Title: Logarithmic Sobolev Inequalities for Quantum Many-Body Systems.

Speakers: Angela Capel

Series: Perimeter Institute Quantum Discussions

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Abstract: The mixing time of Markovian dissipative evolutions of open quantum many-body systems can be bounded using optimal constants of certain quantum functional inequalities, such as the logarithmic Sobolev constant. For classical spin systems, the positivity of such constants follows from a mixing condition for the Gibbs measure, via quasi-factorization results for the entropy.

Inspired by the classical case, we present a strategy to derive the positivity of the logarithmic Sobolev constant associated to the dynamics of certain quantum systems from some clustering conditions on the Gibbs state of a local, commuting Hamiltonian. In particular we address this problem for the heat-bath dynamics in 1D and the Davies dynamics, showing that the first one is positive under the assumptions of a mixing condition on the Gibbs state and a strong quasi-factorization of the relative entropy, and the second one under some strong clustering of correlations.

Based on:

1. A. Capel, A. Lucia and D. Pérez-García, **Superadditivity of Quantum Relative Entropy for General States**, *IEEE Trans. on Inf. Theory*, 64 (7) (2018), 4758–4765.

2. A. Capel, A. Lucia and D. Pérez-García, **Quantum Conditional Relative Entropy and Quasi-Factorization of the Relative Entropy**, *J. Phys. A: Math. Theor.*, 51 (2018), 484001.

3. I. Bardet, A. Capel, A. Lucia, D. Pérez-García and C. Rouzé, **On the modified logarithmic Sobolev inequality for the heat-bath dynamics for 1D systems**, preprint, arXiv: 1908.09004.

4. I. Bardet, A. Capel and C. Rouzé, **Positivity of the modified logarithmic Sobolev constant for quantum Davies semigroups: the commuting case**, in preparation.

Q. information theory $\longleftrightarrow$ Q. many-body physics

Communication channels $\longleftrightarrow$ Physical interactions

Quantum

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Tools and ideas $\longrightarrow$ Solve problems

Storage and transmision of information $\longleftarrow$ Models

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Main topic of this talk

Field of study

Dissipative evolutions of quantum many-body systems

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Velocity of convergence of certain quantum dissipative evolutions to their thermal equilibriums.

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Concrete problem

Provide sufficient static conditions on a Gibbs state which imply the existence of a positive log-Sobolev constant.

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Contents

1. Quantum dissipative systems

Open quantum systems

No experiment can be executed at zero temperature or be completely shielded from noise.

$\Rightarrow$ Open quantum many-body systems.

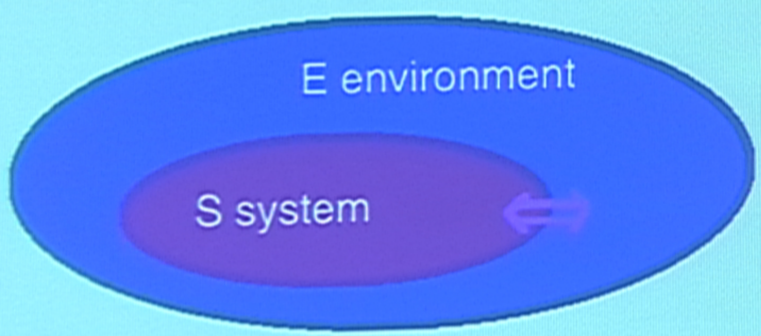

Figure: An open quantum many-body system.

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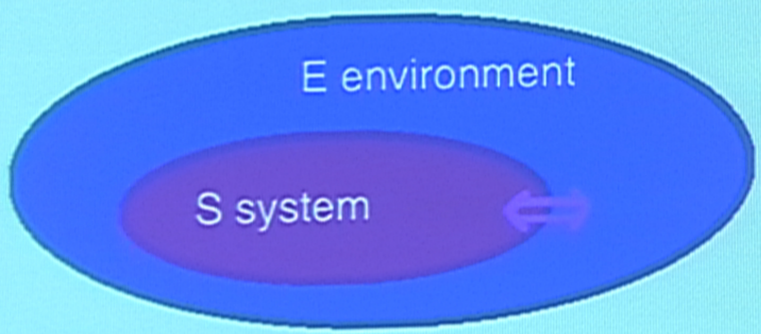

Figure: An open quantum many-body system.

- Dynamics of S is dissipative!
- The continuous-time evolution of a state on S is given by a q. Markov semigroup (Markovian approximation).

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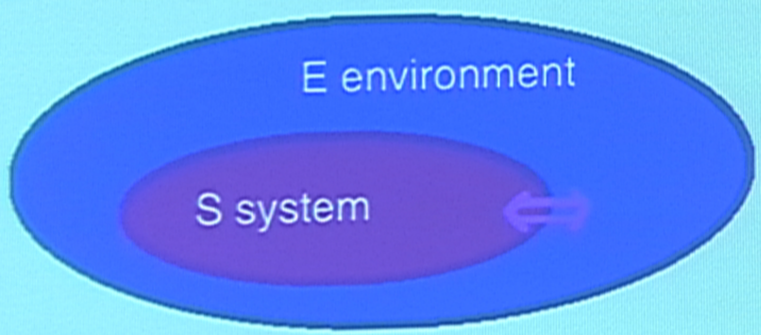

Figure: An open quantum many-body system.

- Dynamics of S is dissipative!
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Notation

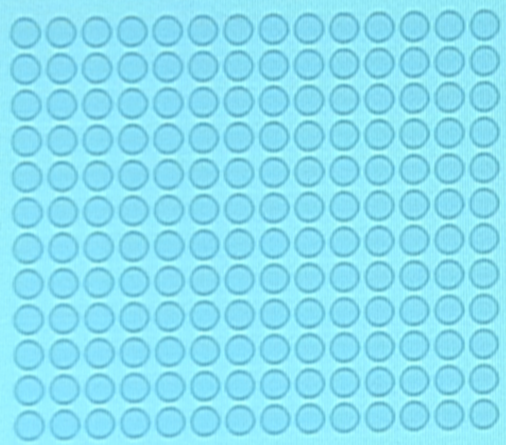

Figure: A quantum spin lattice system.

- Finite lattice $\Lambda \subset\subset \mathbb{Z}^d$.
- To every site $x \in \Lambda$ we associate $\mathcal{H}_x$ $(= \mathbb{C}^D)$.
- The global Hilbert space associated to Λ is $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$.
- The set of bounded linear endomorphisms on $\mathcal{H}_\Lambda$ is denoted by $\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$.
- The set of density matrices is denoted by $\mathcal{S}_\Lambda := \mathcal{S}(\mathcal{H}_\Lambda) = \{\rho_\Lambda \in \mathcal{B}_\Lambda : \rho_\Lambda \geq 0 \text{ and } \operatorname{tr}[\rho_\Lambda] = 1\}$.

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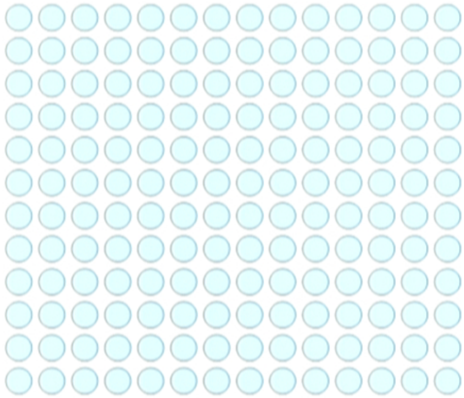

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Evolution of a system

Isolated system.

Physical evolution: $\rho \mapsto U\rho U^* \rightsquigarrow$ Reversible

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Isolated system.

Physical evolution: $\rho \mapsto U\rho U^* \rightsquigarrow$ Reversible

Dissipative quantum system (non-reversible evolution)

$$\mathcal{T} : \rho \mapsto \mathcal{T}(\rho)$$

- States to states $\Rightarrow$ Linear, positive and trace preserving.

  $\rho \otimes \sigma \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}')$, σ with trivial evolution

$$\begin{array}{rccc} \hat{\mathcal{T}}: & \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') & \rightarrow & \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') \\ & \hat{\mathcal{T}}(\rho \otimes \sigma) & = & \mathcal{T}(\rho) \otimes \sigma \end{array} \Rightarrow \hat{\mathcal{T}} = \mathcal{T} \otimes \mathbb{1}$$

- Completely positive.

$\mathcal{T}$ **quantum channel**

Open systems

Open systems $\Rightarrow$ Environment and system interact.

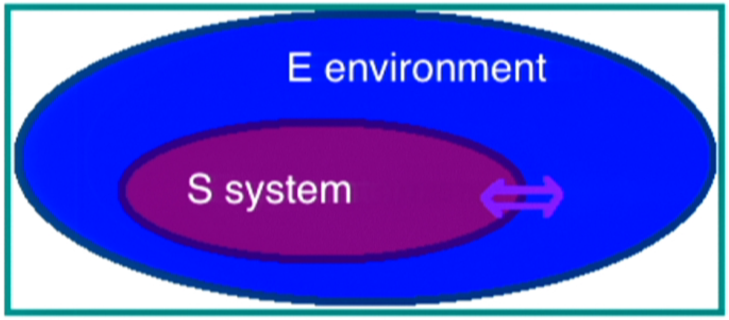
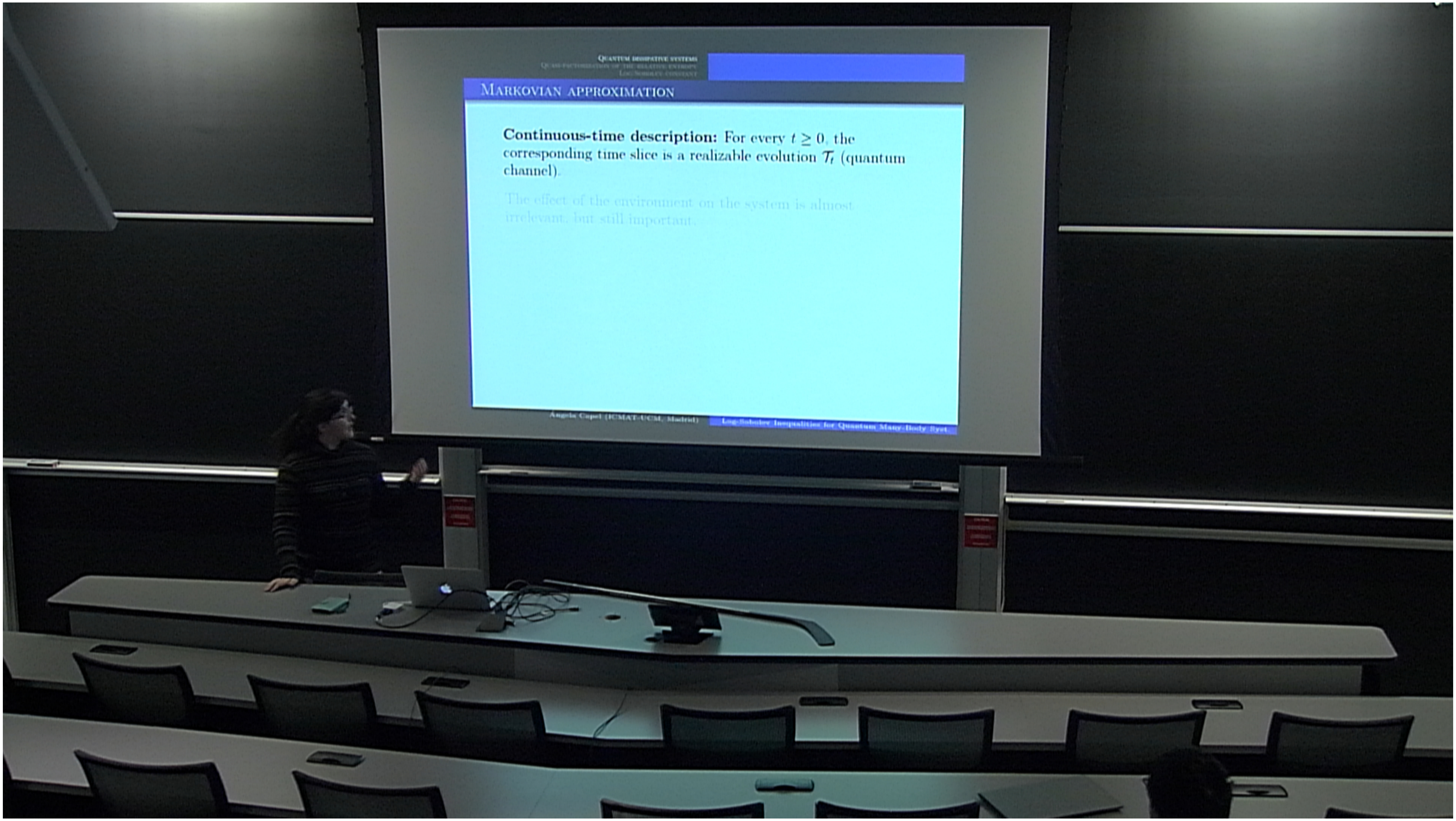


Figure: Environment + System form a closed system.

State for the environment: $|\psi\rangle\langle\psi|_E$

$$\rho \mapsto \rho \otimes |\psi\rangle\langle\psi|_E \mapsto U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^* \mapsto \mathrm{tr}_E[U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^*] = \tilde{\rho}$$

$$\begin{array}{rccc} \mathcal{T}: & \mathcal{S}(\mathcal{H}) & \to & \mathcal{S}(\mathcal{H}) \\ & \rho & \mapsto & \tilde{\rho} \end{array} \quad \text{quantum channel}$$


Quantum dissipative systems
Quasi-factorization of the relative entropy
Log-Sobolev constant
Markovian approximation
Continuous-time description: For every $t \geq 0$, the corresponding time slice is a realizable evolution $\mathcal{T}_t$ (quantum channel).
The effect of the environment on the system is almost irrelevant, but still important.
Angela Capel (ICMAT-UCM, Madrid)
Log-Sobolev Inequalities for Quantum Many-Body Syst.

Markovian approximation

Continuous-time description: For every $t \geq 0$, the corresponding time slice is a realizable evolution $\mathcal{T}_t$ (quantum channel).

The effect of the environment on the system is almost irrelevant, but still important.

Assumption: The environment does not evolve

$\Rightarrow$ **Weak-coupling limit**

Environment holds no memory + Future evolution only depends on the present.

Markovian approximation

Dissipative quantum systems

Dissipative quantum systems

A **dissipative quantum system** is a 1-parameter continuous semigroup $\{\mathcal{T}_t^*\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{S}_\Lambda$.

Semigroup:

- $\mathcal{T}_t^* \circ \mathcal{T}_s^* = \mathcal{T}_{t+s}^*$.
- $\mathcal{T}_0^* = \mathbb{1}$.

$$\frac{d}{dt}\mathcal{T}_t^* = \mathcal{T}_t^* \circ \mathcal{L}_\Lambda^* = \mathcal{L}_\Lambda^* \circ \mathcal{T}_t^*.$$

QMS generator

The infinitesimal generator $\mathcal{L}_\Lambda^*$ of the previous semigroup of quantum channels is usually called **Liouvillian**, or **Lindbladian**.

$$\mathcal{T}_t^* = e^{t\mathcal{L}_\Lambda^*} \Leftrightarrow \mathcal{L}_\Lambda^* = \frac{d}{dt}\mathcal{T}_t^* \mid_{t=0}.$$

Notation: $\rho_t := \mathcal{T}_t^*(\rho)$.

$$\rho_\Lambda \xrightarrow{t} \rho_t := \mathcal{T}_t^*(\rho_\Lambda) = e^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$$

Quantum dissipative evolutions useful?

Recent change of perspective ⇒ Resource to exploit

New area:

Quantum dissipative engineering,

to create artificial evolutions in which the dissipative process works in favor (protecting the system from noisy evolutions).

Interesting problems:

- Computational power
- Conditions against noise
- Time to obtain certain states
- ...

Mixing time

We define the **mixing time** of $\{\mathcal{T}_t^*\}$ by

$$\tau(\varepsilon) = \min\left\{ t > 0 : \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\mathcal{T}_t^*(\rho) - \mathcal{T}_\infty^*(\rho)\|_1 \leq \varepsilon \right\}.$$

Rapid mixing

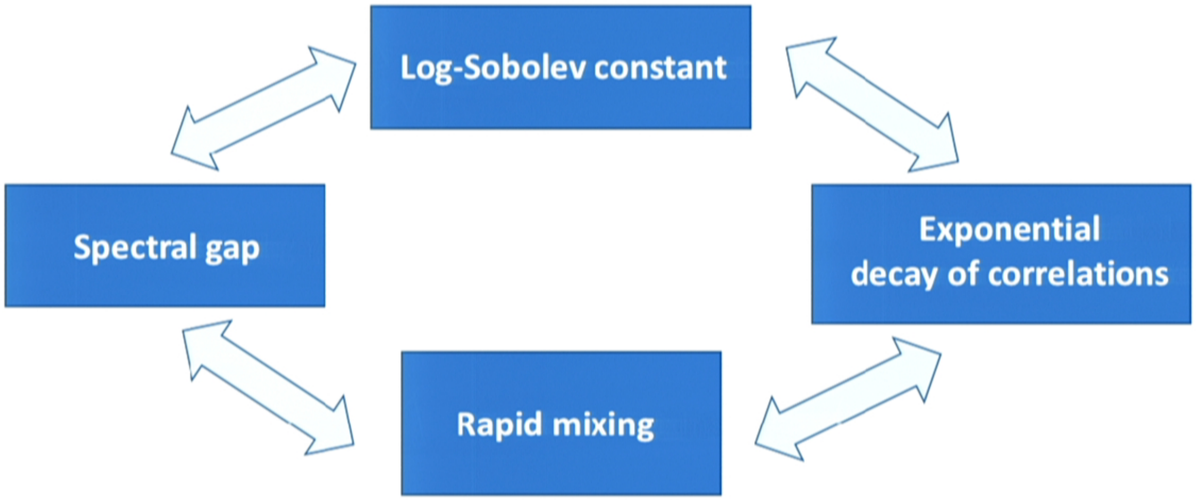
We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\rho_t - \sigma_\Lambda\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}.$$

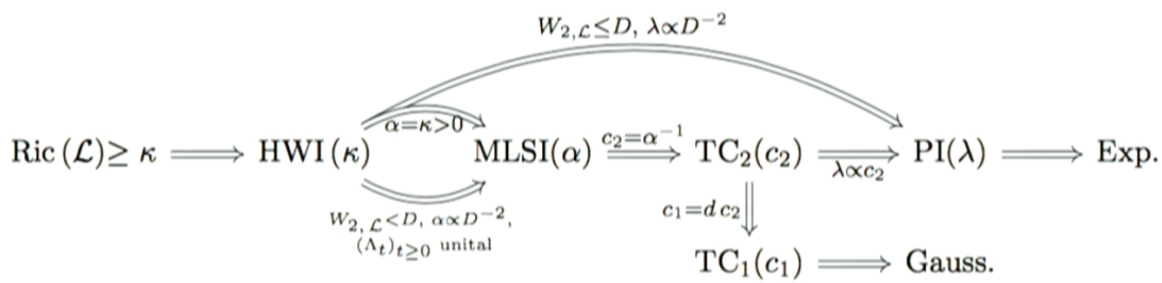
Problem

Find examples of rapid mixing!

Classical spin systems



Quantum spin systems



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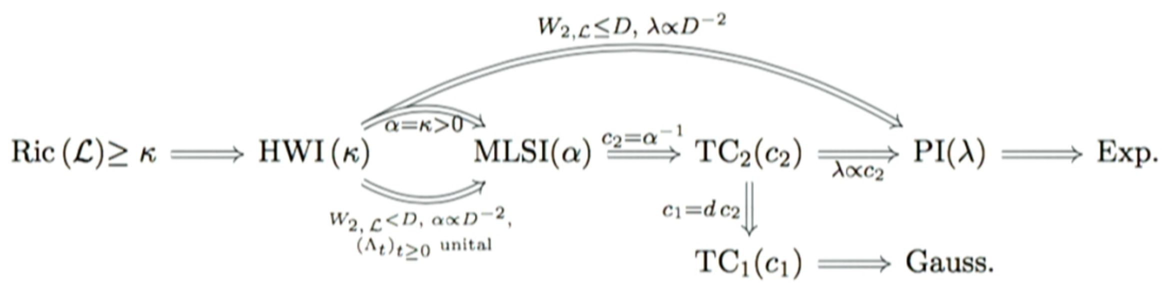
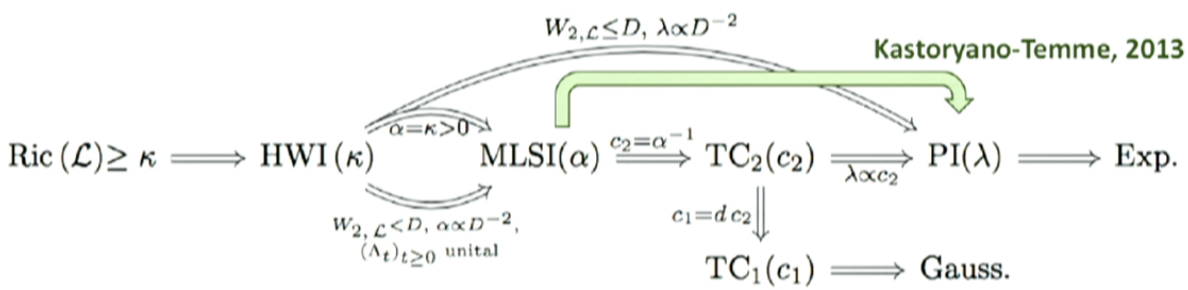


Quantum spin systems



Log-Sobolev inequality (MLSI)

Recall: $\rho_t := \mathcal{T}_t^*(\rho)$.

Liouville's equation:

$$\partial_t \rho_t = \mathcal{L}_\Lambda^*(\rho_t).$$

Relative entropy of ρ_t and σ_Λ:

$$D(\rho_t || \sigma_\Lambda) = \operatorname{tr}[\rho_t(\log \rho_t - \log \sigma_\Lambda)].$$

Differentiating:

$$\partial_t D(\rho_t || \sigma_\Lambda) = \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_t)(\log \rho_t - \log \sigma_\Lambda)]. \tag{1}$$

We want to find a lower bound for the derivative of $D(\rho_t || \sigma_\Lambda)$ in terms of itself:

$$2\alpha D(\rho_t || \sigma_\Lambda) \leq - \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_t)(\log \rho_t - \log \sigma_\Lambda)]. \tag{2}$$

Log-Sobolev constant

Log-Sobolev constant

The **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ is defined as:

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

If $\alpha(\mathcal{L}_\Lambda^*) > 0$:

$$D(\rho_t || \sigma_\Lambda) \leq D(\rho_\Lambda || \sigma_\Lambda) e^{-2\,\alpha(\mathcal{L}_\Lambda^*)\,t},$$

and with **Pinsker's inequality**, we have:

$$\| \rho_t - \sigma_\Lambda \|_1 \leq \sqrt{2D(\rho_\Lambda || \sigma_\Lambda)}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t} \leq \sqrt{2\log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t}.$$

Log-Sobolev constant $\Rightarrow$ Rapid mixing.

Problem

Find positive log-Sobolev constants!

Main problem of this talk

Develop a strategy to find positive log Sobolev constants.

Concrete problem

Provide sufficient static conditions on a Gibbs state which imply the existence of a positive log-Sobolev constant.

Classical spin systems

(Cesi, Dai Pra-Paganoni-Posta, '02)

(1) Quasi-factorization of the entropy (in terms of a conditional entropy).

+

(2) Recursive geometric argument.
Lower bound for the global log-Sobolev constant in terms of the log-Sobolev constant of a size-fixed region.

+

(3) Decay of correlations on the Gibbs measure.

⇓

Positive log-Sobolev constant.

Conditional log-Sobolev constant

Log-Sobolev constant

Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ. We define the **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ by

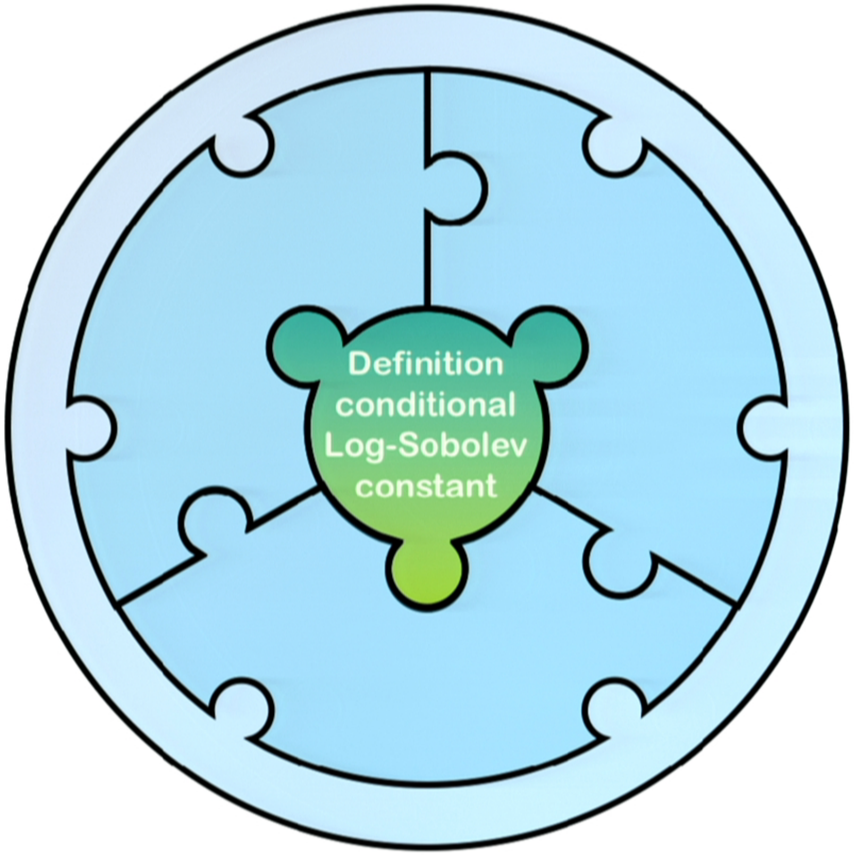
$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

Conditional log-Sobolev constant

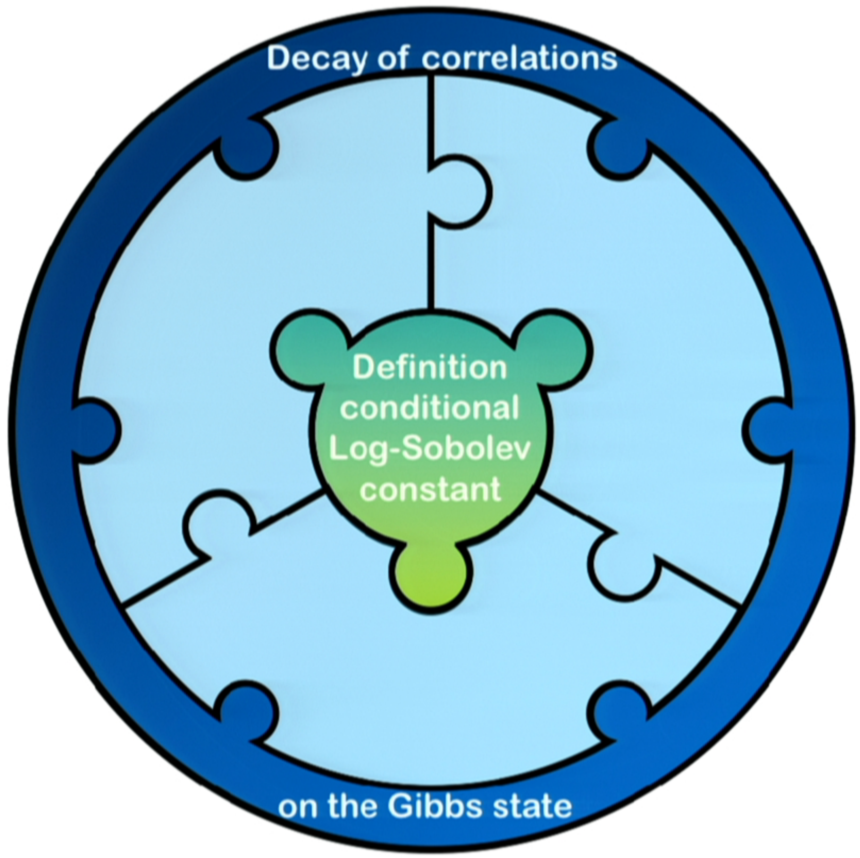
Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ, $A \subseteq \Lambda$. We define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ on A by

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda || \sigma_\Lambda)}$$

Strategy



Strategy


Decay of correlations
Definition
conditional
Log-Sobolev
constant
on the Gibbs state

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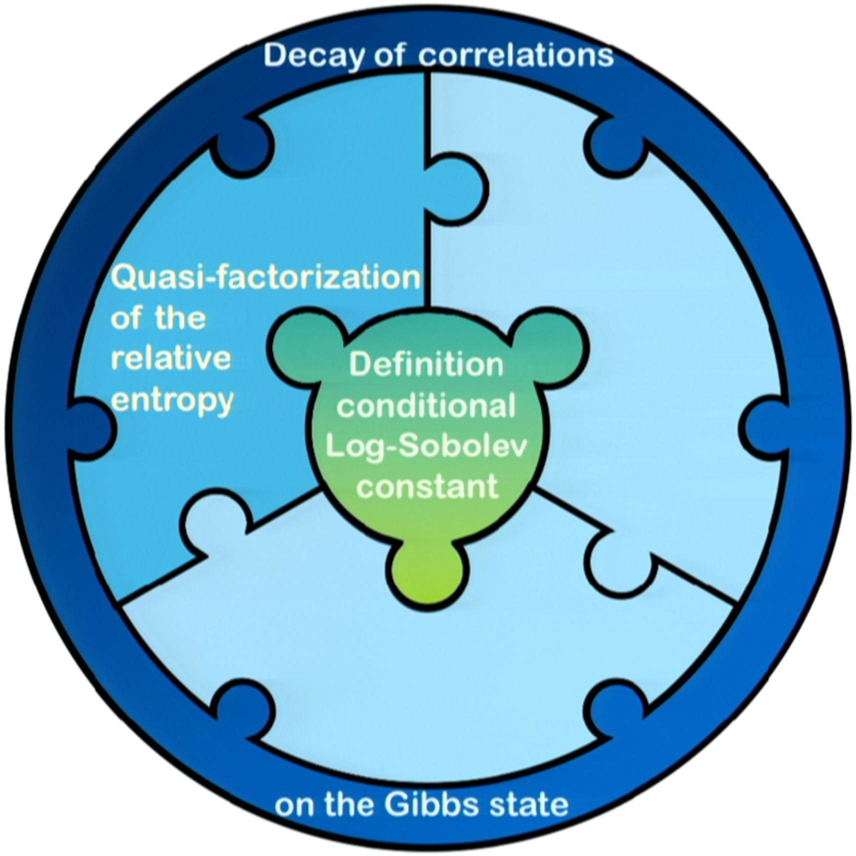
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Quasi-factorization of the relative entropy
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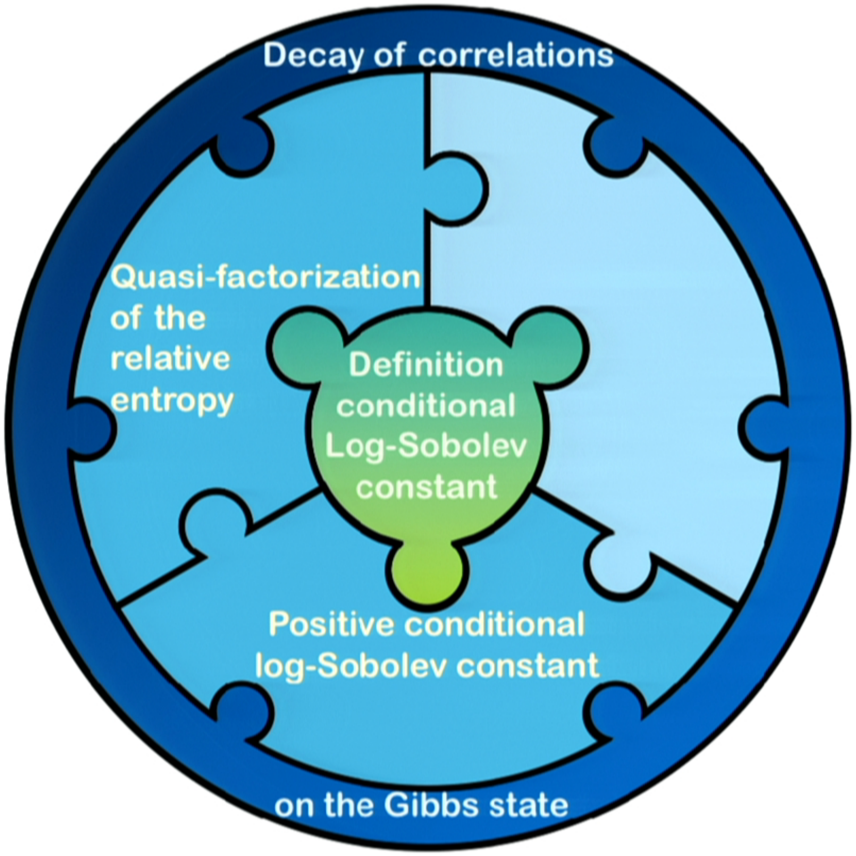
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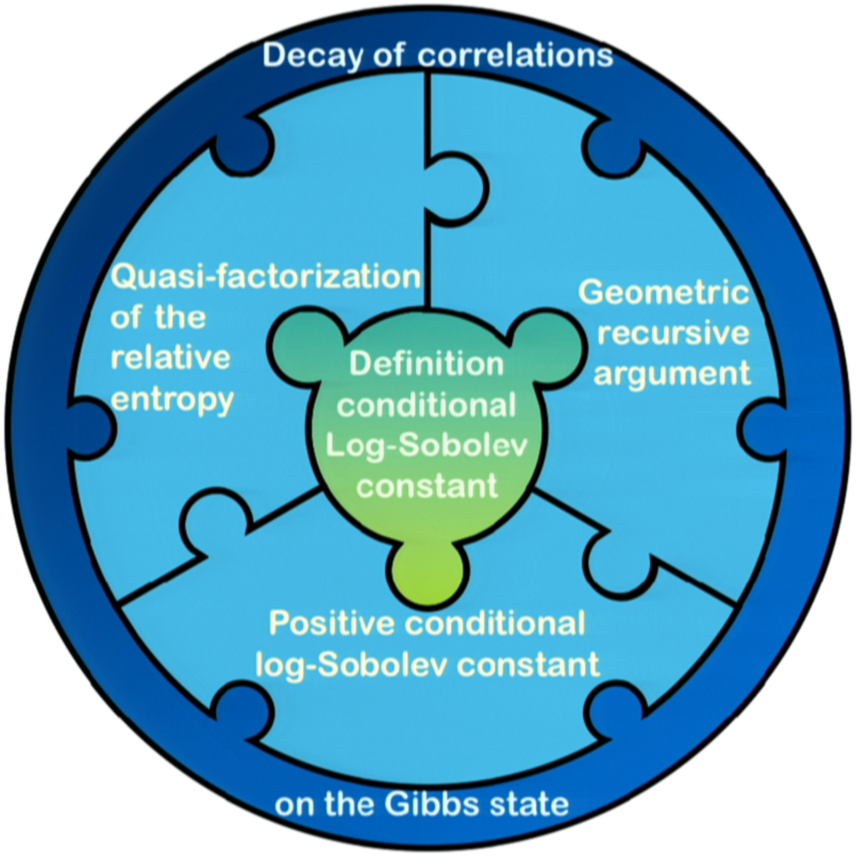
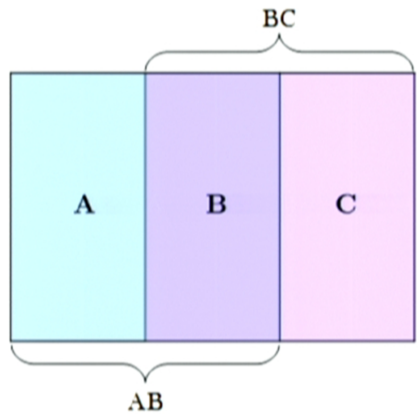

Decay of correlations
on the Gibbs state
Quasi-factorization of the relative entropy
Geometric recursive argument
Definition conditional Log-Sobolev constant
Positive conditional log-Sobolev constant

2. QUASI-FACTORIZATION OF THE RELATIVE ENTROPY

Statement of the problem



Problem

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in S_{ABC}$. Can we prove something like

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right] \ ?$$

Quantum relative entropy

$$D(\rho||\sigma) = \operatorname{tr}\left[\rho(\log\rho - \log\sigma)\right]$$

Problem

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC})\left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC})\right]$$

Classical case, Dai Pra et al. '02

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{1-4\|h-1\|_\infty}\,\mu\left[\mathrm{Ent}_\mu(f \mid \mathcal{F}_1) + \mathrm{Ent}_\mu(f \mid \mathcal{F}_2)\right],$$

where $h = \dfrac{d\mu}{d\bar{\mu}}$.

Classical entropy and conditional entropy

Entropy:

$$\mathrm{Ent}_\mu(f) = \mu(f \log f) - \mu(f)\log\mu(f).$$

Conditional entropy:

$$\mathrm{Ent}_\mu(f \mid \mathcal{G}) = \mu(f \log f \mid \mathcal{G}) - \mu(f \mid \mathcal{G})\log\mu(f \mid \mathcal{G}).$$

Relative entropy

Quantum relative entropy

Let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$. The **quantum relative entropy** of ρ_Λ and σ_Λ is defined by:

$$D(\rho_\Lambda || \sigma_\Lambda) = \operatorname{tr}\left[\rho_\Lambda(\log \rho_\Lambda - \log \sigma_\Lambda)\right].$$

Properties of the relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** $\rho_{AB} \mapsto D(\rho_{AB} || \sigma_{AB})$ is continuous.
2. **Additivity.** $D(\rho_A \otimes \rho_B || \sigma_A \otimes \sigma_B) = D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B)$.
3. **Superadditivity.** $D(\rho_{AB} || \sigma_A \otimes \sigma_B) \geq D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B)$.
4. **Monotonicity.** $D(\rho_{AB} || \sigma_{AB}) \geq D(\mathcal{T}(\rho_{AB}) || \mathcal{T}(\sigma_{AB}))$ for every quantum channel $\mathcal{T}$.

Characterization of the RE, Wilming et al. '17, Matsumoto '10

If $f : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$ satisfies $1 - 4$, then f is the relative entropy.

Conditional relative entropy

Conditional relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a **conditional relative entropy** in A as a function

$$D_A(\cdot||\cdot) : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$$

verifying the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity:** $D_A(\rho_{AB}||\sigma_{AB}) \geq 0$ and
   (2.1) $D_A(\rho_{AB}||\sigma_{AB})$=0 if, and only if, $\rho_{AB} = \sigma_{AB}^{1/2}\sigma_B^{-1/2}\rho_B\sigma_B^{-1/2}\sigma_{AB}^{1/2}$.
3. **Semi-superadditivity:** $D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A)$ and
   (3.1) **Semi-additivity:** if $\rho_{AB} = \rho_A \otimes \rho_B$,
   $D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A)$.
4. **Semi-motonicity:** For every quantum channel $\mathcal{T}$,

$$D_A(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) + D_B((\operatorname{tr}_A \circ \mathcal{T})(\rho_{AB})||(\operatorname{tr}_A \circ \mathcal{T})(\sigma_{AB}))$$
$$\leq D_A(\rho_{AB}||\sigma_{AB}) + D_B(\operatorname{tr}_A(\rho_{AB})||\operatorname{tr}_A(\sigma_{AB})).$$

Remark

Consider for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$

$$D^+_{A,B}(\rho_{AB}||\sigma_{AB}) = D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB}).$$

Then, $D^+_{A,B}$ verifies the following properties:

1. **Continuity:** $\rho_{AB} \mapsto D^+_{A,B}(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Additivity:** $D^+_{A,B}(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
3. **Superadditivity:** $D^+_{A,B}(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

However, it does not satisfy the property of monotonicity.

Axiomatic characterization of the CRE (C-Lucia-Pérez García, '18)

The only possible conditional relative entropy is given by:

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B)$$

for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

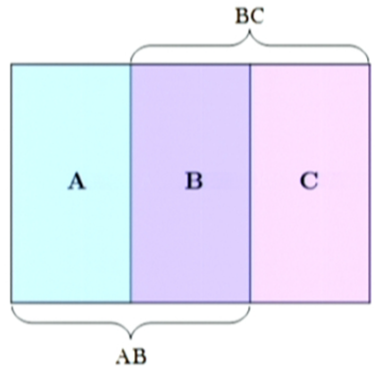


Figure: Choice of indices in $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Result of **quasi-factorization** of the relative entropy, for every $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$:

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$.

Quasi-factorization for the CRE (C-Lucia-Pérez García, '18)

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \frac{1}{1 - 2\|H(\sigma_{AC})\|_\infty} \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \, \sigma_{AC} \, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Note that $H(\sigma_{AC}) = 0$ if σ_{AC} is a tensor product between A and C.

This result is equivalent to:

$$\boxed{(1+2\|H(\sigma_{AB})\|_\infty) D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

Due to:

- **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.

we have

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

This result is equivalent to:

$$\boxed{(1 + 2\|H(\sigma_{AB})\|_\infty)D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

Due to:

- **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.

we have

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

Relation with the classical case

	STATES		OBSERVABLES
QUANTUM SETTING	$D(\rho_{AB}\|\|\sigma_{AB})$ $D(\rho_{AB}\|\|\sigma_{AB}) - D(\rho_B\|\|\sigma_B)$	$f_{AB} = \Gamma^{-1}_{\sigma_{AB}}(\rho_{AB})$ $\leftrightarrow$ $f_B = \Gamma^{-1}_{\sigma_B}(\rho_B)$	$\mathrm{tr}[\sigma_{AB} f_{AB} \log f_{AB}]$ $\mathrm{tr}[\mathrm{tr}_A[\sigma_{AB} f_{AB} \log f_{AB}] - \sigma_B f_B \log f_B]$
	$\updownarrow$ $\rho_{AB} \equiv \nu$ $\sigma_{AB} \equiv \mu$		$\updownarrow$ $\mathrm{tr}[\sigma_{AB}\,\cdot] \equiv \mu(\cdot)$ $\mathrm{tr}_A[\cdot] \equiv \mu(\cdot\,\|\mathcal{G})$
CLASSICAL SETTING	$H(\nu,\mu)$ $H_{\mathcal{G}}(\nu,\mu)$	$f = \frac{d\nu}{d\mu}$ $\leftrightarrow$	$\mu(f \log f)$ $\mu(\mu(f \log f\|\mathcal{G}) - \mu(f\|\mathcal{G}) \log \mu(f\|\mathcal{G}))$

Figure: Identification between classical and quantum quantities when the states considered are classical.

3. Log-Sobolev constant

Quantum spin lattices

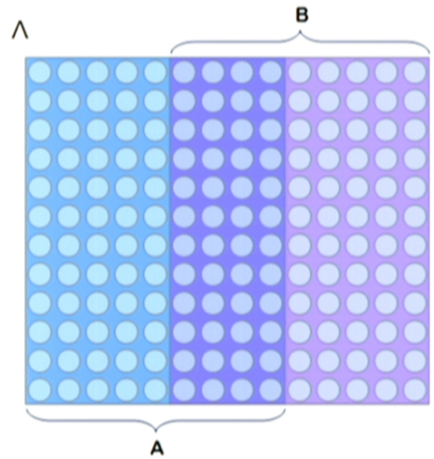


Figure: A quantum spin lattice system Λ and $A, B \subseteq \Lambda$ such that $A \cup B = \Lambda$.

Problem

For a certain $\mathcal{L}_\Lambda^*$, can we prove $\alpha(\mathcal{L}_\Lambda^*) > 0$ using the result of quasi-factorization of the relative entropy?

Example 1

Heat-bath dynamics with tensor product fixed point

Heat-bath with tensor product fixed point

Theorem (C-Lucia-Pérez García, '18)

The **heat-bath dynamics**, with tensor product fixed point, has a positive log-Sobolev constant.

Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \; \mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*$$

Since

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

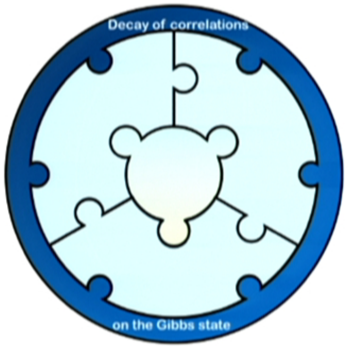
for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x \in \Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda).$$

Heat-bath with tensor product fixed point

Assumption

$$\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x .$$



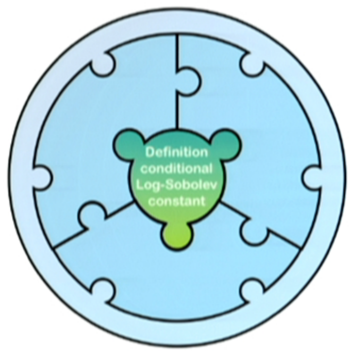
Heat-bath with tensor product fixed point

Conditional log-Sobolev constant

For $x \in \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in x by

$$\alpha_\Lambda(\mathcal{L}_x^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2 D_x(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and $D_x(\rho_\Lambda||\sigma_\Lambda)$ is the conditional relative entropy.

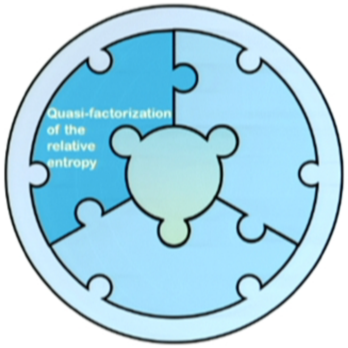


Heat-bath with tensor product fixed point

General quasi-factorization for σ a tensor product

Let $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ and $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x\in\Lambda} \sigma_x$. The following inequality holds:

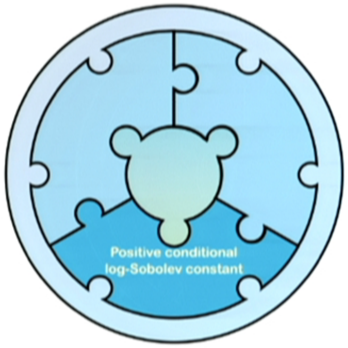
$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x\in\Lambda} D_x(\rho_\Lambda || \sigma_\Lambda).$$



Heat-bath with tensor product fixed point

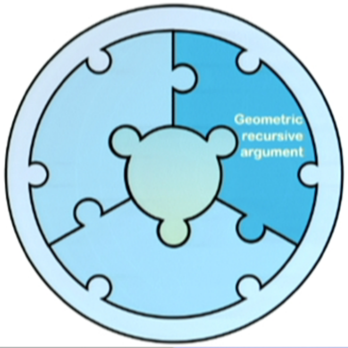
Lemma (Positivity of the conditional log-Sobolev constant)

$$\alpha_\Lambda(\mathcal{L}_x^*) \geq \frac{1}{2}.$$



Heat-bath with tensor product fixed point

$$\begin{aligned} D(\rho_\Lambda || \sigma_\Lambda) &\leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) \\ &\leq \sum_{x \in \Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}_x^*)} \\ &\leq \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \sum_{x \in \Lambda} - \operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \\ &= \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]\right) \\ &\leq \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]\right). \end{aligned}$$



Example 2

Heat-bath dynamics in 1D

Heat-bath dynamics in 1D

Conditional log-Sobolev constant

For $A \subset \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in A by

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2 D_A(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and

$$D_A(\rho_\Lambda || \sigma_\Lambda) = D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{A^c} || \sigma_{A^c}).$$

Heat-bath dynamics in 1D

Assumption 1

In a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$, A and B not connected, we have

$$\left\| h(\sigma_{AB}) \right\|_\infty = \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty \leq K < \frac{1}{2}.$$

In particular, classical Gibbs states satisfy this.

Assumption 2

For any $B \subset \Lambda$, $B = B_1 \cup B_2$, it holds:

$$D_B(\rho_\Lambda || \sigma_\Lambda) \leq f(\sigma_{B\partial}) \left(D_{B_1}(\rho_\Lambda || \sigma_\Lambda) + D_{B_2}(\rho_\Lambda || \sigma_\Lambda) \right).$$

In particular, tensor products satisfy this (with $f = 1$).

Heat-bath dynamics in 1D

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In particular, tensor products satisfy this (with $f = 1$).

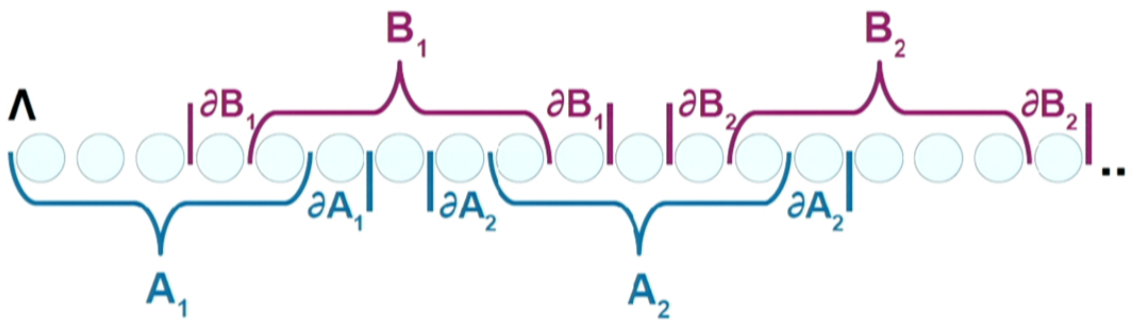
Examples of positive log-Sobolev constants

Theorem (Bardet-C-Lucia-Pérez García-Rouzé, '19)

In 1D, if Assumptions 1 and 2 hold, for a k-local commuting Hamiltonian, the heat-bath dynamics has a positive log-Sobolev constant.

Sketch of the proof

STEP 1



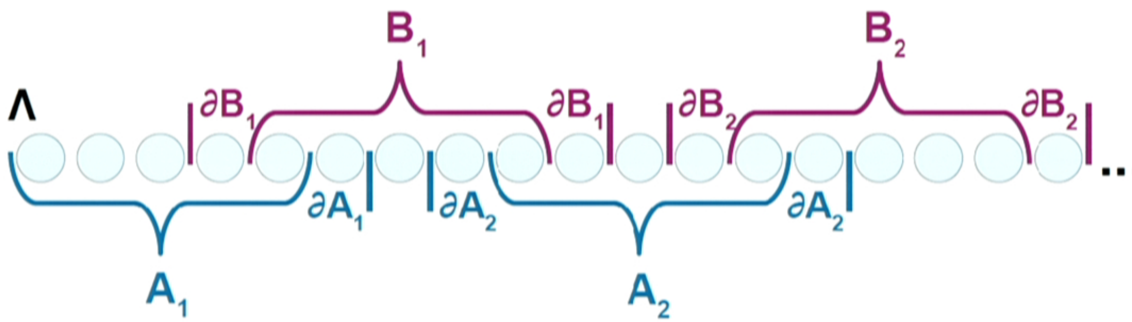
$$A = \bigcup_{i=1}^{n} A_i \text{ and } B = \bigcup_{j=1}^{n} B_j$$

$$D(\rho_\Lambda||\sigma_\Lambda) \leq \frac{1}{1-2\|h(\sigma_{A^cB^c})\|_\infty}\left[D_A(\rho_\Lambda||\sigma_\Lambda) + D_B(\rho_\Lambda||\sigma_\Lambda)\right].$$

$$h(\sigma_{A^cB^c}) := \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \sigma_{A^cB^c} \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} - 1_{A^cB^c}.$$

Sketch of the proof

STEP 1



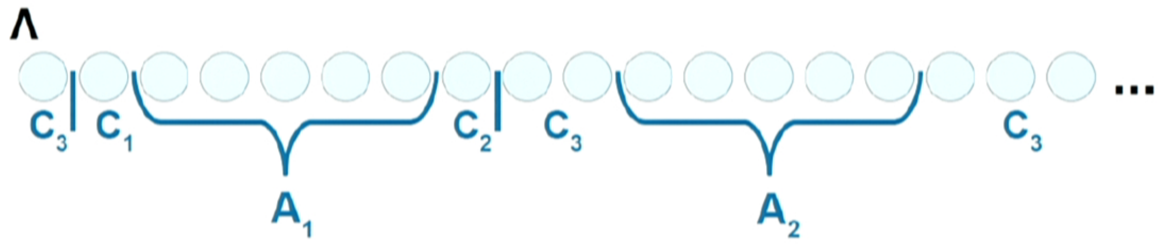
$$A = \bigcup_{i=1}^{n} A_i \text{ and } B = \bigcup_{j=1}^{n} B_j$$

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \frac{1}{1 - 2\|h(\sigma_{A^c B^c})\|_\infty} \left[D_A(\rho_\Lambda || \sigma_\Lambda) + D_B(\rho_\Lambda || \sigma_\Lambda) \right],$$

$$h(\sigma_{A^c B^c}) := \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \sigma_{A^c B^c} \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} - \mathbb{1}_{A^c B^c}.$$

Sketch of the proof

STEP 2



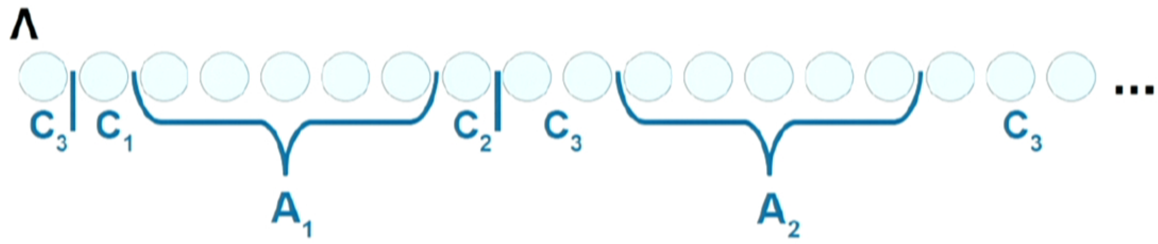
$$D_A(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{i=1}^{n} D_{A_i}(\rho_\Lambda || \sigma_\Lambda)$$

σ_Λ is a QMC between $A_1 \leftrightarrow \partial A_1 \leftrightarrow \Lambda \setminus (A_1 \cup \partial A_1)$

$$\sigma_\Lambda = \bigoplus_{i \in I} \sigma_{A_1 (\partial a_1)_i^L} \otimes \sigma_{(\partial a_1)_i^R \Lambda \setminus (A_1 \cup \partial A_1)}$$

Sketch of the proof

STEP 2



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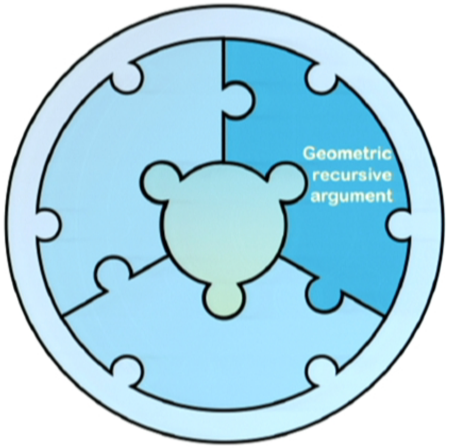
Sketch of the proof

STEP 3

$$\text{Assumption 1} \Rightarrow \alpha(\mathcal{L}_\Lambda^*) \geq \tilde{K} \min_{i \in \{1,\ldots n\}} \{\alpha_\Lambda(\mathcal{L}_{A_i}^*), \alpha_\Lambda(\mathcal{L}_{B_i}^*)\}$$



Davies dynamics

Generator

The generator of the Davies dynamics is of the following form:

$$\mathcal{L}_\Lambda^\beta(X) = i[H_\Lambda, X] + \sum_{k\in\Lambda} \mathcal{L}_k^\beta(X),$$

where

$$\mathcal{L}_k^\beta(X) = \sum_{\omega,\alpha} \chi_{\alpha,k}^\beta(\omega) \left(S_{\alpha,k}^*(\omega) X S_{\alpha,k}(\omega) - \frac{1}{2}\left\{ S_{\alpha,k}^*(\omega) S_{\alpha,k}(\omega), X \right\} \right).$$

Important property: Given $A \subseteq \Lambda$,

$$\mathcal{E}_A^\beta(X) := \mathcal{E}(X|\mathcal{N}) = \lim_{t\to\infty} \mathrm{e}^{t\mathcal{L}_A^\beta}(X).$$

is a conditional expectation onto the subalgebra of fixed points of $\mathcal{L}_A^\beta$.

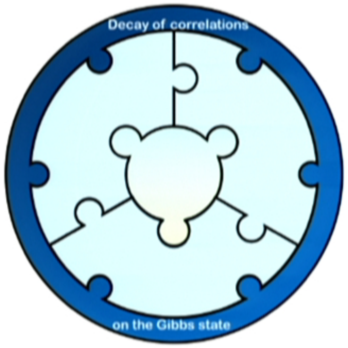
Davies dynamics

Clustering of correlations

The state $\sigma \in \mathcal{S}(\mathcal{H})$ is said to satisfy **exponential conditional $\mathbb{L}_1$-clustering of correlations** with respect to the triple $(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB})$ if there exists a constant $c := c(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB}, \sigma)$ such that, for any $X \in \mathcal{B}(\mathcal{H})$,

$$|\operatorname{Cov}_{\mathcal{N}_{AB},\sigma}(\mathcal{E}_A(X), \mathcal{E}_B(X))| \leq c \, \|X\|^2_{\mathbb{L}_1(\sigma)} e^{-d(A\setminus B, B\setminus A)/\xi} .$$

Moreover, the triple $(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB})$ is said to satisfy **exponential conditional $\mathbb{L}_1$-clustering of correlations** if there exists a constant $c := c(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB}, \sigma)$ such that any state $\sigma = \mathcal{E}^*_{AB}(\sigma)$ satisfies conditional $\mathbb{L}_1$-clustering of correlations with constant c.



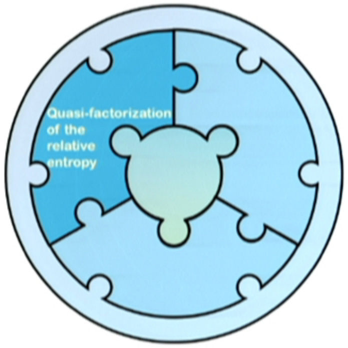
Davies dynamics

Quasi-factorization, Bardet-C-Rouzé '19

Assume that there exists a constant $0 < c < \frac{1}{2(4+\sqrt{2})}$ such that the triple $(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB})$ satisfies the exponential conditional $\mathbb{L}_1$-clustering of correlations with corresponding constant c. Then, the following inequality holds for every $\rho \in \mathcal{S}(\mathcal{H})$:

$$D^{\beta}_{AB}(\rho||\sigma) \leq \frac{1}{1-2(4+\sqrt{2})\,c}\left(D^{\beta}_{A}(\rho||\sigma) + D^{\beta}_{B}(\rho||\sigma)\right), \qquad (3)$$

for every $\sigma = \mathcal{E}^*_{AB}(\sigma)$.

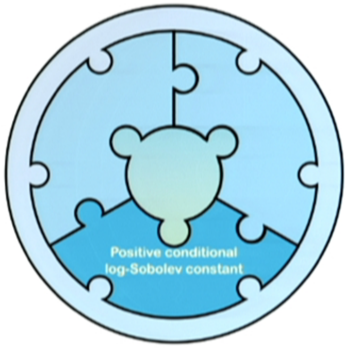


THEOREM, Junge-LaRacuente-Rouzé '19

Given $\Lambda \subset\subset \mathbb{Z}^d$, $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ the Lindbladian associated to the Davies dynamics and a finite lattice and $A \subset \Lambda$, we have

$$\alpha_\Lambda \left(\mathcal{L}_A^{\beta *} \right) \geq \psi(|A|) > 0,$$

where $\psi(|A|)$ might depend on Λ, but is independent of its size.



Open problems

Problem 1

Can we use any of the quasi-factorization results to prove log-Sobolev constants in a more general setting?

Problem 2

Does the heat-bath example hold for greater dimension?

Problem 3

Is there a better definition for conditional relative entropy?


thank you
danke
謝謝
ngiyabonga
شكراً جزيلاً
teşekkür ederim
gracias
спасибо
Баярлалаа
рахмат
dank je
tapadh leat
mochchakkeram
go raibh maith agat
dziękuje
bedankt
obrigado
sukriya
kop khun krap
arigatô
dakujem
merci
grazie
terima kasih
감사합니다
ευχαριστώ
мерси
хвала
hvala
mauruuru
sagolun
takk
taiku
enkosi
mersi
barka
welalin
tack
vinaka
blagodaram
misaotra
matondo
paldies
mahalo
asante
manana
faafetai lava
kiitos
dankie
sobodi
dekuji
mesi
najis tuke
tanemirt
rahmet
diolch
dhanyavadagalu
shukriya
trugarez
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sulpay
chnorakaloutioun
gracies
djiere dieuf
mamnun
ありがとう
kam sah hamnida