Title: The holographic dual of Renyi relative entropy

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Abstract: The relative entropy is a measure of the distinguishability of two quantum states. A great deal of progress has been made in the study of the relative entropy between an excited state and the vacuum state of a conformal field theory (CFT) reduced to a spherical region. For example, when the excited state is a small perturbation of the vacuum state, the relative entropy is known to have a universal expression for all CFTâ€™s. Specifically, the perturbative relative entropy can be written as the symplectic flux of a certain scalar field in an auxiliary AdS-Rindler spacetime. Moreover, if the CFT has a semi-classical holographic dual, the relative entropy is known to be related to conserved charges in the bulk dual spacetime. In this talk, I will introduce a one-parameter generalization of the relative entropy which I will call 'refined' Renyi relative entropy. I will use this quantity to present a one-parameter generalization of the aforementioned known results about the relative entropy. I will also discuss a new family of positive energy theorems in asymptotically locally AdS spacetimes that arises from the holographic dual of the refined RÃ©nyi relative entropy.

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The holographic dual of Rényi relative entropy

arXiv: 1904.08433

with N. Bao and I. Shehzad

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Quantum Fields and Strings Seminar
Perimeter Institute

Oct 11^{th}, 2019

Motivation

Entanglement entropy:

$$S(\rho) = -\operatorname{tr} \rho \log \rho \,.$$

- In AdS-CFT correspondence, entanglement entropy of some spatial region is given by the area of a boundary anchored codimension-2 bulk stationary area surface. [Ryu et. al, 2006; Hubeny et. al, 2007; Wall, 2014]

$$S(\rho_A) = \frac{\text{Area}[\tilde{B}_A]}{4G} \,.$$

- This has provided a connection between a quantum information quantity and a geometric quantity.

Motivation

Rényi entropy:

$$S_n(\rho) = \frac{1}{1-n} \log \operatorname{tr} \rho^n .$$

- Rényi entropy does not yet have a natural holographic dual.
- However, consider a refined version of the Rényi entropy:

$$\widetilde{S}_n(\rho) = n^2 \, \partial_n \left(\frac{n-1}{n} \, S_n(\rho) \right) .$$

- In AdS-CFT correspondence, refined Rényi entropy of some spatial region is given by the minimal area of a cosmic brane with tension given by $T_n = \frac{n-1}{4nG}$. [Dong, 2016]

Motivation

Relative entropy:

$$S_{\rm rel}(\rho||\sigma) = {\rm tr}\rho\log\rho - {\rm tr}\rho\log\sigma\,.$$

- Relative entropy is a measure of the distinguishability of two quantum states.
- Relative entropy has been studied in detail both in field theories and in holography.

Motivation

Relative entropy:

$$S_{\rm rel}(\rho||\sigma) = {\rm tr}\rho\log\rho - {\rm tr}\rho\log\sigma\,.$$

- Relative entropy is a measure of the distinguishability of two quantum states.
- Relative entropy has been studied in detail both in field theories and in holography.

Motivation

Relative entropy has proven to be very important in the context of quantum gravity, such as,

- Entanglement wedge reconstruction,
  [Jafferis *et. al.*, 2015; Dong *et. al*, 2016; Faulkner *et. al*, 2017; Cotler *et. al* 2017]
- Proofs of the Bousso bound, [Bousso *et. al.*, 2014]
- Proofs of the generalized second law, [Wall, 2012]
- Proof of the average and quantum null energy condition,
  [Faulkner *et al.* 2011; Leichenauer *et. al.* 2018]
- Distinguishability of black hole microstates, [Bao *et. al.*, 2017]
- Proofs of the holographic positive energy theorems, [Lashkari *et. al.*, 2016]
- And etc.

Therefore, it is advisable to study the Rényi generalization of the relative entropy.

Rényi relative entropy

Sandwiched Rényi relative entropy:

- Several Rényi generalizations of relative entropy have appeared in the literature.
- The quantity that preserves most of the properties of the relative entropy is called *sandwiched* Rényi relative entropy

$$S_n(\rho||\sigma) \equiv \frac{1}{n-1} \log \mathrm{tr} \left\{ \left(\sigma^{\frac{1-n}{2n}} \rho \, \sigma^{\frac{1-n}{2n}} \right)^n \right\} .$$

- In the limit $n \to 1$,

$$\lim_{n\to 1} S_n(\rho||\sigma) = S_{\rm rel}(\rho||\sigma) .$$

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Rényi relative entropy

Refined Rényi relative entropy:

- We introduce a refined version of this quantity, which we call *refined* Rényi relative entropy

$$\widetilde{S}_n(\rho||\sigma) \equiv n^2 \partial_n \left(\frac{n-1}{n} S_n(\rho||\sigma) \right) .$$

- In the limit $n \to 1$,

$$\lim_{n\to 1} \widetilde{S}_n(\rho||\sigma) = S_{\text{rel}}(\rho||\sigma) .$$

- In the rest of this talk, we will discuss the properties of this quantity.

Review of relative entropy in CFT and holography

Relative entropy

Vacuum state

- Consider a CFT in $R^{1,d-1}$ and consider the vacuum state reduced to a ball-shaped region, B.

$$\sigma = \mathrm{tr}_{\bar{B}} \left|\mathrm{vac}\right\rangle \left\langle\mathrm{vac}\right| .$$

- The domain of dependence of B can be mapped to $R \times \mathbb{H}^{d-1}$ by a conformal transformation. [Casini *et. al.*, 2011]
- Under this transformation, the state σ maps to a thermal state of temperature $T = 1/2\pi$: [Casini *et. al.*, 2011]

$$\sigma = \frac{e^{-2\pi H}}{Z}, \qquad Z = \mathrm{tr}\, e^{-2\pi H} .$$

Relative entropy

Excited state

- The vacuum state σ on B can be constructed as a path integral over $S^1 \times \mathbb{H}^{d-1}$.
- An excited state state ρ on B can similarly be constructed as path integral with operator insertions.

$$\langle \phi_- | \, \rho \, | \phi_+ \rangle \sim \int_{\Phi(\tau=0)=\tilde{\phi}_+}^{\Phi(\tau=2\pi)=\tilde{\phi}_-} D\Phi \, e^{-I_0[\Phi]} \;\; \Psi(\tau = \pi + \tau_0) \, \Psi(\tau = \pi - \tau_0) \, ,$$

- Here, $\Psi(x_0)$ is defined as the smearing of a local operator $\mathcal{O}(x)$ in a small neighborhood, $C(x_0)$, around the point x_0.

$$\Psi(x_0) \equiv \exp\left(- \int_{C(x_0)} \lambda(x) \mathcal{O}(x) \right) \, .$$

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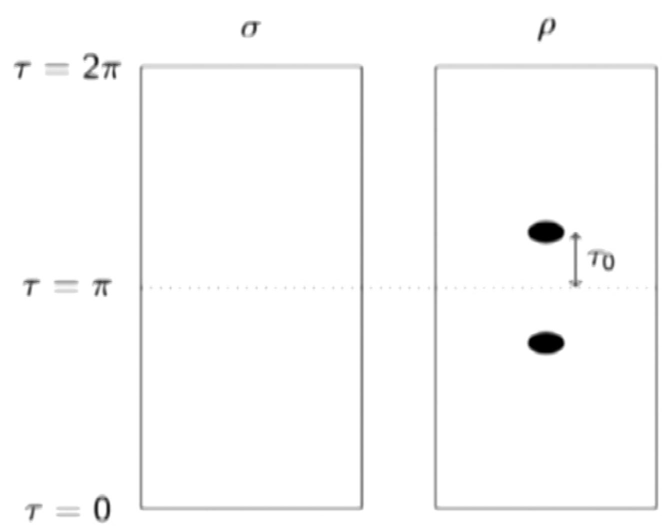
Relative entropy

Excited state



Relative entropy

Perturbative state

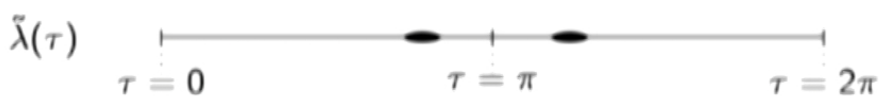
- When $\lambda(x)$ is small, the state ρ can be expanded around σ:

$$\rho = \sigma + \rho^{(1)} + O(\lambda^2)\,,$$

with

$$\rho^{(1)} = -\int_0^{2\pi} d\tau \int_{\mathbb{H}^{d-1}} d^{d-1}y\ \tilde{\lambda}(\tau,y)\,\sigma\,\mathcal{O}(\tau,y)\,.$$

- Here, $\tilde{\lambda}(\tau,y)$ is equal to the smearing function $\lambda(\tau,y)$ inside small neighbourhoods $C(\tau=\pi\pm\tau_0)$ and vanishes everywhere outside these neighborhoods.
- By construction, $\tilde{\lambda}$ is symmetric around $\tau=\pi$.



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Relative entropy

Perturbative relative entropy

- The relative entropy between the state σ and a perturbative state ρ has a universal form for all CFTs.
- It is fixed by the CFT vacuum two-point function.
- It is related to the symplectic flux of a scalar field through a Cauchy slice of an auxillary AdS-Rindler spacetime. [Faulkner *et. al.*, 2017]

$$S_{\rm rel}(\rho||\sigma) = \int_{\Sigma_0} \omega_\phi\Big(\Phi(r,t,y) \,,\, \mathcal{L}_\xi \Phi(r,t,y) \Big) \,.$$

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Relative entropy

Perturbative relative entropy

$$S_{\text{rel}}(\rho||\sigma) = \int_{\Sigma_0} \omega_\phi\Big(\Phi(r,t,y)\,,\,\mathcal{L}_\xi\Phi(r,t,y)\Big)\,.$$

- The mass of the scalar field is fixed by the conformal dimension of the operator $\mathcal{O}$,

$$m^2 = \Delta(\Delta - d)\,.$$

- The boundary conditions of the scalar field is fixed by the smearing function, $\tilde{\lambda}(x)$.
- For holographic CFTs, this scalar field would have been the holographic dual to the relevant operator $\mathcal{O}$.

What is a Rényi generalization of this result?

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Relative entropy

Relative entropy in holographic CFTs

- What if the state ρ is not a small perturbation of the vacuum σ?
- For holographic CFTs, the relative entropy is related to the conserved charges: [Lashkari *et. al.*, 2016]

$$S_{\text{rel}}(\rho||\sigma) = H_{\xi}\left(\mathcal{M}_{\rho}\right) - H_{\xi}\left(\mathcal{M}_{\sigma}\right) .$$

- The conserved charges H_{ξ} have contributions from both the codimension-2 extremal surface and the asymptotic boundary.

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Relative entropy

Quasi-local conserved charges

- Symplectic potential $\Theta(g, \delta g)$:

$$\delta L \equiv E(g)\delta g + d\Theta(g, \delta g) .$$

- Symplectic current $\omega(g, \delta_1 g, \delta_2 g)$:

$$\omega(g, \delta_1 g, \delta_2 g) \equiv \delta_1\Theta(g, \delta_2 g) - \delta_2\Theta(g, \delta_1 g) .$$

- For a codimension-1 hyperspace Σ and a vector field η on Σ, the Hamiltonian conjugate to η is such that

$$\delta H_\eta \equiv \int_\Sigma \omega(g, \delta g, \mathcal{L}_\eta g) .$$

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Relative entropy

Quasi-local conserved charges

- With $E(g) = 0$,

$$\delta H_\eta = \int_\Sigma \delta\Big(\theta(\mathcal{L}_\eta g) - \eta \cdot L\Big) - \int_{\partial\Sigma} \eta \cdot \theta(\delta g) \,.$$

- H_η exists iff the boundary term is a total derivative:

$$\delta\big(\eta \cdot K(g)\big) \equiv \eta \cdot \theta(\delta g) \qquad \text{on} \quad \partial\Sigma \,.$$

- Hamiltonian conjugate to η:

$$H_\eta = \int_\Sigma J_\eta - \int_{\partial\Sigma} \eta \cdot K \,,$$

$$J_\eta \equiv \theta(\mathcal{L}_\eta g) - \eta \cdot L$$

- This is similar to $H = p\dot{q} - L$!

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Relative entropy

Quasi-local conserved charges

- With $E(g) = 0$, J_η can be written in terms of the Noether charge:

$$J_\eta = dQ_\eta \,.$$

- H_η is then a purely boundary integral:

$$H_\eta = \int_{\delta\Sigma} (Q_\eta - \eta \cdot K) \,.$$

- H_η only depends on the details on the vector field η near the boundary $\delta\Sigma$.

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Relative entropy

Relative entropy in holographic CFTs

- Relative entropy can be written as:

$$S_{\rm rel}(\rho||\sigma) = {\rm tr}\rho\log\rho - {\rm tr}\rho\log\sigma\,,$$
$$= {\rm tr}\rho\log\rho - {\rm tr}\sigma\log\sigma + {\rm tr}\sigma\log\sigma - {\rm tr}\rho\log\sigma\,.$$

- Since $\sigma \sim e^{-2\pi H}$,

$$S_{\rm rel}(\rho||\sigma) = S(\sigma) - S(\rho) + 2\pi\,\langle H\rangle_\rho - 2\pi\,\langle H\rangle_\sigma\,.$$

- For holographic CFTs,

$$S_{\rm rel}(\rho||\sigma) = \frac{{\rm Area}(\tilde{B}_\sigma)}{4\,G} - \frac{{\rm Area}(\tilde{B}_\rho)}{4\,G} + 2\pi R\,\langle H\rangle_\rho - 2\pi R\,\langle H\rangle_\sigma\,.$$

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Relative entropy

Relative entropy in holographic CFTs

- The bulk dual to the thermal state of temperature $T = 1/(2\pi)$ on $\mathbb{H}^{d-1}$ is the AdS-Rindler spacetime

$$ds^2 = -\left(r^2 - 1\right) dt^2 + \left(r^2 - 1\right)^{-1} dr^2 + r^2 \, ds^2_{\mathbb{H}^{d-1}} .$$

- The HRT surface, $\tilde{B}_\sigma$, is the bifurcation surface, $r = 1$.
- The vector field $\xi = -2\pi\partial_t$ satisfies the following boundary conditions:

$$\xi\big|_{\tilde{B}_\sigma} = 0 \qquad \nabla^{[a}\xi^{b]}\big|_{\tilde{B}_\sigma} = 4\pi n_1^{[a} n_2^{b]} .$$

- The area of the bifurcation surface is the integral of the Noether charge: [Wald et. al., 1993]

$$\frac{\text{Area}(\tilde{B}_\sigma)}{4\,G} = \int_{\tilde{B}_\sigma} Q_\xi\left(\mathcal{M}_\sigma\right) = \int_{\tilde{B}_\sigma} \left(Q_\xi - \xi \cdot K\right) .$$

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Relative entropy

Relative entropy in holographic CFTs

- There always exists a vector field ξ in $\mathcal{M}_\rho$ that satisfies [Lashkari et. al., 2016]

$$\xi\big|_{\bar{B}_\rho} = 0 \qquad \nabla^{[a}\xi^{b]}\big|_{\bar{B}_\rho} = 4\pi n_1^{[a} n_2^{b]} ,$$

  and

$$\xi\big|_{B} = -\, 2\pi\, \partial_t \, .$$

- With this vector field, one can write [Wald et. al., 1993; Lashkari et. al., 2016]

$$\frac{\text{Area}(\bar{B}_\rho)}{4\,G} = \int_{\bar{B}_\rho} Q_\xi\,(\mathcal{M}_\sigma) = \int_{\bar{B}_\rho} (Q_\xi - \xi \cdot K)\, .$$

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Relative entropy

Relative entropy in holographic CFTs

- The integral of the Noether charge on the boundary gives the boundary Hamiltonian: [Faulkner *et. al.*, 2013]

$$2\pi \langle H\rangle_\rho - 2\pi \langle H\rangle_\sigma = \int_B \left(Q_\xi(\mathcal{M}_\rho) - \xi\cdot K(\mathcal{M}_\rho)\right) - \int_B \left(Q_\xi(\mathcal{M}_\sigma) - \xi\cdot K(\mathcal{M}_\sigma)\right) .$$

- As a result:

$$S_{\text{rel}}(\rho||\sigma) = \int_B \left(Q_\xi(\mathcal{M}_\rho) - \xi\cdot K(\mathcal{M}_\rho)\right) - \int_{\tilde{B}_\rho} \left(Q_\xi(\mathcal{M}_\rho) - \xi\cdot K(\mathcal{M}_\rho)\right) - \int_B \left(Q_\xi(\mathcal{M}_\sigma) - \xi\cdot K(\mathcal{M}_\sigma)\right) + \int_{\tilde{B}_\sigma} \left(Q_\xi(\mathcal{M}_\sigma) - \xi\cdot K(\mathcal{M}_\sigma)\right) .$$

Relative entropy

Relative entropy in holographic CFTs

- Or equivalently:

$$S_{\rm rel}(\rho||\sigma) = H_\xi\left(\mathcal{M}_\rho\right) - H_\xi\left(\mathcal{M}_\sigma\right) \ .$$

- Positivity of the relative entropy leads to a positive energy theorem:

$$H_\xi\left(\mathcal{M}_\rho\right) - H_\xi\left(\mathcal{M}_\sigma\right) \geq 0 \, .$$

What is a Rényi generalization of these results?

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Rényi relative entropy

Rényi relative entropy

Refined Rényi relative entropy:

$$\widetilde{S}_n(\rho||\sigma) \equiv n^2 \partial_n \left(\frac{n-1}{n} S_n(\rho||\sigma) \right) .$$

- An useful identity:

$$\widetilde{S}_n(\rho||\sigma) = S_{\text{rel}} \left(\rho_{(n)} || \sigma \right) .$$

- The *sandwiched* state is defined as

$$\rho_{(n)} \equiv \frac{\left(\sigma^{\frac{1-n}{2n}} \rho \, \sigma^{\frac{1-n}{2n}} \right)^n}{\text{tr} \left(\sigma^{\frac{1-n}{2n}} \rho \, \sigma^{\frac{1-n}{2n}} \right)^n} .$$

We will use this identity to find Rényi generalization of the known properties of the relative entropy.

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Perturbative refined Rényi relative entropy

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Perturbative refined Rényi relative entropy

- Suppose $\rho = \sigma + \epsilon\rho^{(1)} + O(\epsilon^2)$.
- Then relative entropy at the lowest order in ϵ is of the form [Faulkner *et. al.*, 2017]

$$S_{\text{rel}}(\rho||\sigma) = \frac{\epsilon^2}{2} S_{\text{rel}}^{(2)}(\rho||\sigma) + O(\epsilon^3),$$

where

$$S_{\text{rel}}^{(2)}(\rho||\sigma) = -\int_{-\infty}^{\infty} \frac{ds}{4\sinh^2\left(\frac{s+i\delta}{2}\right)} \text{tr}\left(\sigma^{-1-\frac{is}{2\pi}} \rho^{(1)} \sigma^{\frac{is}{2\pi}} \rho^{(1)}\right).$$

Perturbative refined Rényi relative entropy

- When $\rho = \sigma + \epsilon\rho^{(1)} + O(\epsilon^2)$, the sandwiched state is (for integer $n \geq 1$)

$$\rho_{(n)} = \sigma + \epsilon \sum_{k=0}^{n-1} \sigma^{\frac{1-n+2k}{2n}} \rho^{(1)} \sigma^{\frac{n-1-2k}{2n}} + O(\epsilon^2)\,.$$

- The refined Rényi relative entropy at the lowest order in ϵ is then

$$\widetilde{S}_n(\rho||\sigma) = \frac{\epsilon^2}{2}\, \widetilde{S}_n^{(2)}(\rho||\sigma) + O(\epsilon^3)\,,$$

where

$$\widetilde{S}_n^{(2)}(\rho||\sigma) = -\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} \int_{-\infty}^{\infty} \frac{ds}{4\sinh^2\left(\frac{s+i\delta}{2}\right)} \times$$
$$\operatorname{tr}\left(\sigma^{-1}\, \sigma^{-\frac{is}{2\pi}+\frac{(k-j)}{n}}\, \rho^{(1)}\, \sigma^{\frac{is}{2\pi}-\frac{(k-j)}{n}}\, \rho^{(1)}\right).$$

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Perturbative refined Rényi relative entropy

- Consider a CFT and take $\rho^{(1)}$ to be

$$\rho^{(1)} = -\int_0^{2\pi} d\tau \int_{\mathbb{H}^{d-1}} d^{d-1}y \ \tilde{\lambda}(\tau, y) \ \sigma \, \mathcal{O}(\tau, y) \, .$$

- Perturbative refined Rényi relative entropy is given in terms of a time-ordered two point function on $\mathcal{H}' = S^1 \times \mathbb{H}^{d-1}$:

$$\tilde{S}_n^{(2)}(\rho||\sigma) = -\sum_{k,j=0}^{n-1} \left(\prod_{i=1}^{2} \int_0^{2\pi} d\tau_i \int_{\mathbb{H}^{d-1}} d^{d-1}y_i \, \tilde{\lambda}(\tau_i, y_i) \right) \int_{-\infty}^{\infty} \frac{ds}{4\sinh^2\left(\frac{s+i\delta}{2}\right)}$$
$$\times \ \left\langle \mathcal{T} \, \mathcal{O}(\tau_2 + 2\pi k/n, y_2) \, \mathcal{O}(\tau_1 + 2\pi j/n + is, y_1) \right\rangle_{\mathcal{H}'} \, .$$

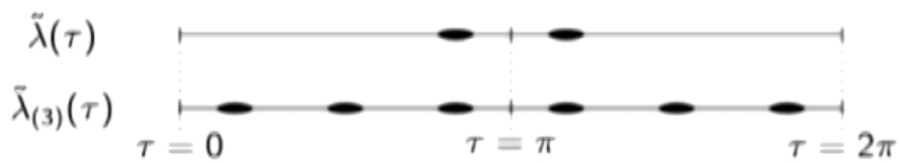
Perturbative refined Rényi relative entropy

- We simplify using the KMS condition

$$\widetilde{S}_n^{(2)}(\rho||\sigma) = -\left(\prod_{i=1}^{2}\int_0^{2\pi} d\tau_i \int_{\mathbb{H}^{d-1}} d^{d-1}y_i\, \tilde{\lambda}_{(n)}(\tau_i, y_i)\right)\int_{-\infty}^{\infty}\frac{ds}{4\sinh^2\left(\frac{s+i\delta}{2}\right)}$$
$$\times \left\langle \mathcal{T}\, \mathcal{O}(\tau_2, y_2)\, \mathcal{O}(\tau_1 + +is, y_1)\right\rangle_{\mathcal{H}'} .$$

- All the n dependence is absorbed in the $\mathbb{Z}_n$-symmetric source function:

$$\tilde{\lambda}_{(n)}(\tau, y) \equiv \sum_{k=0}^{n-1} \tilde{\lambda}(\tau - 2\pi k/n, y)\,.$$



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Perturbative refined Rényi relative entropy

- The expression of the $\widetilde{S}_n^{(2)}$ can be deduced from the expression of the $S_{\text{rel}}^{(2)}$.
- We just need to replace the source function $\tilde{\lambda}$ with the $\mathbb{Z}_n$-symmetric source function $\tilde{\lambda}_{(n)}$.
- Therefore, $\widetilde{S}_n^{(2)}$ is related to the symplectic flux of a scalar field through a Cauchy slice of an *auxillary* AdS-Rindler spacetime.
- The mass of the scalar field is fixed by the conformal dimension of $\mathcal{O}$.
- Whereas, the boundary condition is fixed by the $\mathbb{Z}_n$-symmetric source function $\tilde{\lambda}_{(n)}$.

Perturbative refined Rényi relative entropy

- The CFT two-point function can be written as [Faulkner *et. al.*, 2017]

$$\left\langle \mathcal{T}\; \mathcal{O}(\tau, y_1)\mathcal{O}(is, y_2)\right\rangle_{\mathcal{H}'} = -\int_{-\infty}^{\infty} dt \int_{\mathbb{H}^{d-1}} d^{d-1}y \\ \times \omega_\phi\Big(K_E\left(r_0, it, y|\tau, y_1\right), K_R\left(r_0, t, y|s, y_2\right)\Big)\,.$$

- Here K_E (K_R) is the Euclidean (retarded) bulk-to-boundary propagator in the AdS-Rindler spacetime.
- This is true in for *all* CFTs! We are not assuming AdS-CFT correspondence.

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Perturbative refined Rényi relative entropy

- With this identity,

$$\widetilde{S}_n^{(2)}(\rho||\sigma) = \int_{\Sigma_0} \omega_\phi\Big(\Phi(r,t,y)\,,\,\mathcal{L}_\xi\Phi(r,t,y)\Big)\,,$$

where $\xi = -2\pi\partial_t$ and

$$\Phi(r,t,y) = \int_{\mathcal{H}'} dX\;\bar{\lambda}_{(n)}(X)\;K_E(r,t,y\,|\,X)\,.$$

This is a Rényi generalization of the known result about perturbative relative entropy.

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Refined Rényi relative entropy in holographic CFTs

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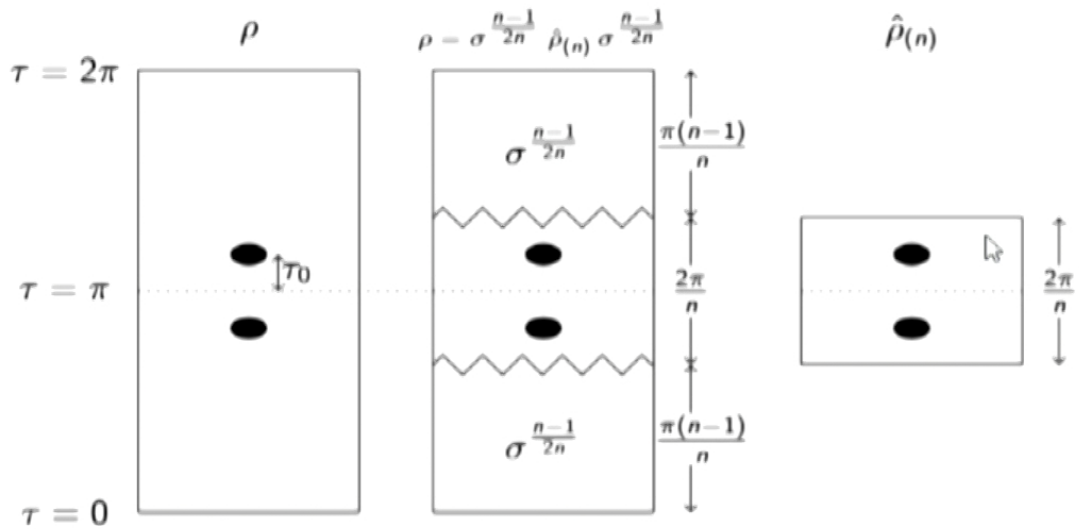
Holographic refined Rényi relative entropy

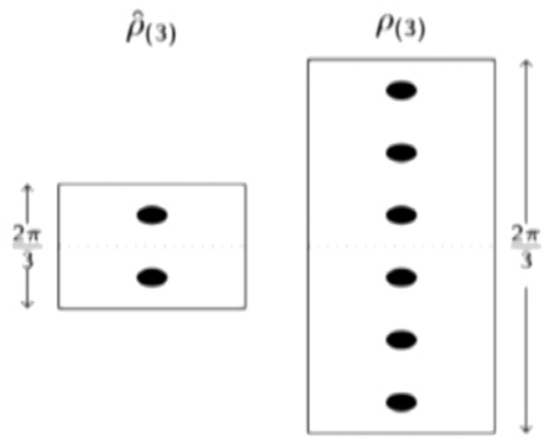
Sandwiched state $\rho_{(n)}$:

- Recall that $\rho_{(n)} \sim \hat{\rho}_{(n)}^{n}$, where $\hat{\rho}_{(n)} = \sigma^{\frac{1-n}{2n}} \rho \sigma^{\frac{1-n}{2n}}$.
- We can write ρ as a product $\rho = \sigma^{\frac{n-1}{2n}} \hat{\rho}_{(n)} \sigma^{\frac{n-1}{2n}}$.



Holographic refined Rényi relative entropy

- $\rho_{(n)}$ is obtained by taking n copies of $\hat{\rho}_{(n)}$ and gluing them together.



- For integer $n \geq 1$, the sandwiched state $\rho_{(n)}$ can be written as a path integral over $S^1 \times \mathbb{H}^{d-1}$ with $2n$ operators insertions.

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Holographic refined Rényi relative entropy

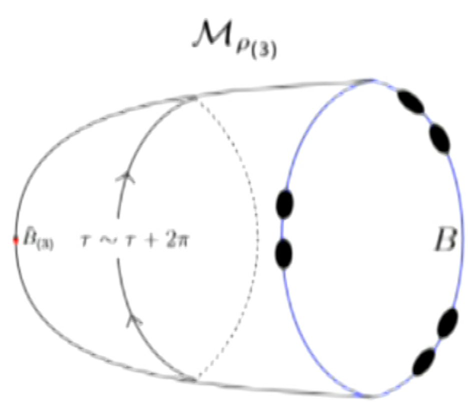
Bulk dual of the sandwiched state:

- The Euclidean bulk dual of $\rho_{(n)}$ can be solved by solving the Euclidean equations of motions with the following conditions:
  1. The bulk metric approaches the metric of $S^1 \times \mathbb{H}^{d-1}$ near the asymptotic boundary.
  2. The Euclidean time, τ, has of periodicity of 2π.
- The sources for the $2n$ operators inserted on the boundary provide the boundary conditions for the matter fields.

Holographic refined Rényi relative entropy

Bulk dual of the sandwiched state:

- The Euclidean circle contracts as we go into the bulk.
- It shrinks to zero size on a codimension-2 surface, which we call $\tilde{B}_{(n)}$.



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Holographic refined Rényi relative entropy

Bulk dual of the sandwiched state:

- A general ansatz for the metric near $\tilde{B}_{(n)}$ is [Lashkari et. al., 2016]

$$ds^2 = \alpha^2_{(n)}(\tau,\hat{r})\,d\tau^2 + d\hat{r}^2 + 2\beta^{(n)}_i(\tau,\hat{r})\,d\tau dy^i + h_{ij}dy^i dy^j\,,$$

with $\alpha_{(n)}(\tau,\hat{r}) = \hat{r} + O(\hat{r}^2)$ and $\beta^{(n)}_i(\tau,\hat{r}) = O(\hat{r}^2)$.

- It is easy to check that the vector field $\xi = 2\pi(\partial_\tau)$ satisfies the boundary conditions:

$$\xi\big|_{\tilde{B}_{(n)}} = 0 \qquad \nabla^{[a}\xi^{b]}\big|_{\tilde{B}_{(n)}} = 4\pi n_1^{[a} n_2^{b]}\,.$$

Holographic refined Rényi relative entropy

- Holographic refined Rényi relative entropy is related to the conserved charges:

$$\widetilde{S}_n(\rho||\sigma) = H_\xi\left(\mathcal{M}_{\rho_{(n)}}\right) - H_\xi\left(\mathcal{M}_\sigma\right) .$$

This is a Rényi generalization of the known result about holographic relative entropy.

Holographic refined Rényi relative entropy

- Our result is, in some sense, similar to the holographic formula for the refined Rényi entropy.
- The refined Rényi entropy of a boundary state ρ is given by the area of a codimension-2 surface in some bulk spacetime other than the spacetime dual to the boundary state ρ.

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Positive energy theorems

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Positive energy theorems

- Sandwiched Rényi relative entropy is monotonic in n:

$$\partial_n S_n(\rho||\sigma) \geq 0 .$$

- This implies

$$\widetilde{S}_n(\rho||\sigma) \geq S_{\text{rel}}(\rho||\sigma) \qquad \text{for } n \geq 1 .$$

- This further implies

$$H_\xi(\mathcal{M}_{\rho_{(n)}}) - H_\xi(\sigma) \geq H_\xi(\mathcal{M}_\rho) - H_\xi(\sigma) \qquad \text{for } n \geq 1 .$$

- Equivalently,

$$H_\xi(\mathcal{M}_{\rho_{(n)}}) - H_\xi(\mathcal{M}_\rho) \geq 0 \qquad \text{for } n \geq 1 .$$

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Positive energy theorems

$$H_\xi\left(\mathcal{M}_{\rho_{(n)}}\right) - H_\xi\left(\mathcal{M}_\rho\right) \geq 0 \qquad \text{for } n \geq 1\,.$$

- This condition is interesting as it compares the conserved charges in two non-vacuum bulk geometries.
- This positive energy condition is another nontrivial bulk consequence of a quantum information theoretic property.

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Future directions

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Future directions

- Is $\widetilde{S}_n$ monotonic in the Rényi parameter n?
  This will lead to stronger positive energy theorems.
- Does $\widetilde{S}_n$ satisfy data-processing inequality?
- Are there any constraints on $\widetilde{S}_n$ which are are specific to holographic states?
- Is there an analogue of JLMS formula?
  If yes, what are the implications for the bulk reconstructions?
- If the boundary of an entangling region is on a null place, the QNEC can be written as $S''_{\text{rel}}(\rho||\sigma) \geq 0$. Do null deformations of $\widetilde{S}_n$ satisfy similar bounds?

Thank you!