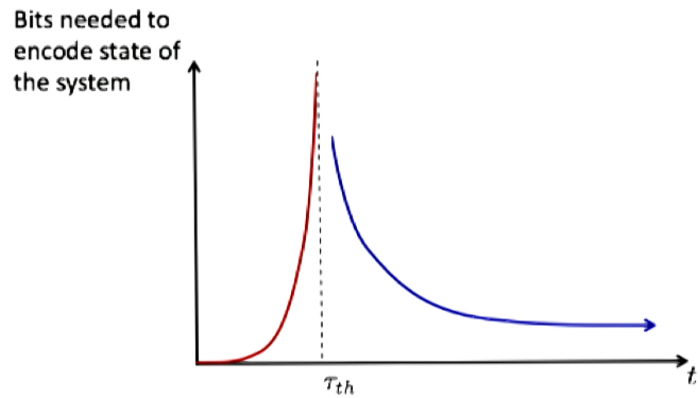


Xiangyu Cao

Alex Avdoshkin

Ehud Altman

Short-time and late-time behavior

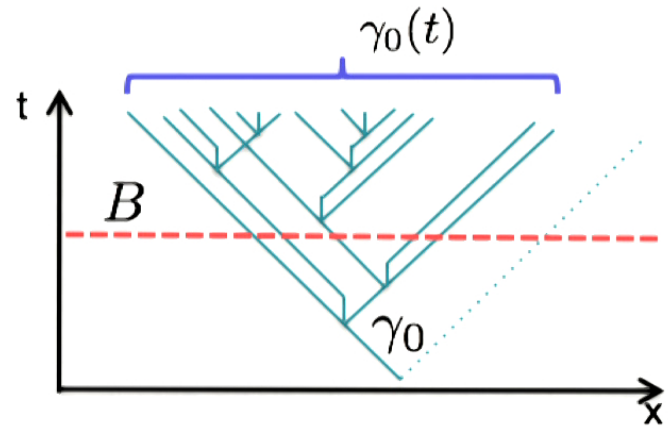


Thermalization
Chaos
Lyapunov exponents

Hydrodynamic regime
Diffusion constants

Recent focus: operator dynamics

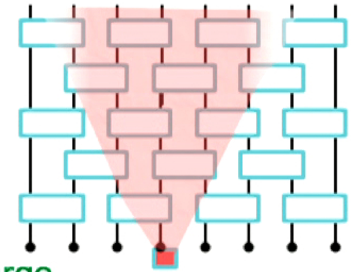
- Operator size; Lyapunov exponents (measured by OTOC)
- Operator complexity
- Emergent Hydrodynamics (Diffusion constants)



Is there a universal structure that governs and relates these quantities in generic systems?

Can we utilize such a structure to enable computation of dynamics? (e.g. compute transport coefficients of strongly coupled systems.)

Recent progress from special models



Random unitary networks:

Nahum (2018), von Keyserlingk (2018) Khemani (2018)

✔ Local model and finite N per site
Universal operator front propagation.
Emergent dissipation / diffusion of conserved charge

✗ Non Hamiltonian dynamics.
No energy conservation or notion of temperature.
Lyapunov exponents not well defined

SYK model

Sachdev, Kitaev, Stanford-Maldacena, …

✔ Hamiltonian dynamics, Lyapunov exponents;
Some connections to energy transport.

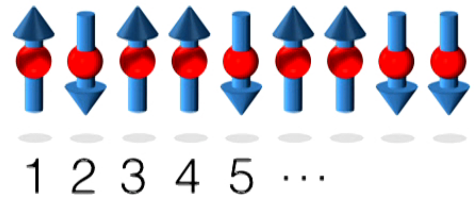
✗ Non generic features: Large-N / non locality

This talk

- **Preliminaries: operator dynamics, recursion methods**
- **A hypothesis for universal operator growth**
- **Evidence for the hypothesis:**
  (i) Numerical (Spin chains)
  (ii) Analytical (SYK models)
  (iii) Physical arguments
- **Application: new bound on chaos**
- **Application: computational approach**
  Transport coefficients in strongly coupled systems

Focus on models with finite-dimensional Hilbert space per site

Example: Quantum Ising chain

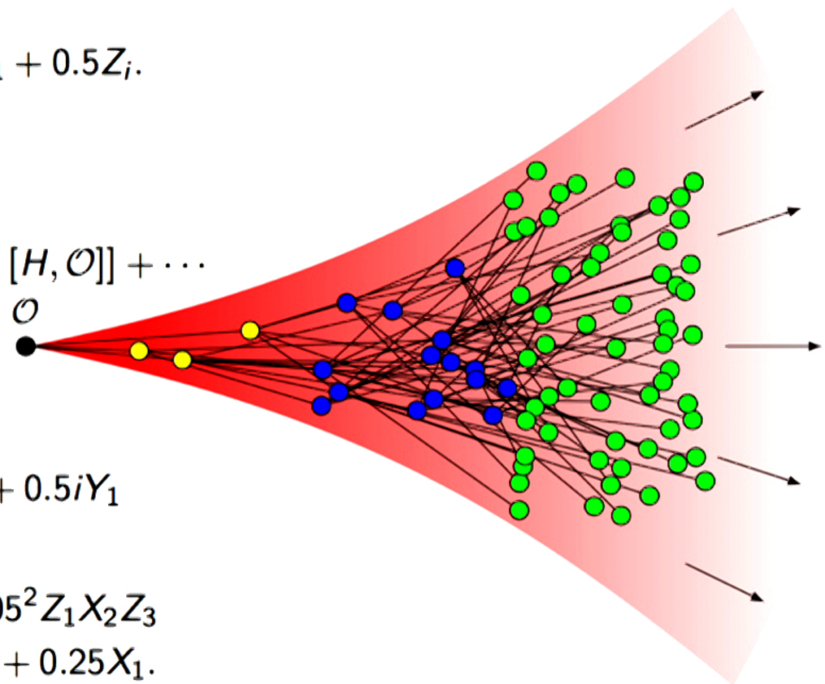


Let's consider an example. Suppose $\mathcal{O} = X_1$,

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i.$$

We know

$$\mathcal{O}(t) = e^{-iHt}\mathcal{O}e^{iHt}$$
$$= \mathcal{O} - it[H,\mathcal{O}] + (-it)^2[H,[H,\mathcal{O}]] + \cdots$$



Let's compute!

$$\mathcal{O} = X_1$$
$$[H,\mathcal{O}] = 1.05iY_1Z_2 + 1.05iZ_1Y_2 + 0.5iY_1$$
$$[H,[H,\mathcal{O}]] = 2.1Z_1Z_2 - 2.1Y_1Y_2$$
$$+ 1.05^2X_1 + 1.05^2X_2 + 1.05^2Z_1X_2Z_3$$
$$+ 0.525X_1Z_2 + 0.525Z_1X_2 + 0.25X_1.$$

$\cdots$

Space of operators

operators are "rounded" kets $|\mathcal{O})$

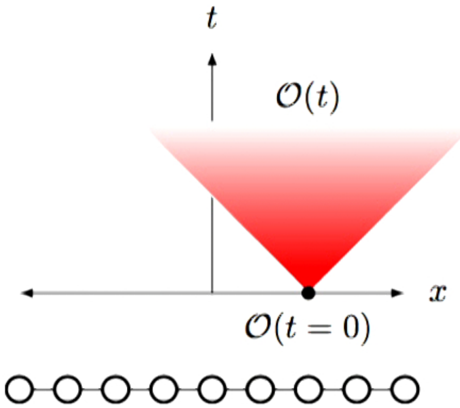
an example is $|O) = X_1 \otimes Y_2 \otimes Z_3 + 0.3 Y_1 \otimes X_2$

the inner product is $(A|B) := \mathrm{Tr}[A^\dagger B] / \mathrm{Tr}[1]$

the Liouvillian generalizes the Hamiltonian $\mathcal{L} = [H, \cdot]$.

time-evolution from Heisenberg EOM $-i \frac{d|\mathcal{O})}{dt} = \mathcal{L}\,|\mathcal{O})$.

solution $|\mathcal{O}(t)) = e^{i\mathcal{L}t}\,|\mathcal{O})$



A convenient basis in the space of operators: the Krylov basis

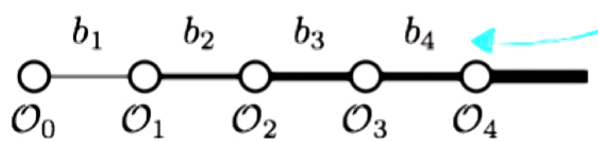
$$\{|\mathcal{O}_n)\} = \text{Gram Schmidt}\,[\mathcal{L}^n\,|\mathcal{O})]$$

$\{\mathcal{O}_n\}$ orthonormal,
$\forall\, n$, $\text{span}(\mathcal{O}_0, \ldots, \mathcal{O}_n)$
$=\text{span}(O, \mathcal{L}O, \ldots, \mathcal{L}^n\mathcal{O})$

The Liouvillian is *tridiagonal* under the Krylov basis $\{\mathcal{O}_n\}_{n=0}^{\infty}$

$$(\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Lanczos coefficients

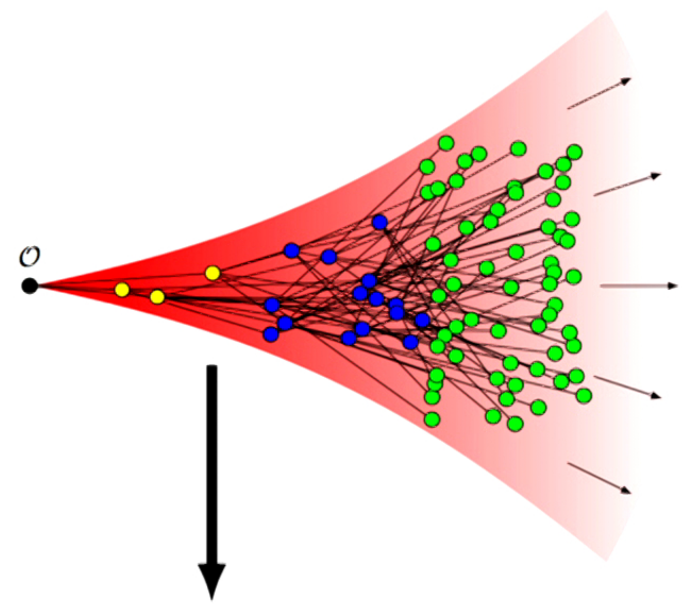


Mattis, 1981
Physics in 1d

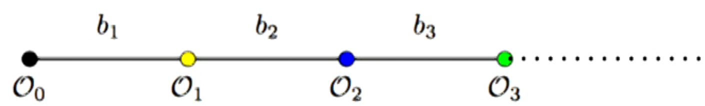
How to reduce practically any problem to one dimension, was the title of the first talk (given by D.C. Mattis) at the 1980 International Conference on Physics in One Dimension. That title refers, of course, to the recursion method as an instrument by means of which a large set of possible problems in condensed matter physics can be reduced to a pseudo-1D problem, known as the *chain model*

A convenient basis in the space of operators: the Krylov basis

“String of Pauli’s” basis

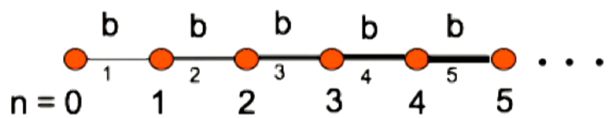


Krylov basis

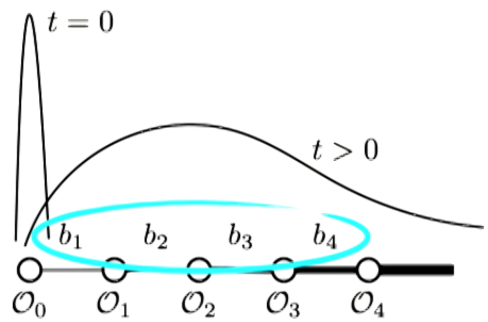


"Operator wavefunction" in Krylov space

$$\varphi_n(t) = (\mathcal{O}_n|\mathcal{O}(t))$$



$$\partial_t \varphi_n = -b_{n+1}\varphi_{n+1} + b_n \varphi_{n-1}, \quad \varphi_n(0) = \delta_{n0}$$



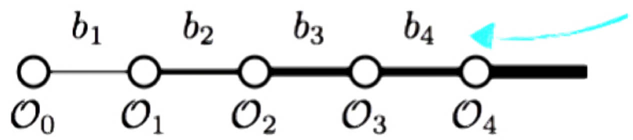
The autocorrelation function: $C(t) = \mathrm{tr}\,[\mathcal{O}(t)\mathcal{O}] = \varphi_0(t)$

Operator complexity: $\langle n(t) \rangle = \sum_{n=0}^{\infty} |\varphi_n(t)|^2 n$

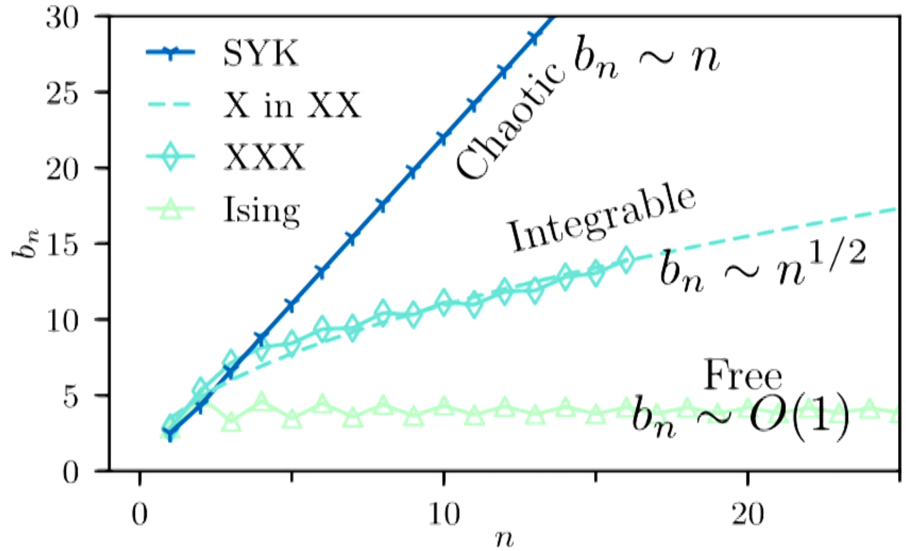
"K-complexity"

We have mapped the time evolution of any operator to a 1D hopping problem.

Now the question is: what do the hopping amplitudes b_n look like?



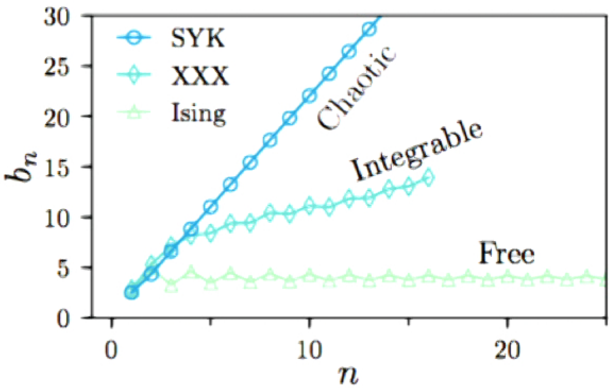
Empirical evidence:



Hypothesis: In a chaotic quantum system, the Lanczos coefficients b_n are asymptotically linear, i.e. for $\alpha, \gamma \geq 0$,

$$b_n \xrightarrow{n \gg 1} \alpha n + \gamma.$$

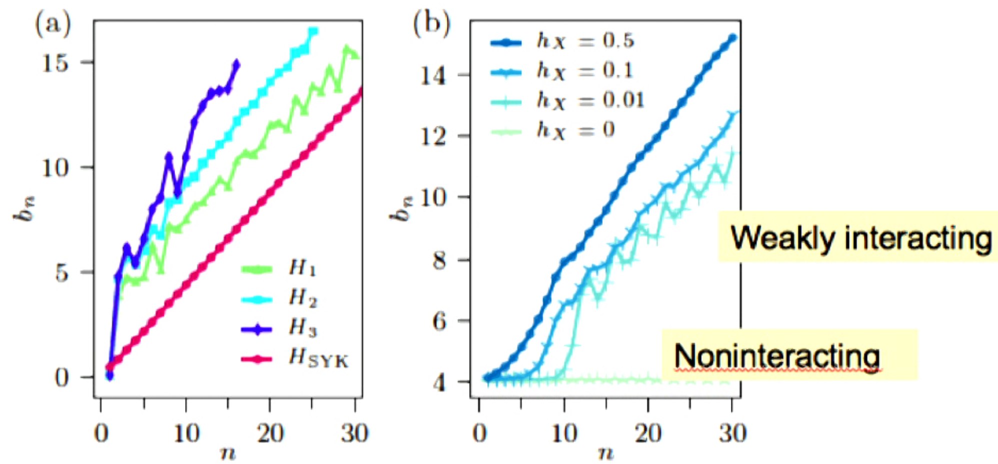
Asymptotic	Growth Rate	System Type
$b_n \sim O(1)$	Constant	Free models
$b_n \sim O(\sqrt{n})$	Square-root	Integrable models
$b_n \sim O(n)$	Linear	Chaotic models
$b_n \gtrsim O(n)$	Superlinear	Disallowed



We term the slope α, the "growth rate" of the operator for reasons that will become clear.

The evidence

Numerical: Many distinct nonintegrable spin chains, SYK model



Analytical: SYK model in the limit of large q

$$b_n \to \sqrt{q\, n(n-1)/2} \qquad n \geq 2$$

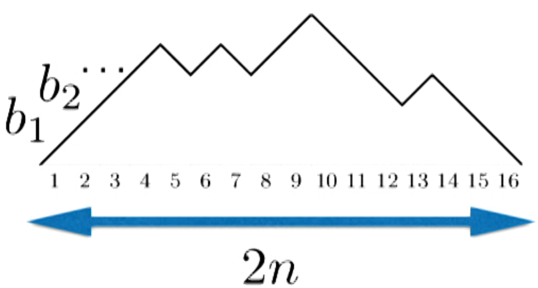
Linear growth is the fastest growth possible

Sketch of proof

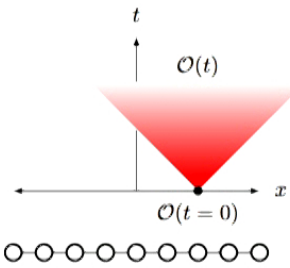
$$b_1^2 \dots b_n^2 \leq ||\mathcal{L}^n \mathcal{O}||^2 \leq n!^2 \; e^{2n} \implies b_n \leq \alpha n$$

$$||\mathcal{L}^n \mathcal{O}||^2 = (\mathcal{O}|\mathcal{L}^{2n}|\mathcal{O})$$

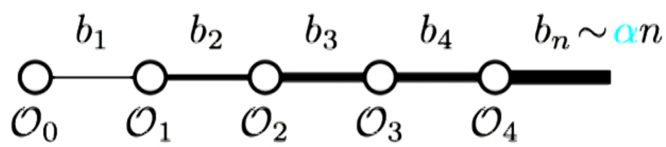
Sum over hopping histories that start at the origin and return to it after 2n steps => Dyck paths



Combinatorics of how many terms arise by successive commutation with H, using k-locality



Solution of 1D QM with linearly growing hopping



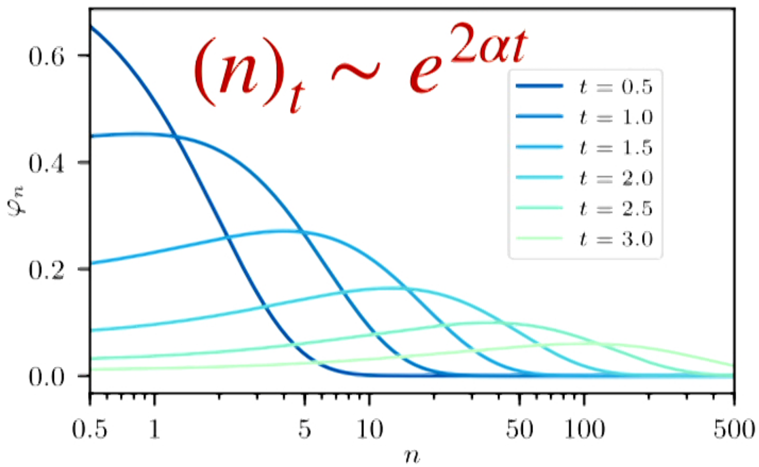
$$\partial_t \varphi_n = -b_{n+1}\varphi_{n+1} + b_n \varphi_{n-1}$$

Continuum limit

$$\partial_t \varphi_x = -2\alpha x \partial_x \varphi_n$$

Method of Characteristics

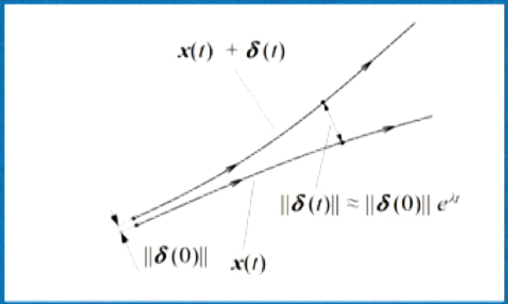
$$x(t) = e^{2\alpha t}$$



K(rylov)-complexity grows exponentially in time!

Relation to chaos

Classical and quantum chaos



$$\left|\frac{\partial x(t)}{\partial x(0)}\right|^2 = |\{x(t), p(0)\}|^2 \sim e^{2\lambda_L t}$$

Quantum systems:
Out-of-time-order correlation

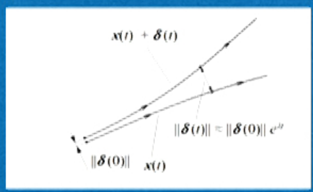
$$OTOC(t) = \sum_j ([\mathcal{O}(t), V_j]|[\mathcal{O}(t), V_j]) \sim e^{\lambda_L t}$$

Larkin, Kitaev, Maldacena, Shenker, Stanford,…

Out-of-time order correlators (OTOC)

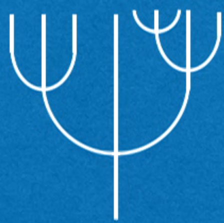
$$\sum_j \langle [V_j, \mathcal{O}(t)]^\dagger [V_j, \mathcal{O}(t)] \rangle$$

Classical systems



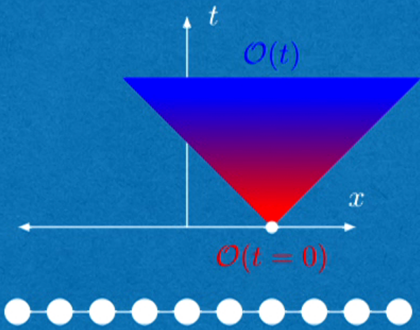
$\sim e^{\lambda_L t}$

Non-local quantum models (SYK-like)



$\sim e^{\lambda_L t}$

Local quantum systems

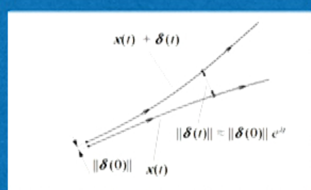
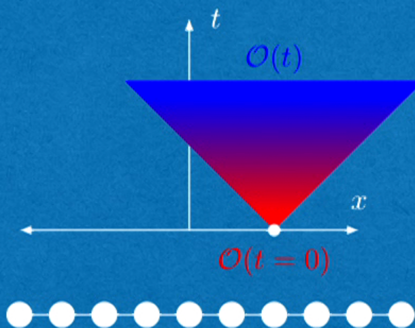


~~$\sim e^{\lambda_L t}$~~

(more involved)

[Huse, Khemani, Nahum, Swingle, ...]

OTOC and K-complexity

	Classical systems	Non-local quantum models (SYK-like)	Local quantum systems
			
OTOC:	$\sim e^{\lambda_L t}$	$\sim e^{\lambda_L t}$	~~$\sim e^{\lambda_L t}$~~
K-complexity:	$\sim e^{2\alpha t}$	$\sim e^{2\alpha t}$	$\sim e^{2\alpha t}$

Can they be compared?

Bound on chaos at infinite-T

$$\lambda_L \leq 2\alpha$$

Empirical evidence:

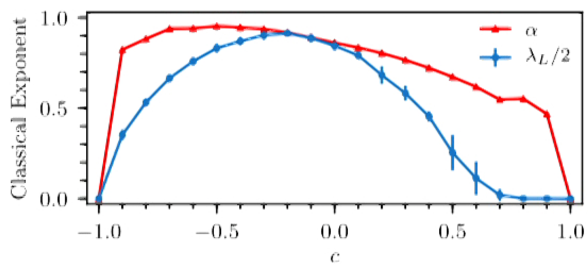
SYK_q

q	2	3	4	7	10	∞
$\alpha/\mathcal{J}$	0	0.461	0.623	0.800	0.863	1
$\lambda_L/(2\mathcal{J})$	0	0.454	0.620	0.799	0.863	1

[Roberts, Stanford, Streicher]

Classical two-spin model

$$H = (1+c)(S_1^x + S_2^x) + 4(1-c)S_1^z S_2^z$$



Bound on chaos at infinite T $\lambda_L \leq 2\alpha$

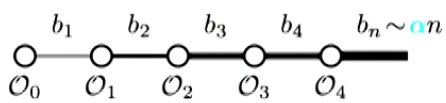
Sketch of proof for quantum systems $OTOC(t) = \sum_j ([\mathcal{O}(t), V_j]|[\mathcal{O}(t), V_j])$

$$OTOC(\mathcal{O}_n) \leq Cn \;\; => \;\; OTOC(t) \leq C' \, n(t) \;\; => \;\; \lambda_L \leq 2\alpha$$

$|\mathcal{O}_n) = \mathrm{GramSchmidt}[\mathcal{L}^n|\mathcal{O})]$

$OTOC(t) \sim e^{\lambda_L t}$

$(n)_t \sim e^{2\alpha t}$



Relation to Maldacena, Shenker, Stanford bound?

$$\lambda_L \leq 2\pi T$$

Easy to generalize Lanczos formalism to finite T,
but need to decide on an inner product:

- The "standard" inner product (for linear response theory) is given by $g(\lambda) = [\delta(\lambda) + \delta(\lambda - \beta)]/2$:

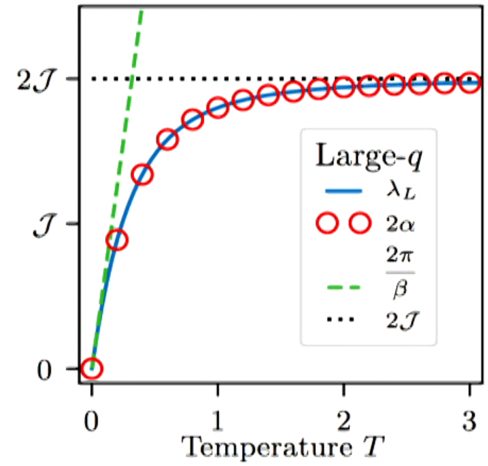
$$(A|B)^S_\beta := \frac{1}{2Z}\operatorname{Tr}[y^\beta A^\dagger B + A^\dagger y^\beta B] \quad (54)$$

- The Wightman inner product corresponds to $g(\lambda) = \delta(\lambda - \beta/2)$:

$$(A|B)^W_\beta := \frac{1}{Z}\operatorname{Tr}[y^{\beta/2} A^\dagger y^{\beta/2} B]\,. \quad (55)$$

$$\lambda_{L,T} \leq 2\alpha_T^{(W)} \quad \text{(conjecture)}$$

$$2\alpha_T^{(W)} \leq 2\pi T \quad \text{(proved)}$$



This talk

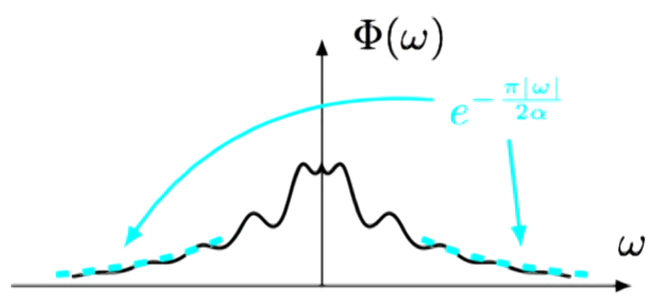
- **Preliminaries: operator dynamics, recursion methods**
- **A hypothesis for universal operator growth**
- **Evidence for the hypothesis:**
  (i) Numerical (Spin chains)
  (ii) Analytical (SYK models)
  (iii) Physical arguments
- **Application: new bound on chaos**
- **Application: computational approach**
  Transport coefficients in strongly coupled systems

α is measurable in linear response: It gives the high-frequency decay of the spectral function

$$C(t) = \mathrm{Tr}\left[\mathcal{O}(t)\mathcal{O}\right]$$

$$\Phi(\omega) = \int C(t) e^{i\omega t} dt$$

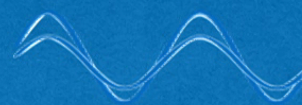
$$b_n \sim \alpha n \Leftrightarrow \Phi(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$



Such exponential tails have been measured for nuclear spins with NMR in CaF2 [Lundin et al, J. Phys.: Condens. Matter 2 (1990) 10131]

Application to hydrodynamics: diffusive transport

$$\mathcal{O} = \sum_x q_x e^{ikx}$$



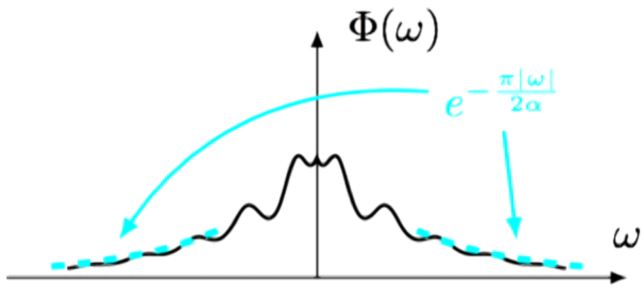
Diffusive transport means:

$$(\mathcal{O}|\mathcal{O}(t)) \sim e^{-\Omega(k)t} \qquad \Omega(k) = Dk^2 + \ldots$$

Pole of Green function

Can the knowledge of universal operator growth help us compute D?

Cautionary remark:
This work is concerned with high-frequency behavior



There is no a priori relation with long time behavior!

Example:

$$C(t) = \frac{1}{\cosh(t)}$$

$$C(t) = \frac{1}{1+t^2}$$

$$\Phi(\omega) \sim e^{-\frac{\omega}{\omega_0}} \Leftrightarrow b_n \sim \alpha n$$

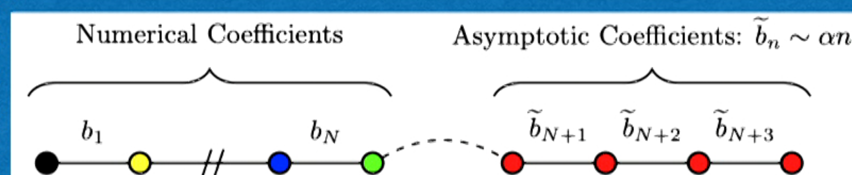
Mattis, 1981
V. Vishwanath G. Müller 2008

Recursion method

is a well known technique that uses the Lanczos coefficients to calculate the auto-correlation Green function via a continued fraction expansion

$$G(z) := \left(\mathcal{O}\left|\frac{1}{z-\mathcal{L}}\right|\mathcal{O}\right) = \cfrac{1}{z - \cfrac{b_1^2}{z - \cfrac{b_2^2}{z - \ddots}}}$$

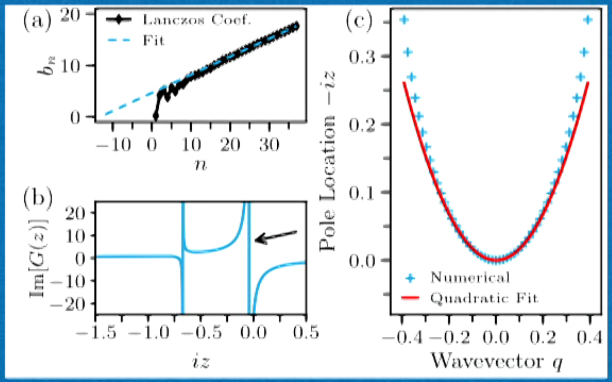
But usually, only a finite number of Lanzcos coefficients are known. A proper asymptotic extrapolation is then required.



Application to mixed field Ising chain

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i.$$

- We apply the method to calculate energy diffusion constant (thermal conductivity) in a mixed field Ising model.
- The result compares favorably with independent numerical methods Density Matrix Truncation [Bingtian Ye, Francisco Machado, in progress].



We calculate Green function for energy density wave operator $\mathcal{O}_q = \sum_j e^{ijq}\epsilon_j$

$$G(z) = \frac{1}{z - i\frac{Dq^2}{2} + O(q^4)}$$

$$D = 3.3(5)$$

Summary

- Hypothesis for universal operator dynamics supported by extensive evidence. Linear growth of Lanczos coefficients

$$b_n = \alpha n + \beta + o(1), \quad n \to \infty$$

- Implies exponential growth of Krylov-complexity with the exponent α
- Unlike the Lyapunov exponent, α is well defined in any generic model (no need for large N or non-locality)
- α bounds from above the Lyapunov exponent, when the latter is well defined

PS: There are log corrections in 1D