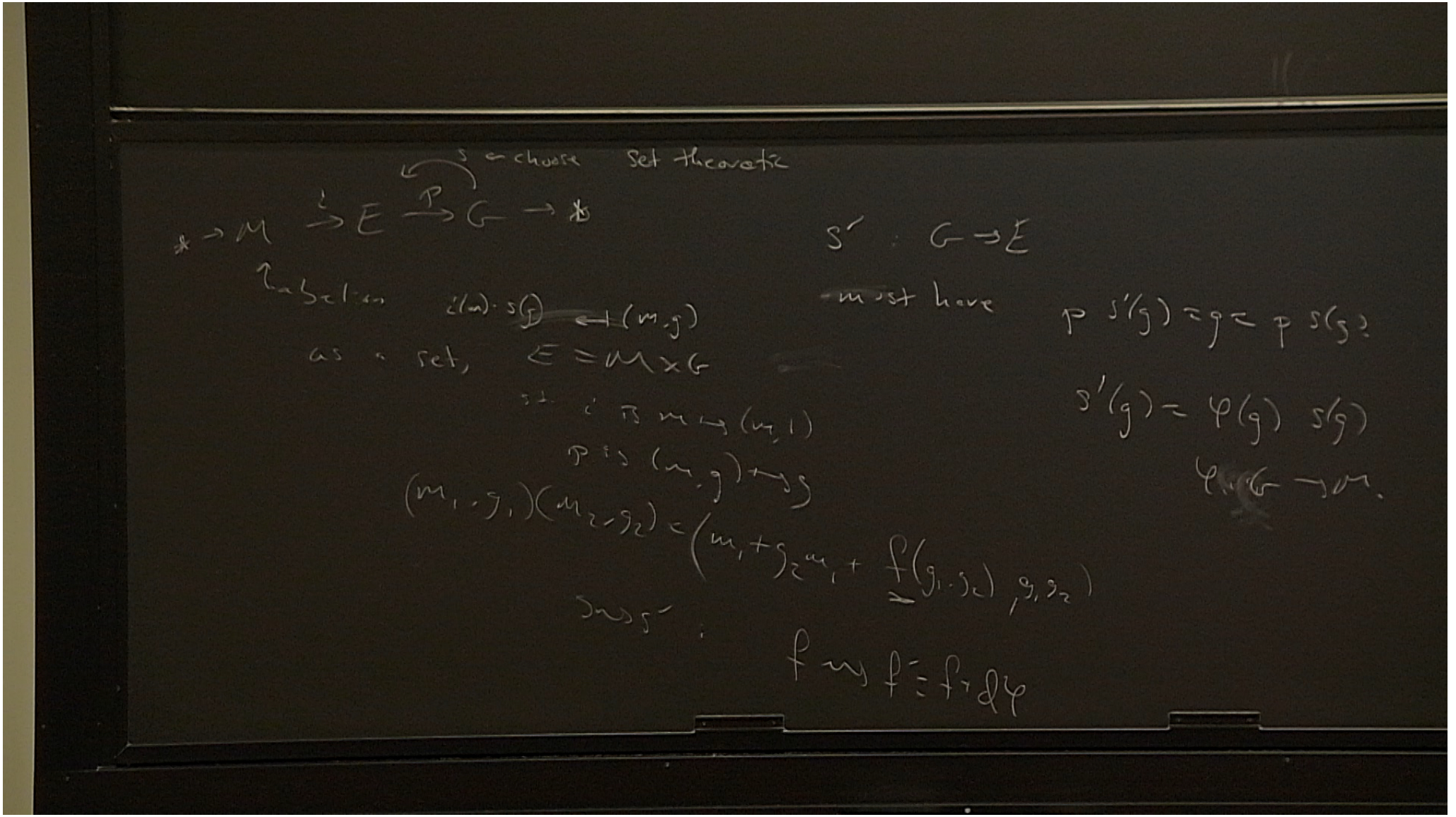
Title: The Cohomology of Groups (Johnson-Freyd/Guo) - Lecture 5

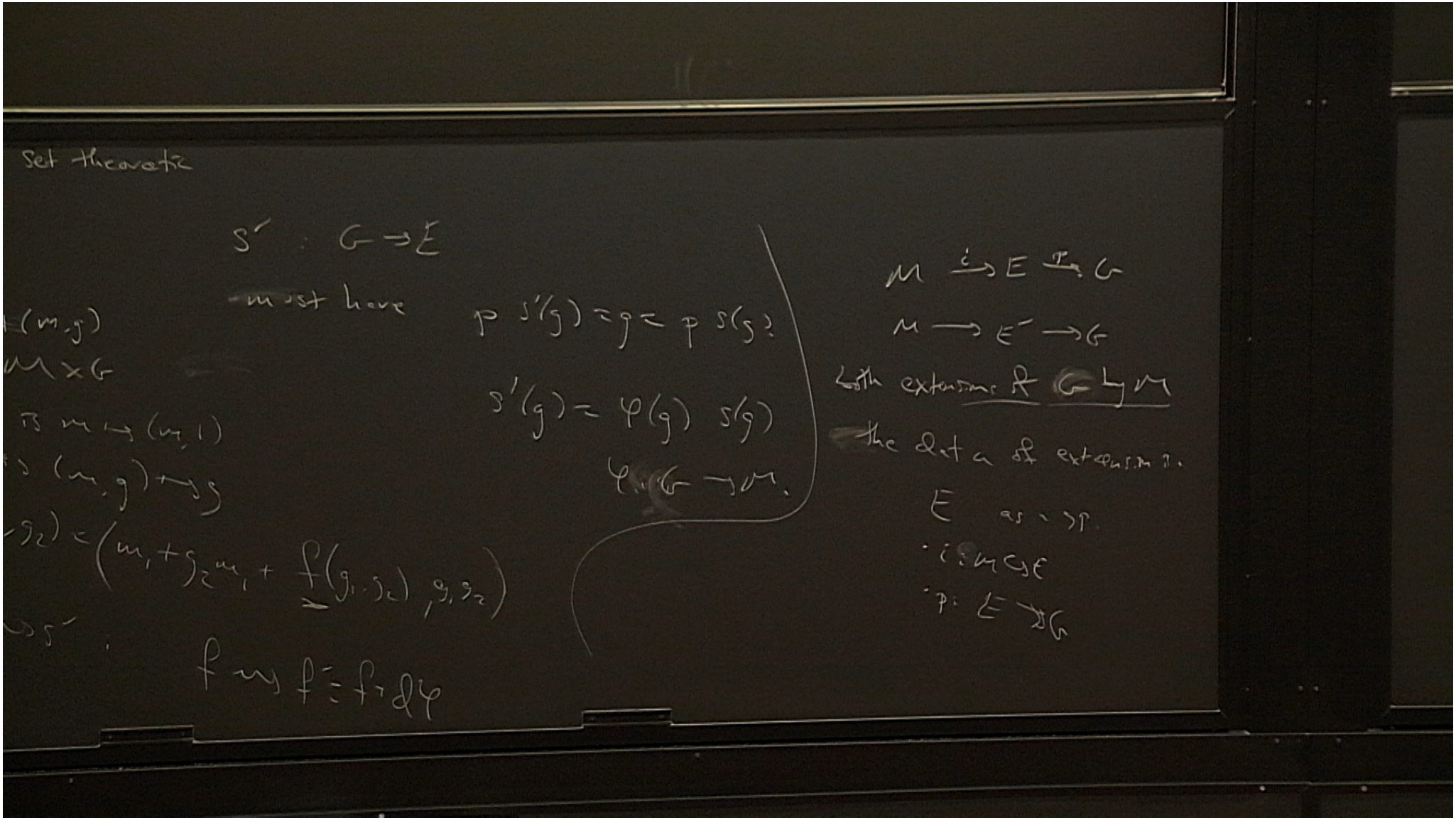
Speakers: Meng Guo, Theo Johnson-Freyd

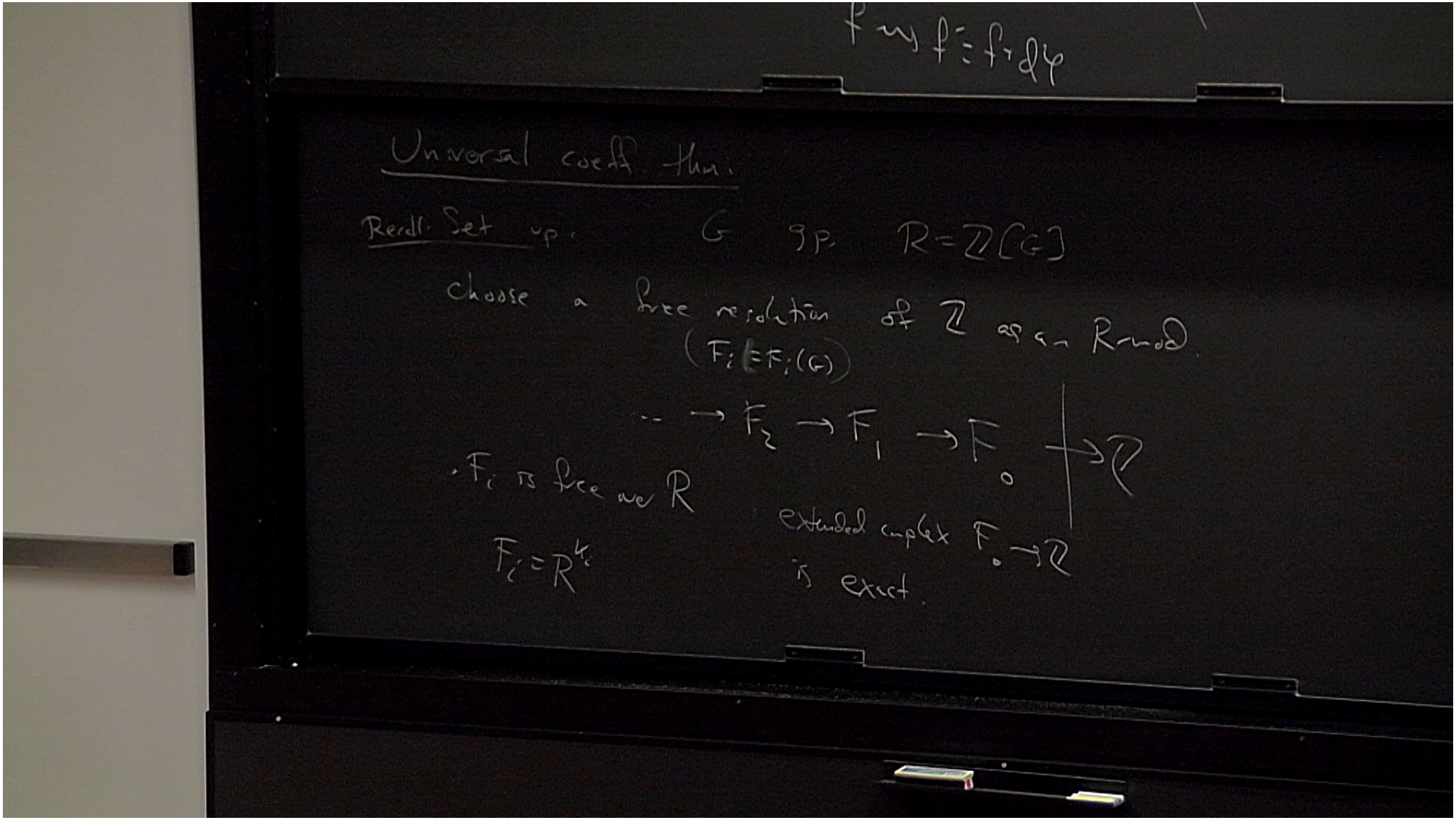
Collection: The Cohomology of Groups (Johnson-Freyd/Guo)

Date: October 30, 2019 - 10:00 AM

URL: http://pirsa.org/19100049


s = choose
Set theoretic
$* \to M \xrightarrow{i} E \xrightarrow{p} G \to *$
$s' : G \to E$
must have
$p\,s'(g) = g = p\,s(g)$
$s'(g) = \varphi(g)\, s(g)$
$\varphi : G \to M$.
Labelling $c(m) \cdot s(g) \longleftarrow (m, g)$
as a set, $E = M \times G$
$i : m \mapsto (m, 1)$
$p : (m, g) \mapsto g$
$(m_1, g_1)(m_2, g_2) = (m_1 + g_1 m_2 + f(g_1, g_2), g_1 g_2)$
$f \sim f' : f - f' = d\varphi$


Set theoretic
$s' : G \to E$
must have
$p\, s'(g) = g = p\, s(g)$
$s'(g) = \varphi(g)\, s(g)$
$\varphi : G \to M$
$M \xrightarrow{i} E \xrightarrow{p} G$
$M \to E \to G$
both extensions of G by M
the set of extensions of
E as
$i : M \to E$
$p : E \to G$
$f \sim f' \Leftrightarrow f - f' = d\varphi$


f ↦ f : f · dφ
Universal coeff. thm.
Recall: Set up: G gp. R = ℤ[G]
Choose a free resolution of ℤ as an R-module
(F_i = F_i(G))
... → F_2 → F_1 → F_0 → ℤ
· F_i is free over R
F_i = R^{k_i}
extended complex F_• → ℤ
is exact.

$f \sim_{\mathcal{G}} f' := f \circ d\varphi$

Thm.

G gp. $R = \mathbb{Z}[G]$

free resolution of $\mathbb{Z}$ as an R-mod

$(F_i = F_i(G))$

$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z}$

R

extended complex $F_\bullet \to \mathbb{Z}$ is exact.

M a G-module:

$H_i(G, M) \cong H^i(G, M)$

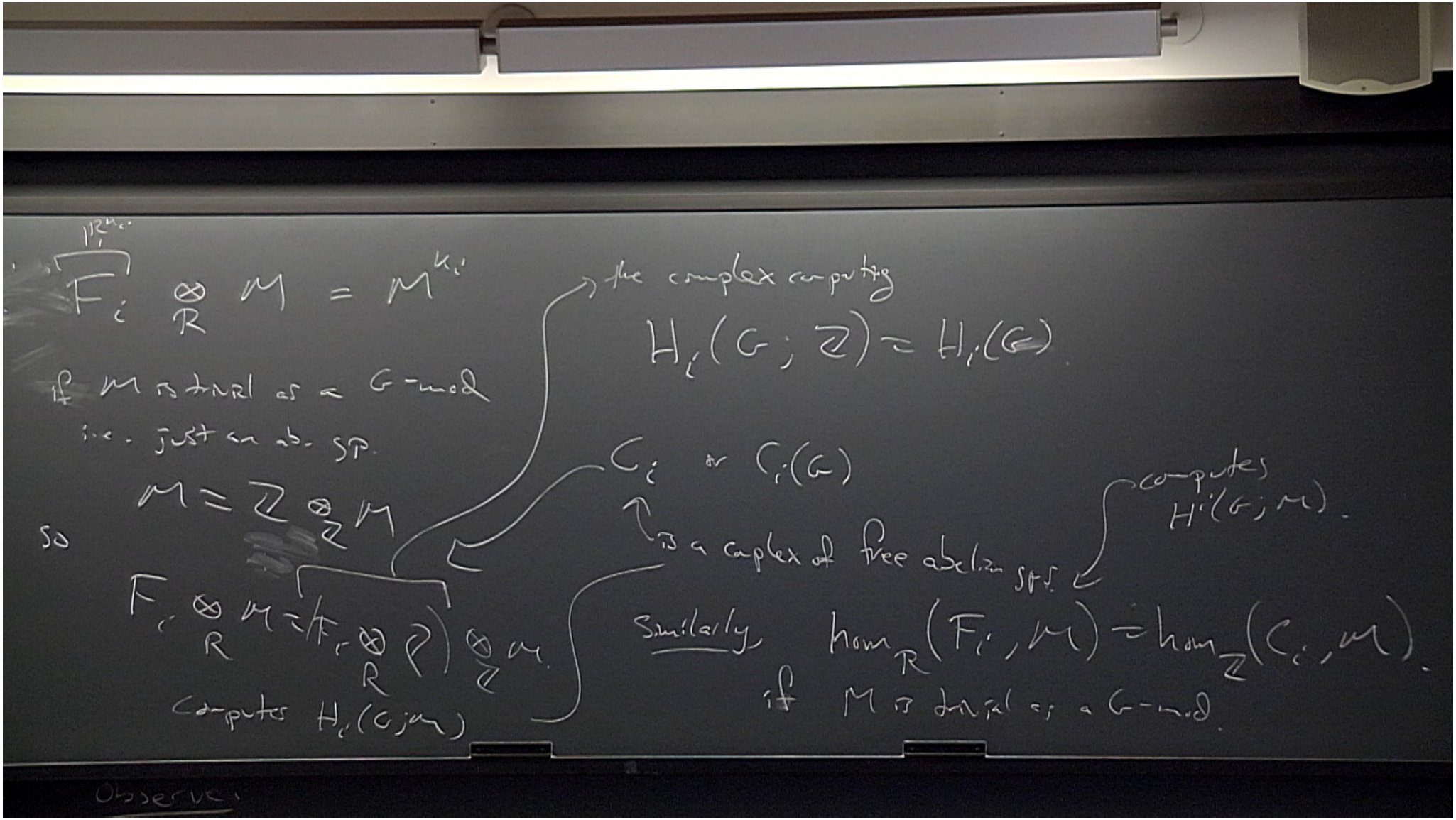
apply: $H_i(F_\bullet \otimes_R M)$ $\quad$ $H^i(\hom(F_\bullet, M))$

$(-) \otimes_R M \cong \hom_R(-, M)$

then take (co)homology.

UCT focuses in case where G-action on M is trivial.

$R^$


$F_i \otimes_R M = M^{k_i}$
if M is trivial as a G-mod
i.e. just an ab. gp.
$M = \mathbb{Z} \otimes_{\mathbb{Z}} M$
so
$F_i \otimes_R M = (F_i \otimes_R \mathbb{Z}) \otimes_{\mathbb{Z}} M$
computes $H_i(G; M)$
the complex computing
$H_i(G; \mathbb{Z}) = H_i(G)$
C_i or $C_i(G)$
is a complex of free abelian gps
computes $H^i(G; M)$.
Similarly, $\hom_R(F_i, M) = \hom_{\mathbb{Z}}(C_i, M)$.
if M is trivial as a G-mod.
Observe:

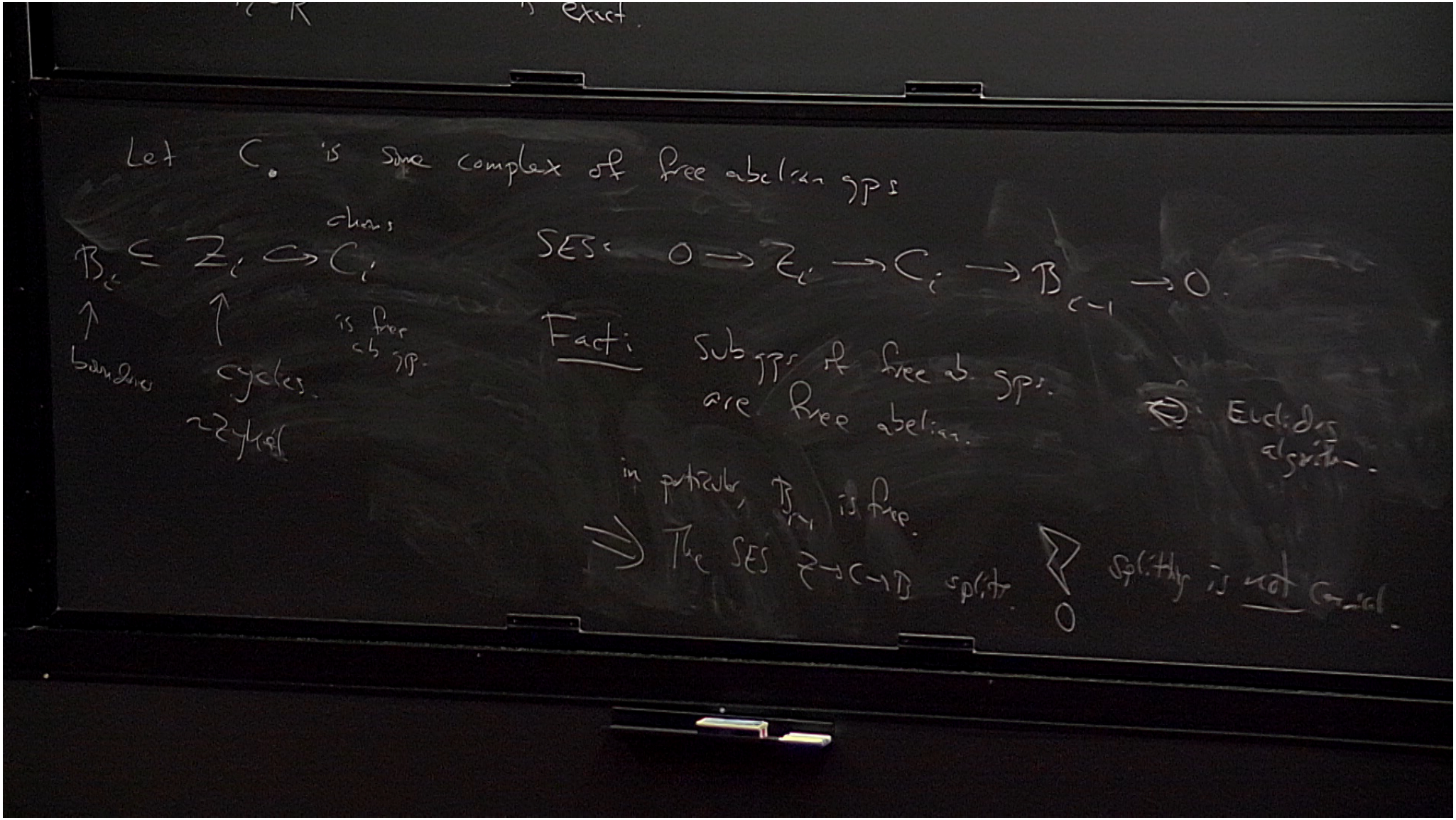
computes $H_i(G;M)$

Observe:

$H_i(G;M)$ and $H^i(G;M)$

only depend on $\mathbb{Z}M$ as an ab. gp.

· C_* as a complex of free ab. gps.


exact.
Let $C_\bullet$ is some complex of free abelian gps
$B_i \subseteq Z_i \hookrightarrow C_i$
chains
boundaries
cycles
is free ab gp.
SES: $0 \to Z_i \to C_i \to B_{i-1} \to 0$.
Fact: Subgps of free ab. gps are free abelian.
Euclid's algorithm.
in particular, B_{i-1} is free.
$\Rightarrow$ The SES $Z \to C \to B$ splits.
splitting is not canonical.
0

$\otimes_R M = F_i \otimes_R P \otimes_{\mathbb{Z}} M$.

computes $H_i(G; M)$

Similarly, $\hom_R(F_i, M) = \hom_{\mathbb{Z}}(C_i, M)$.

if M is trivial as a G-mod.

Up to noncanonical isomorphism,

$$\cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots$$

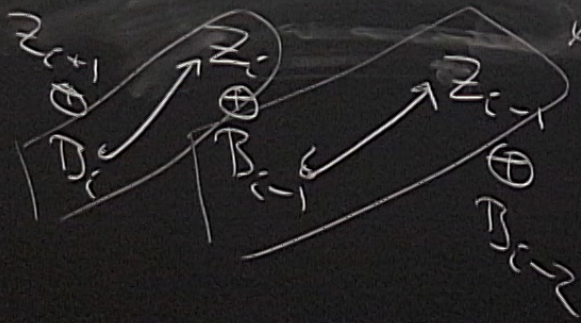
$C_i \cong$ a $\oplus$ of 2-dim complexes

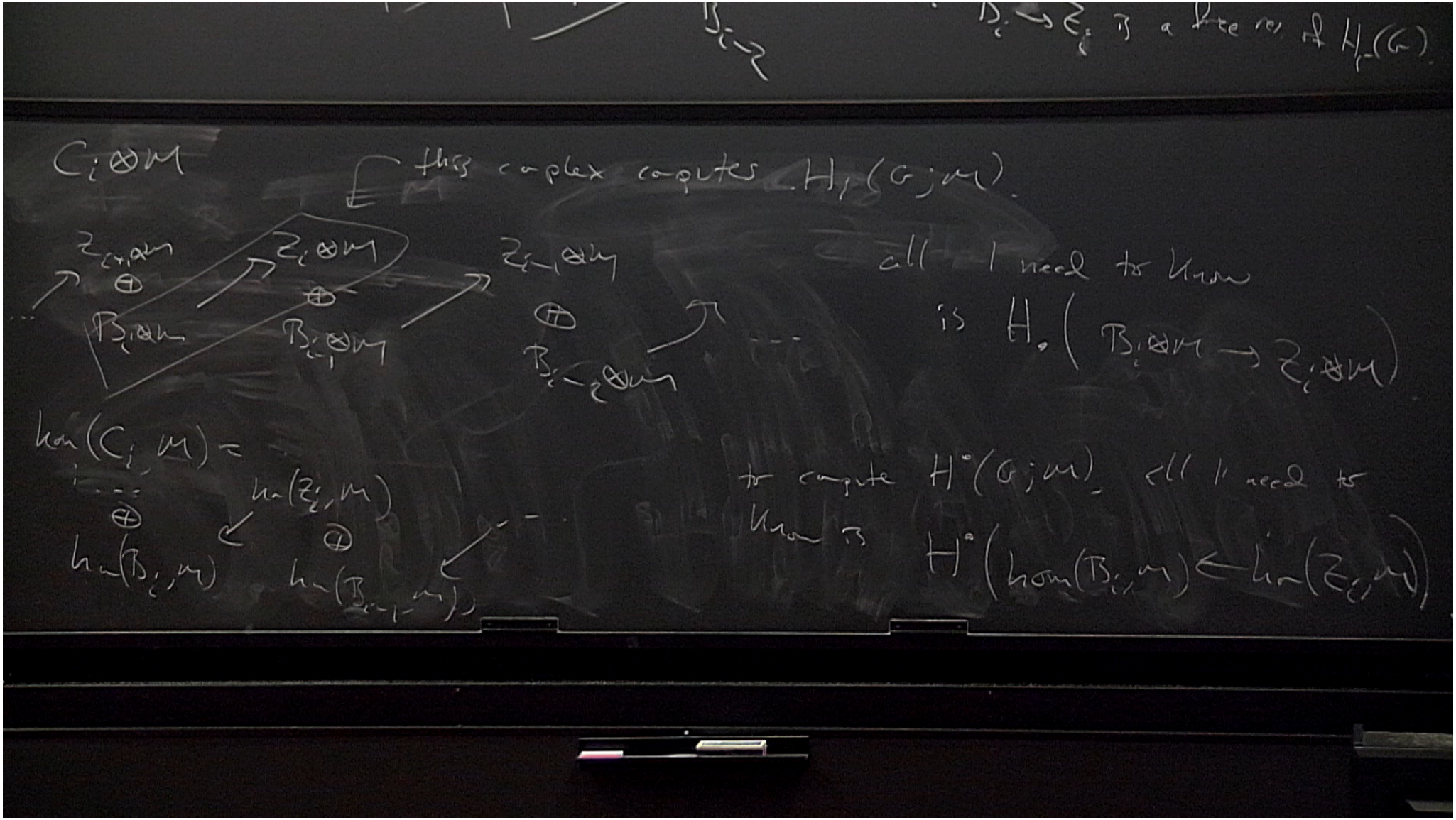
$$0 \to Z_i \to B_i \to 0$$

are free abs.

H_* of this little complex is $H_i(G)$.

i.e. $B_i \to Z_i$ is a free res of $H_i(G)$.




$B_i \to Z_i$ is a free res of $H_i(G)$
$C_i \otimes M$
this complex computes $H_i(G;M)$
$Z_{i+1} \otimes M$
$Z_i \otimes M$
$Z_{i-1} \otimes M$
$B_i \otimes M$
$B_{i-1} \otimes M$
$B_{i-2} \otimes M$
all I need to know is $H_0(B_i \otimes M \to Z_i \otimes M)$
$\hom(C_i, M) =$
$\hom(Z_i, M)$
$\hom(B_i, M)$
$\hom(B_{i-1}, M)$
to compute $H^0(G;M)$ all I need to know is $H^0(\hom(B_i, M) \leftarrow \hom(Z_i, M))$

$F_i = R^{4^i}$ is exact.

know $B_i \to Z_i$ is a free res of $H_i(G)$ — different choices are homotopy equiv.

Let A be any abelian gp. Choose a free resolution of ab. gps.

$\cdots \to P_1 \to P_0 \twoheadrightarrow A$

M is some other ab gp.

Defn.

Fact about Euclid's algs. $\Rightarrow$ I can choose $P_i = 0$ $\forall i > 1$.

basic fact.

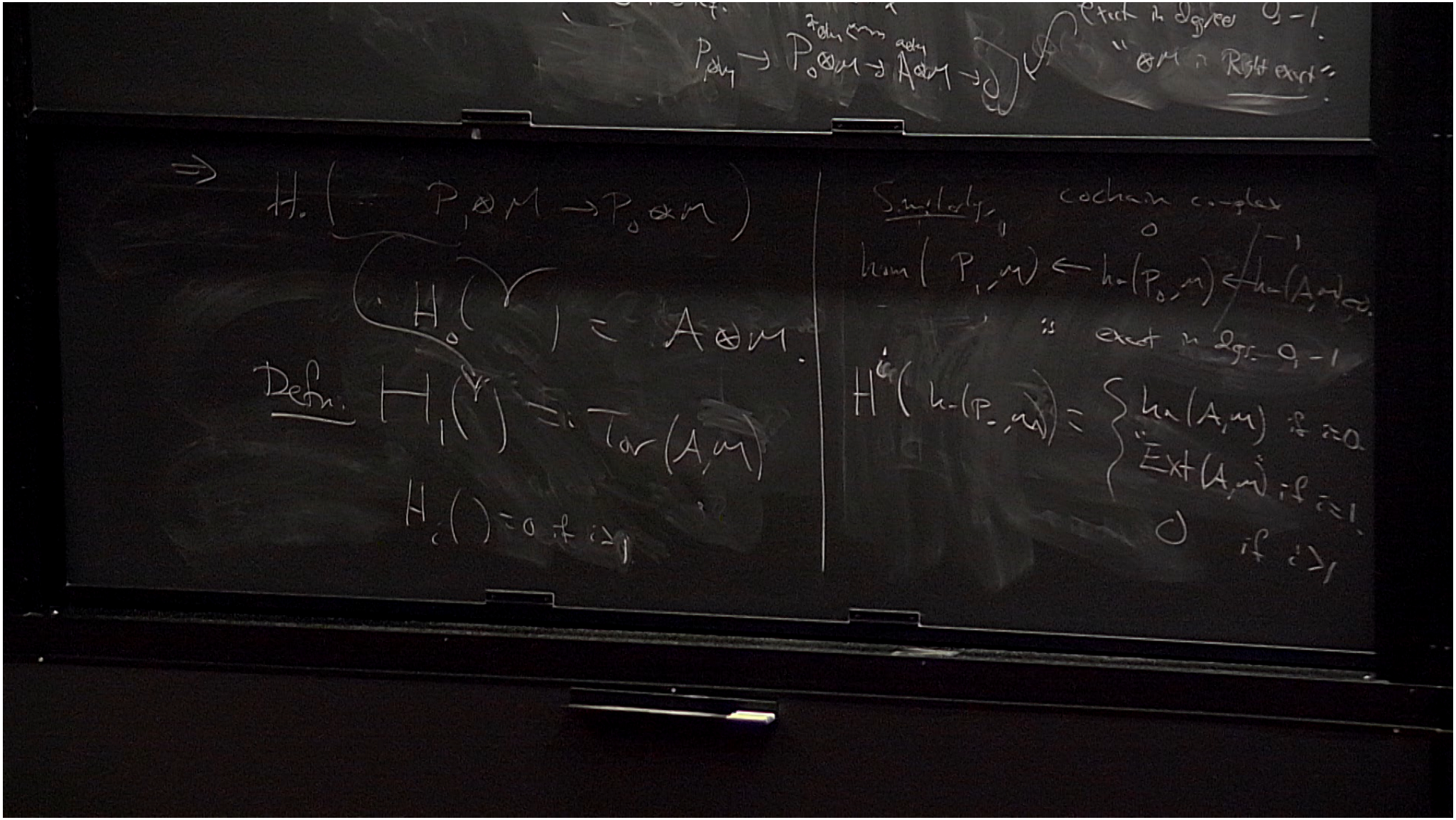
$S \in S_1$

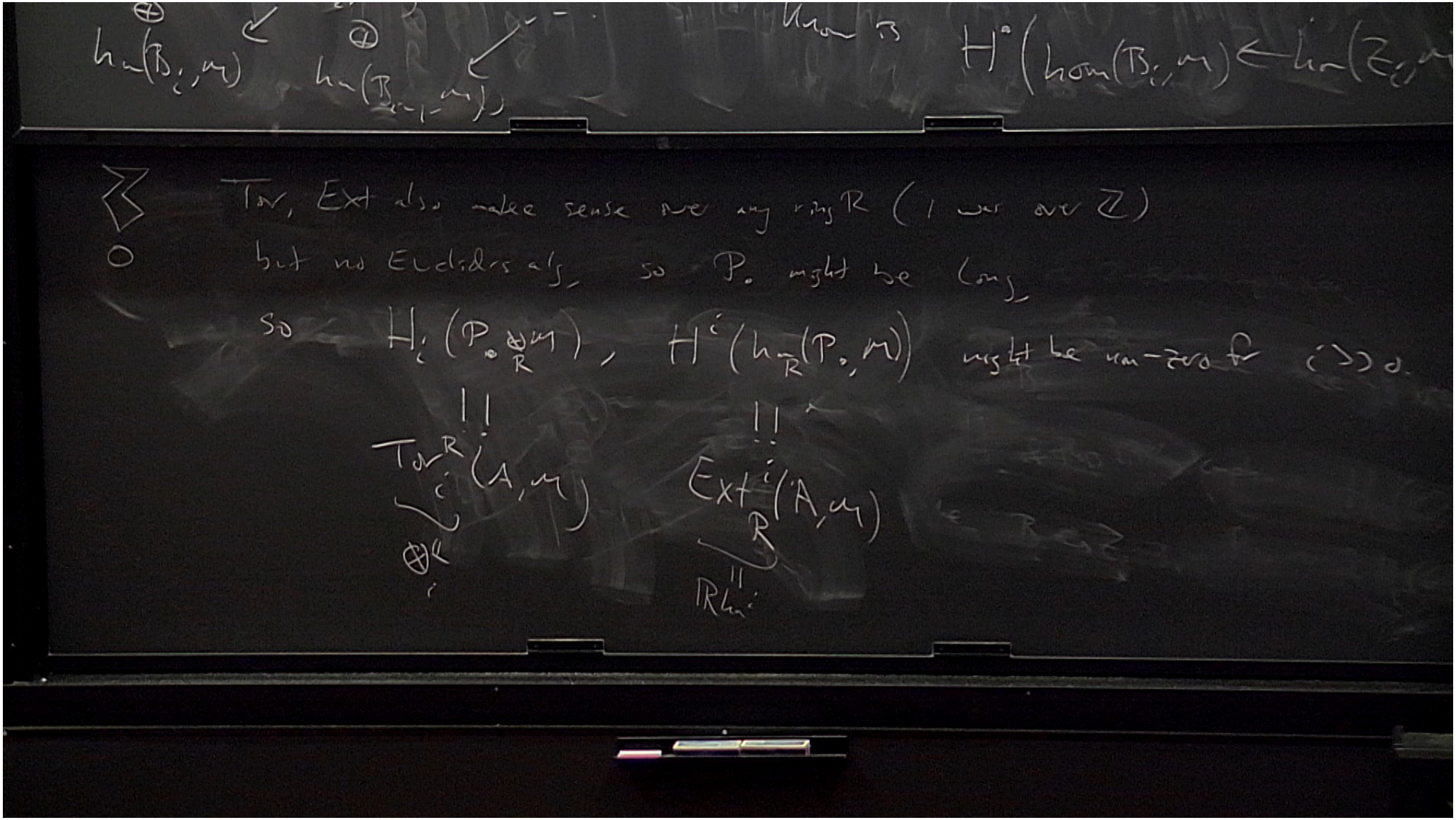
$0 \to P_1 \to P_0 \twoheadrightarrow A \to 0$

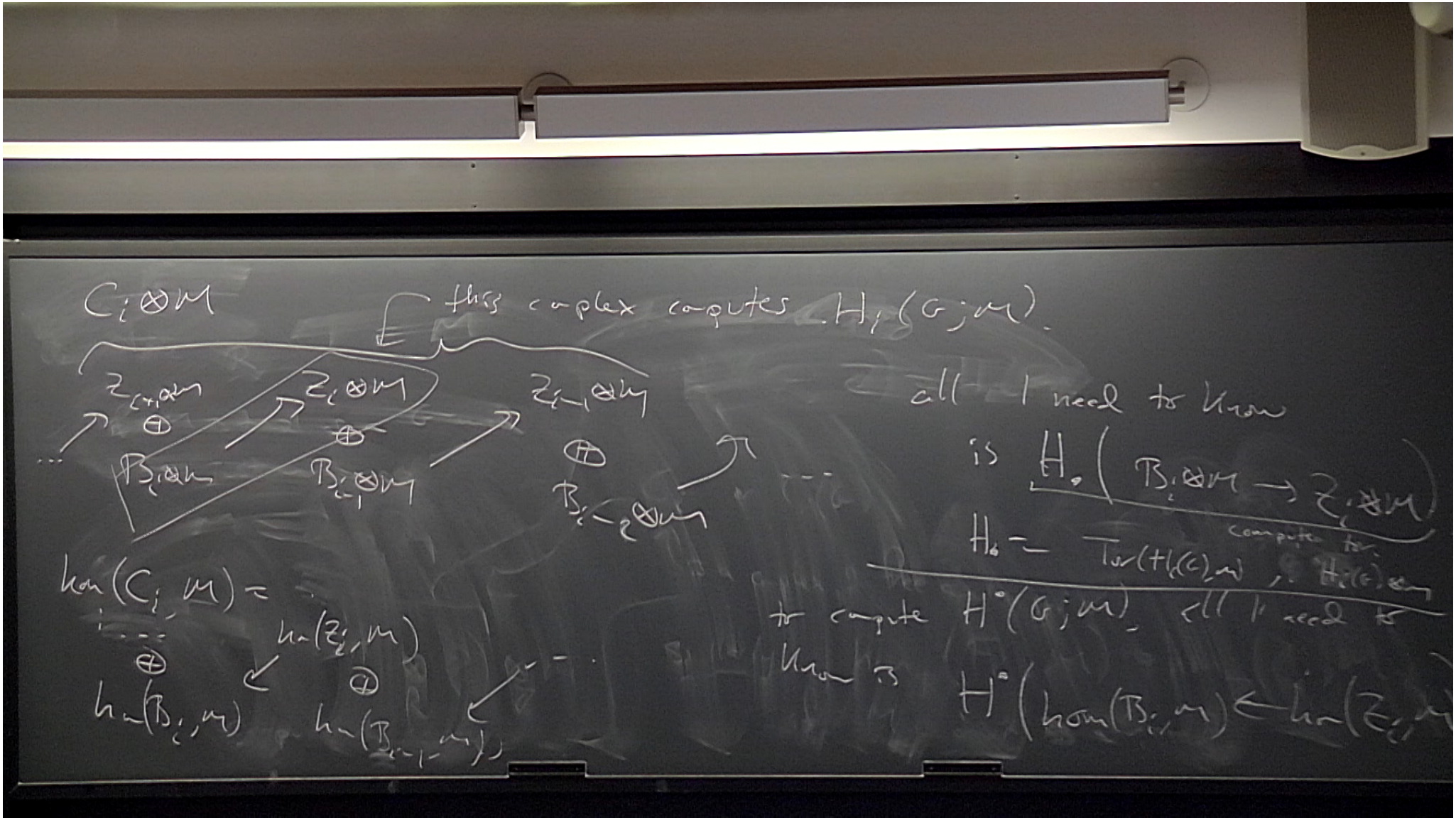
$\otimes M$ is right exact.

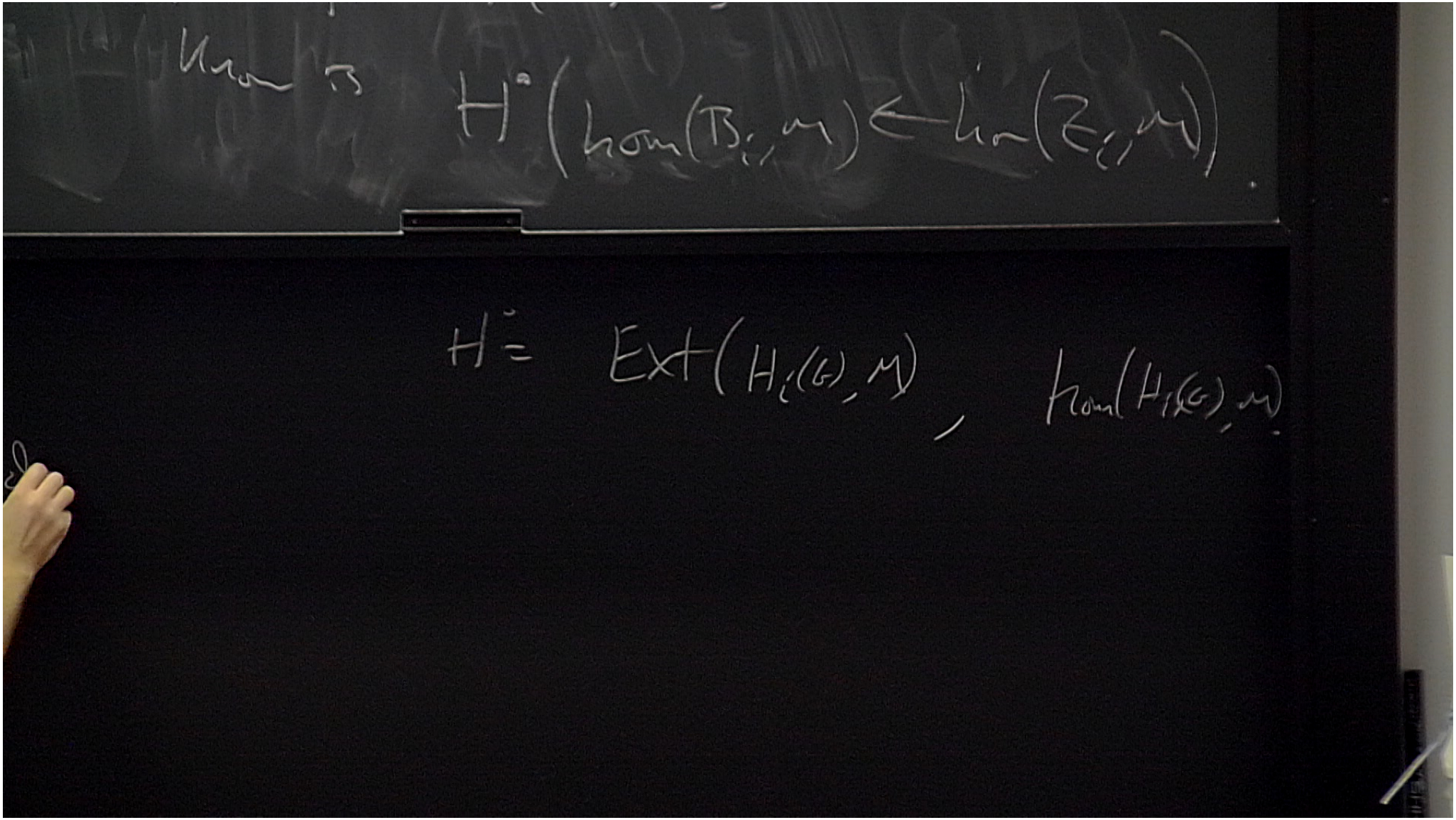
$P_1 \otimes M \to P_0 \otimes M \to A \otimes M \to 0$

$\otimes M$ is Right exact.


$P_1 \otimes M \to P_0 \otimes M \to A \otimes M \to 0$
"$\otimes M$: Right exact"
$\Rightarrow H_\cdot(\cdots P_1 \otimes M \to P_0 \otimes M)$
$H_0(\quad) = A \otimes M$.
Defn. $H_1(\quad) =: \mathrm{Tor}(A, M)$
$H_i(\quad) = 0$ if $i \geq 1$
Similarly, cochain complex
$\mathrm{Hom}(P_1, M) \leftarrow \mathrm{Hom}(P_0, M) \leftarrow \mathrm{Hom}(A, M) \leftarrow 0$
exact in deg $\leq 0, -1$
$H^i(\mathrm{Hom}(P_\cdot, M)) = \begin{cases} \mathrm{Hom}(A, M) & \text{if } i = 0 \\ \mathrm{Ext}(A, M) & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$


Tor, Ext also make sense over any ring R (1 was over ℤ)
but no Euclidean alg, so $P_\bullet$ might be long
so $H_i(P_\bullet \otimes_R M)$, $H^i(\hom_R(P_\bullet, M))$ might be non-zero for $i \gg 0$.
$\mathrm{Tor}_i^R(A, M)$
$\mathrm{Ext}^i_R(A, M)$


$C_i \otimes M$
this complex computes $H_i(G;M)$.
$Z_{i+1} \otimes M$
$Z_i \otimes M$
$Z_{i-1} \otimes M$
$B_i \otimes M$
$B_{i-1} \otimes M$
$B_{i-2} \otimes M$
all I need to know
is $H_0(B_i \otimes M \to Z_i \otimes M)$
$H_0 = \mathrm{Tor}(H_i(G), M)$, $H_i(G) \otimes M$
to compute $H^*(G;M)$, all I need to know is
$H^*(\hom(B_i, M) \leftarrow \hom(Z_i$


$H^n(\hom(B_i, M) \leftarrow \hom(Z_i, M))$
$H^* = \mathrm{Ext}(H_i(G), M)$, $\hom(H_i(G), M)$

Universal coef. thm:

if M is trivial as a G-module, then there are non canonical isomorphisms

$$H_i(G;M) \cong H_i(G) \otimes M \oplus \mathrm{Tor}(H_{i-1}(G), M)$$

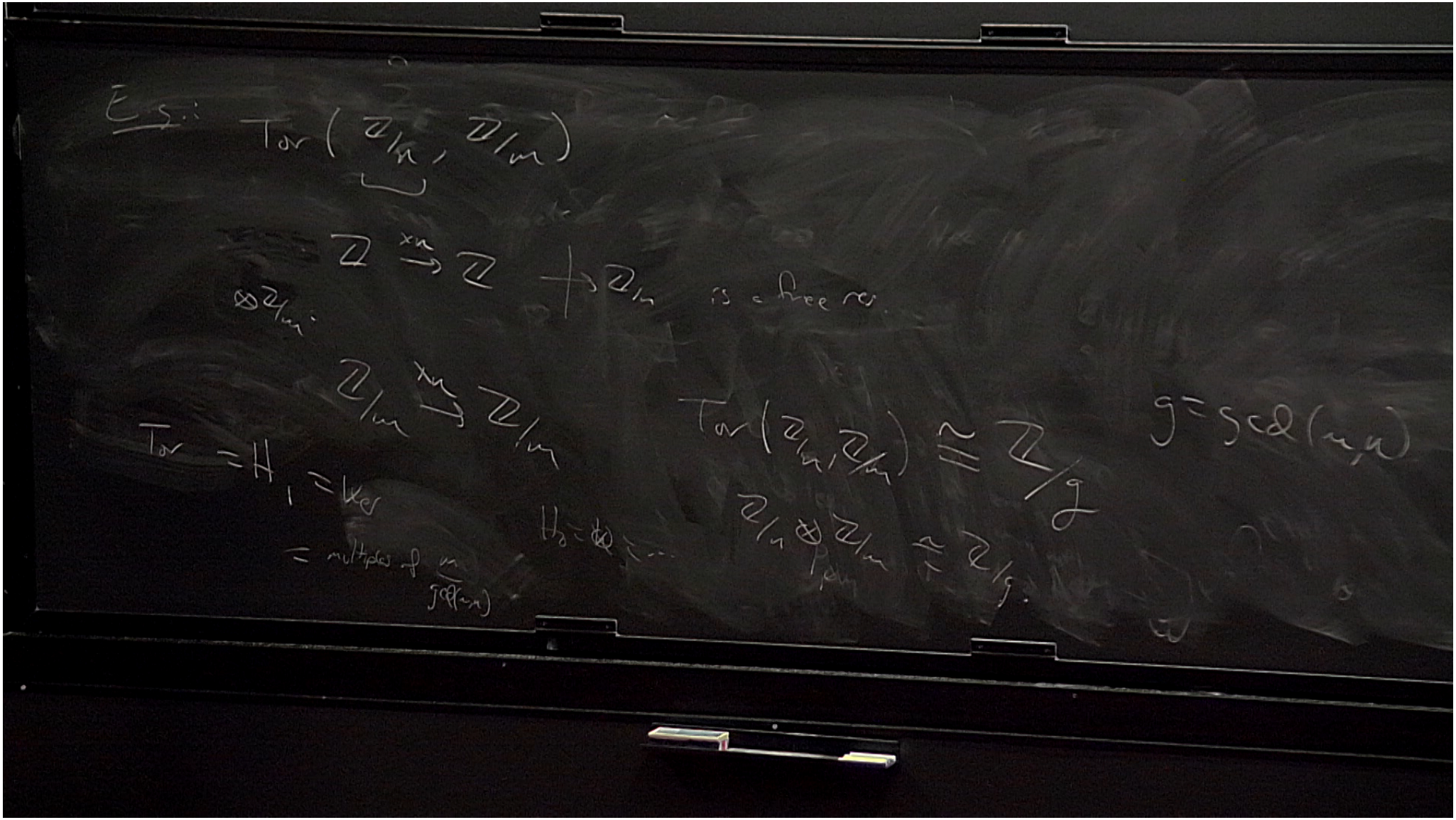
$$H^i(G;M) \cong \mathrm{Hom}(H_i(G), M) \oplus \mathrm{Ext}(H_{i-1}(G), M)$$

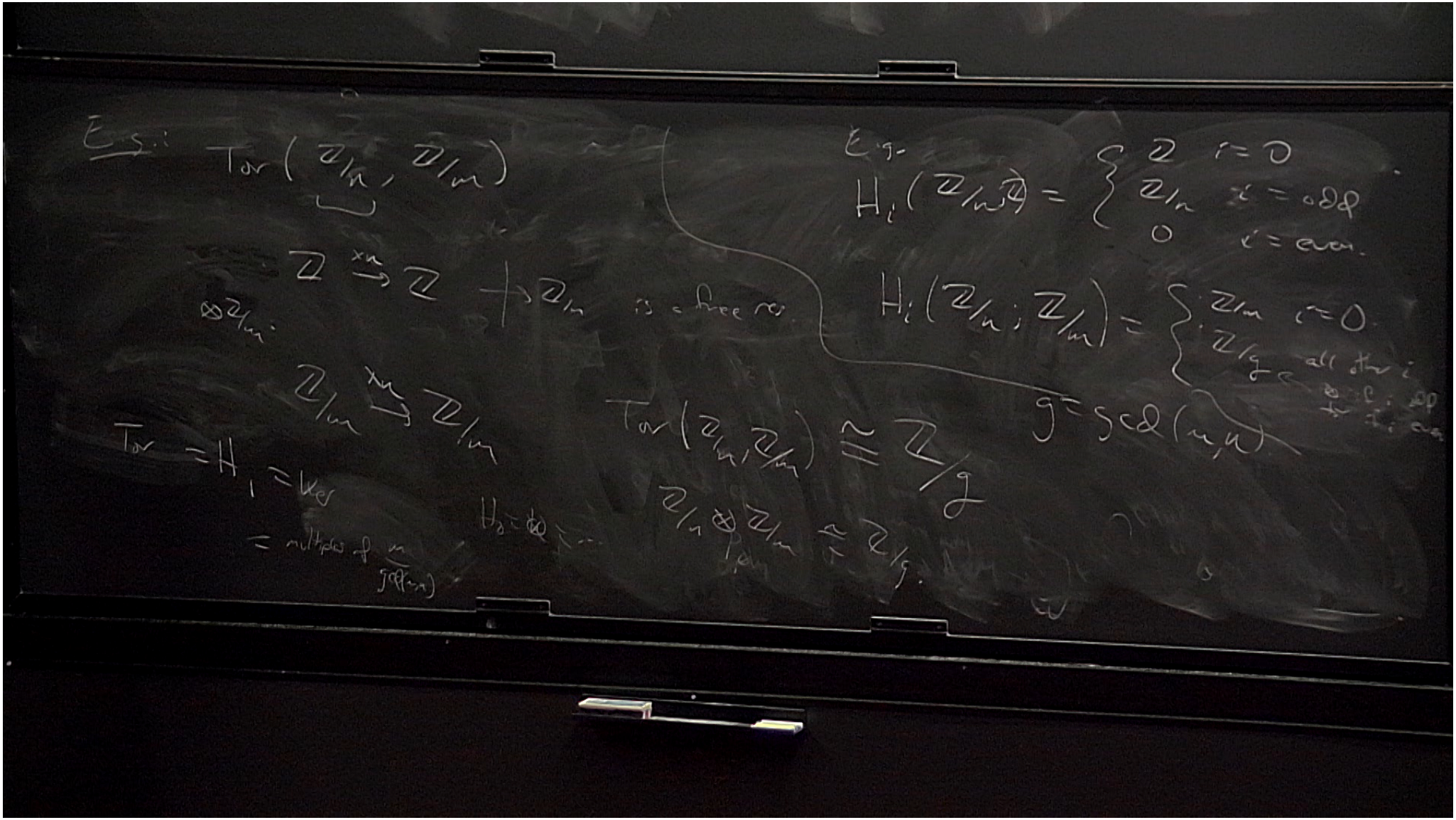
What is canonical is $Z_i \hookrightarrow C_i \to B_{i-1}$

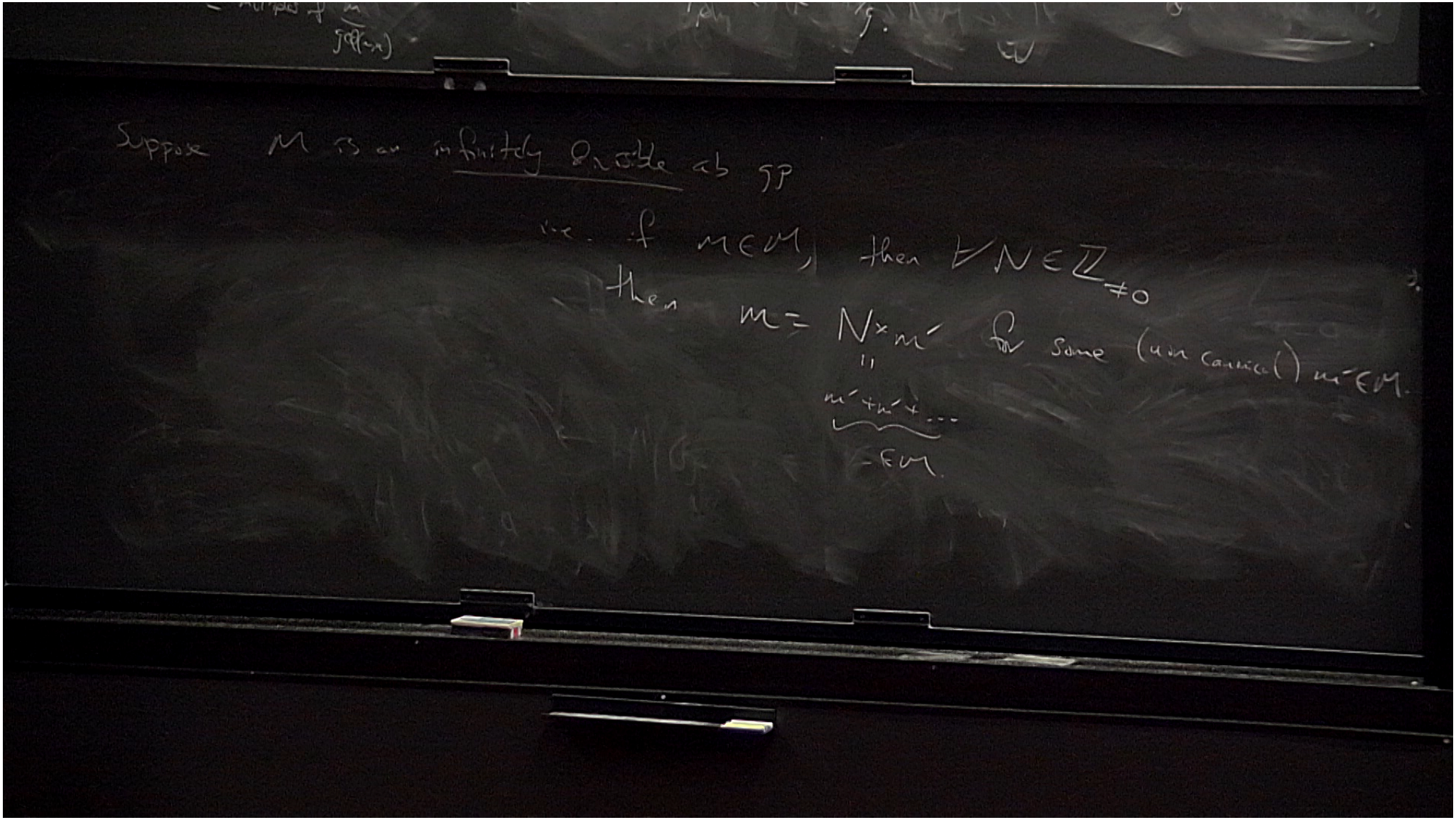
So the SES:

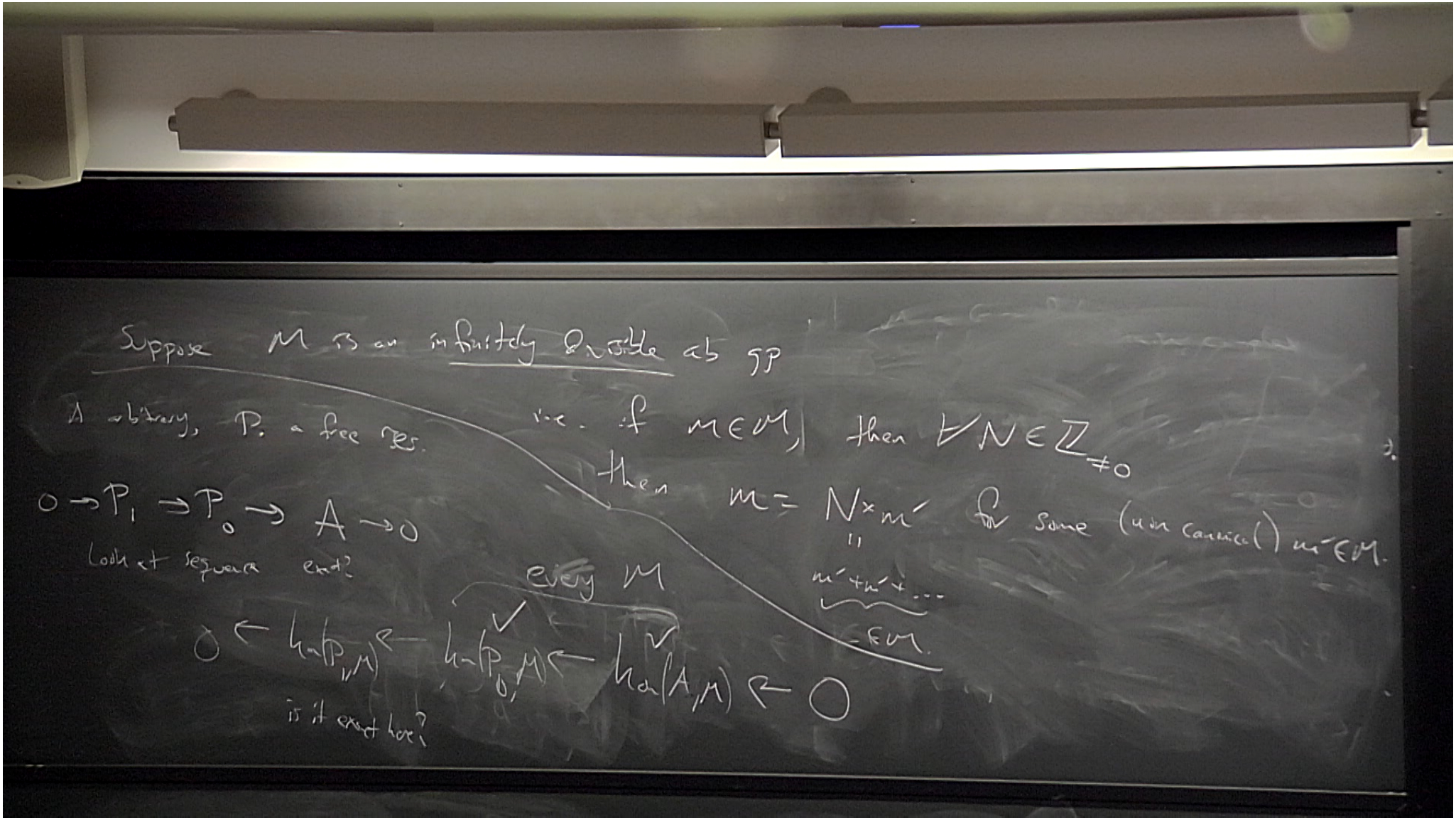
$$0 \to H_i(G) \otimes M \to H_i(G; M) \to Tor(H_{i-1}(G), M) \to 0$$

$$0 \leftarrow hom(H_i G, M) \leftarrow H^i(G; M) \leftarrow Ext(H_{i-1} G, M) \leftarrow 0.$$


Ex: Tor(Z/n, Z/m)
Z →(×n) Z → Z/n is a free res.
⊗Z/m:
Z/m →(×n) Z/m
Tor = H1 = ker
= multiples of n
gcd(n,m)
H0 = ⊗ = ...
Tor(Z/n, Z/m) ≅ Z/g
g = gcd(n,m)
Z/n ⊗ Z/m ≅ Z/g.


Ex:
$\mathrm{Tor}(\mathbb{Z}/n, \mathbb{Z}/m)$
$\mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n$ is a free res.
$\otimes \mathbb{Z}/m$
$\mathbb{Z}/m \xrightarrow{\times n} \mathbb{Z}/m$
$\mathrm{Tor} = H_1 = \ker$
= multiples of n in $\mathbb{Z}/m$
$H_2 = 0$
$\mathrm{Tor}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/g$
$\mathbb{Z}/n \otimes \mathbb{Z}/m \cong \mathbb{Z}/g$.
e.g.
$H_i(\mathbb{Z}/n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/n & i = \text{odd} \\ 0 & i = \text{even} \end{cases}$
$H_i(\mathbb{Z}/n; \mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m & i=0 \\ \mathbb{Z}/g & \text{all other } i \end{cases}$
$g = \gcd(n, m)$


Suppose M is an infinitely divisible ab gp
i.e. if $m \in M$, then $\forall N \in \mathbb{Z}_{\neq 0}$
then $m = N \times m'$ for some (non canonical) $m' \in M$.
$m' + m' + \cdots$
$\in M$

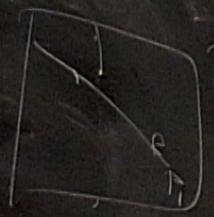

Suppose M is an infinitely divisible ab gp
i.e. if $m \in M$, then $\forall N \in \mathbb{Z}_{\neq 0}$
then $m = N \times m'$ for some (non canonical) $m' \in M$.
$m' + m' + \cdots$
$\in M$.
A arbitrary, $P_\bullet$ a free res.
$0 \to P_1 \to P_0 \to A \to 0$
Look at sequence exact?
every M
$0 \leftarrow \hom(P_1, M) \leftarrow \hom(P_0, M) \leftarrow \hom(A, M) \leftarrow 0$
is it exact here?

is it exact here? Yes

i.e. is

$$\mathrm{Hom}(P_1, M) \leftarrow \mathrm{Hom}(P_0, M) \quad \text{surjective?}$$

$f : P_1 \to M$.



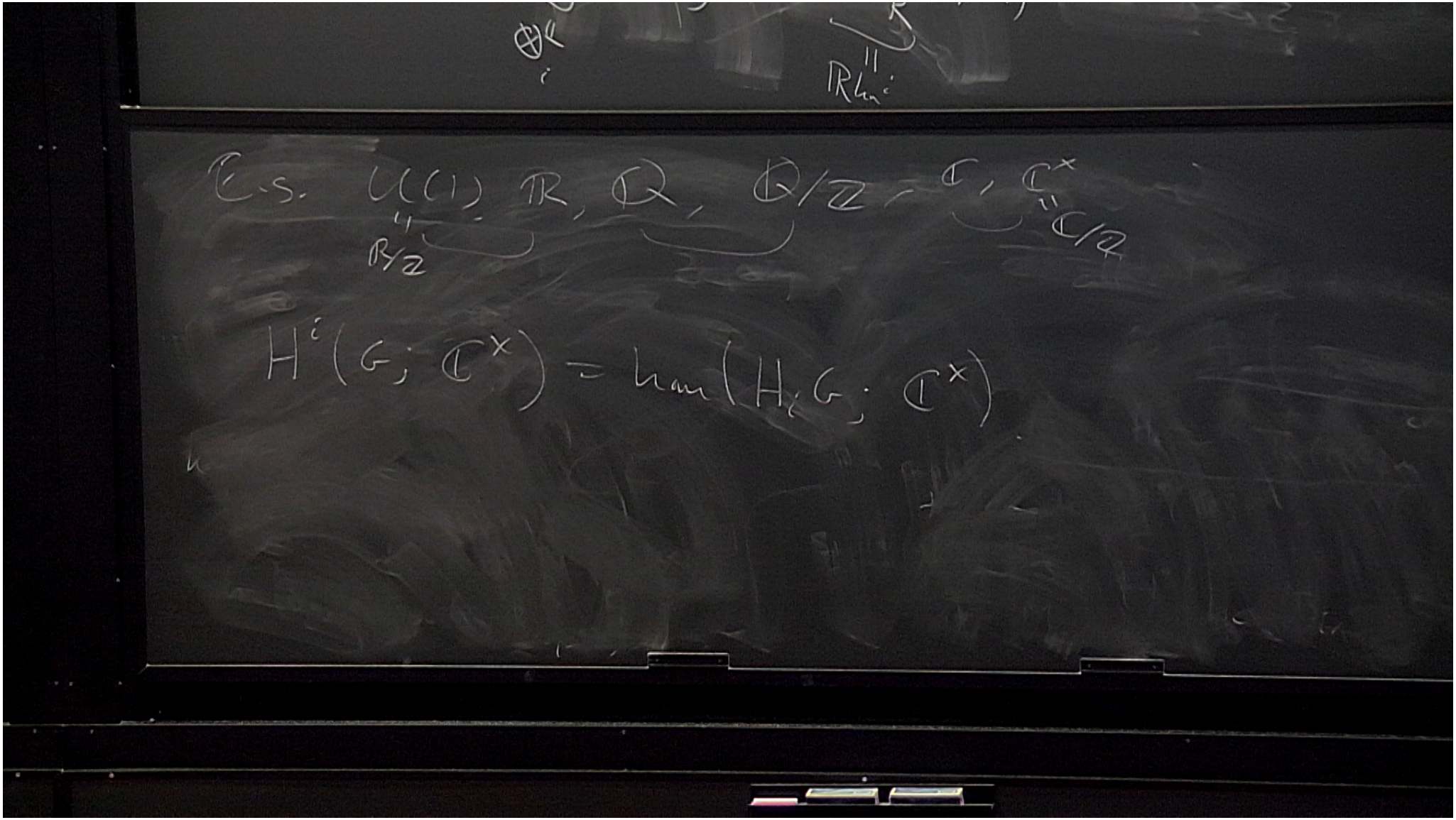
pick $p \in P_0 \setminus P_1$

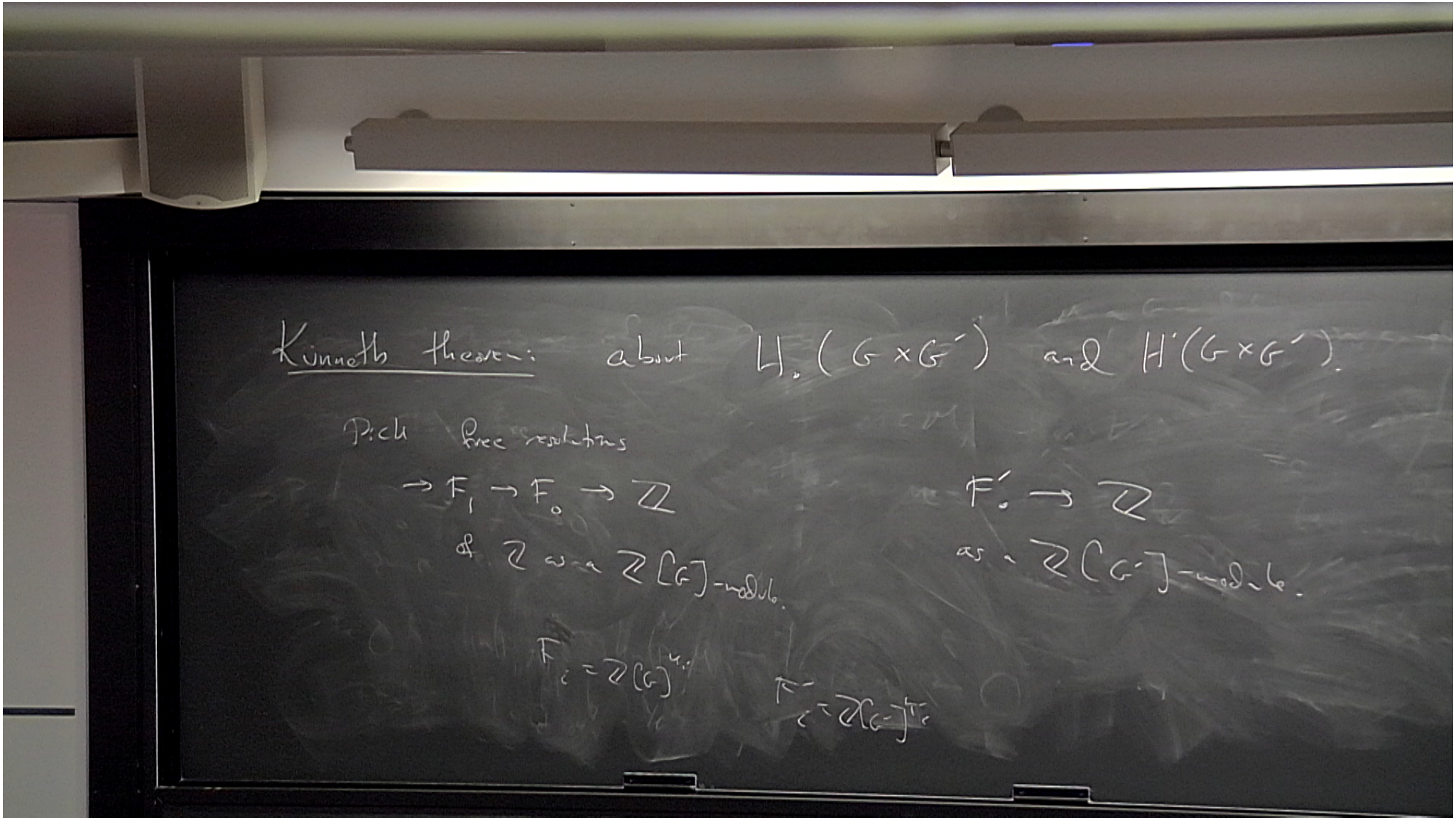
either $Np \in P_1$ & some N (minimal) for p

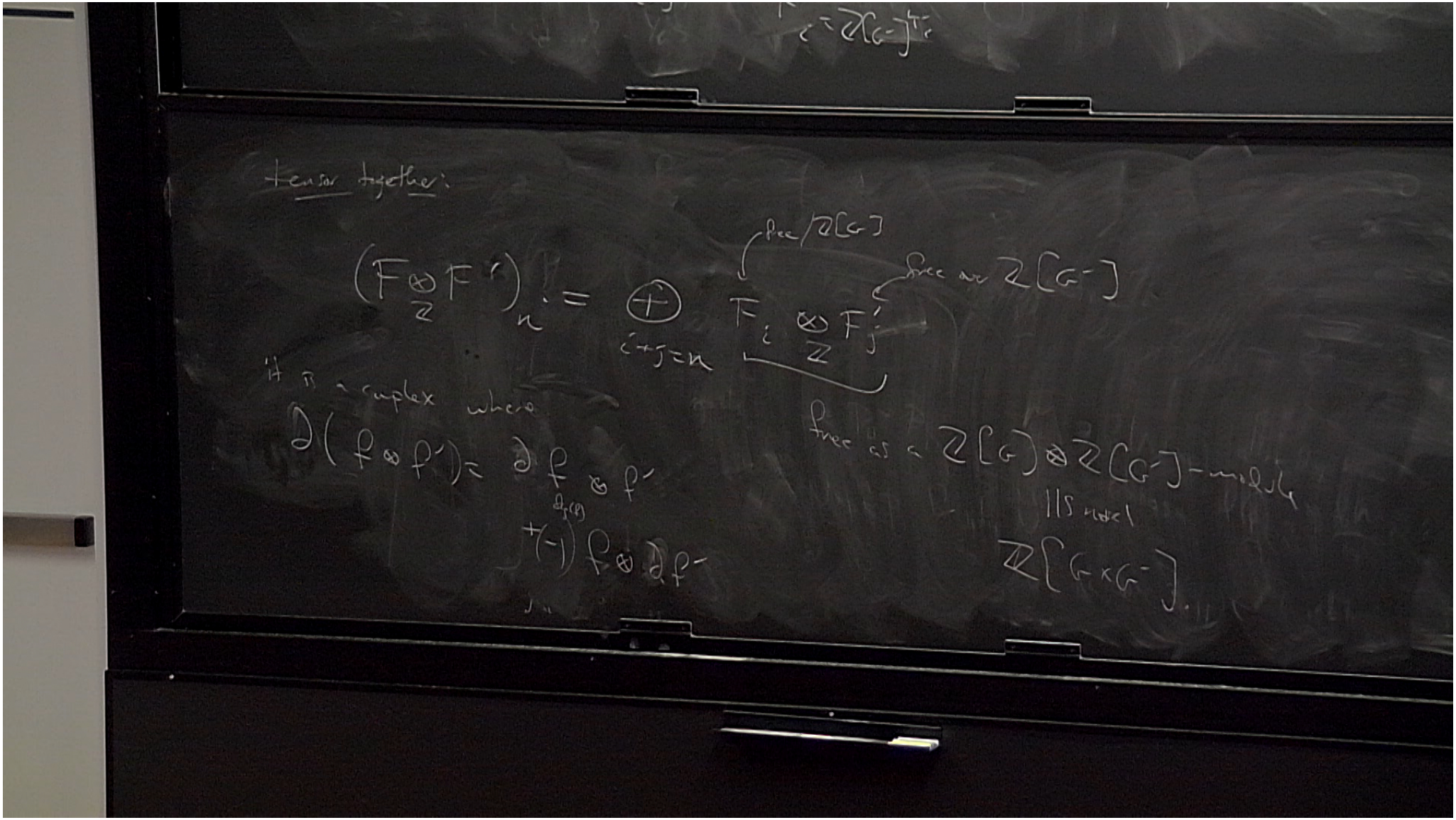
$M = f(Np)$ divided by N

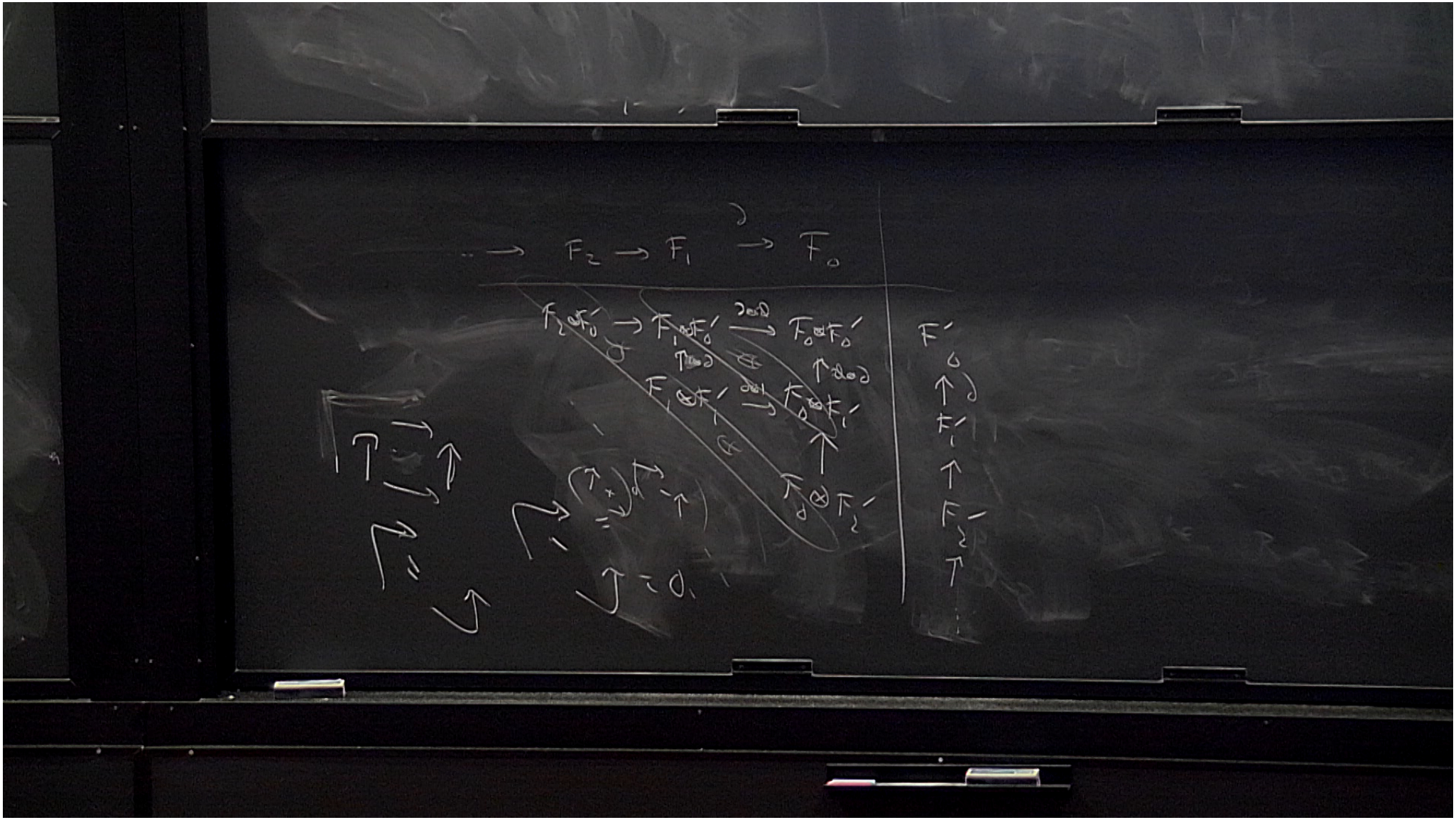
choose $f(p) = \frac{f(Np)}{N}$;

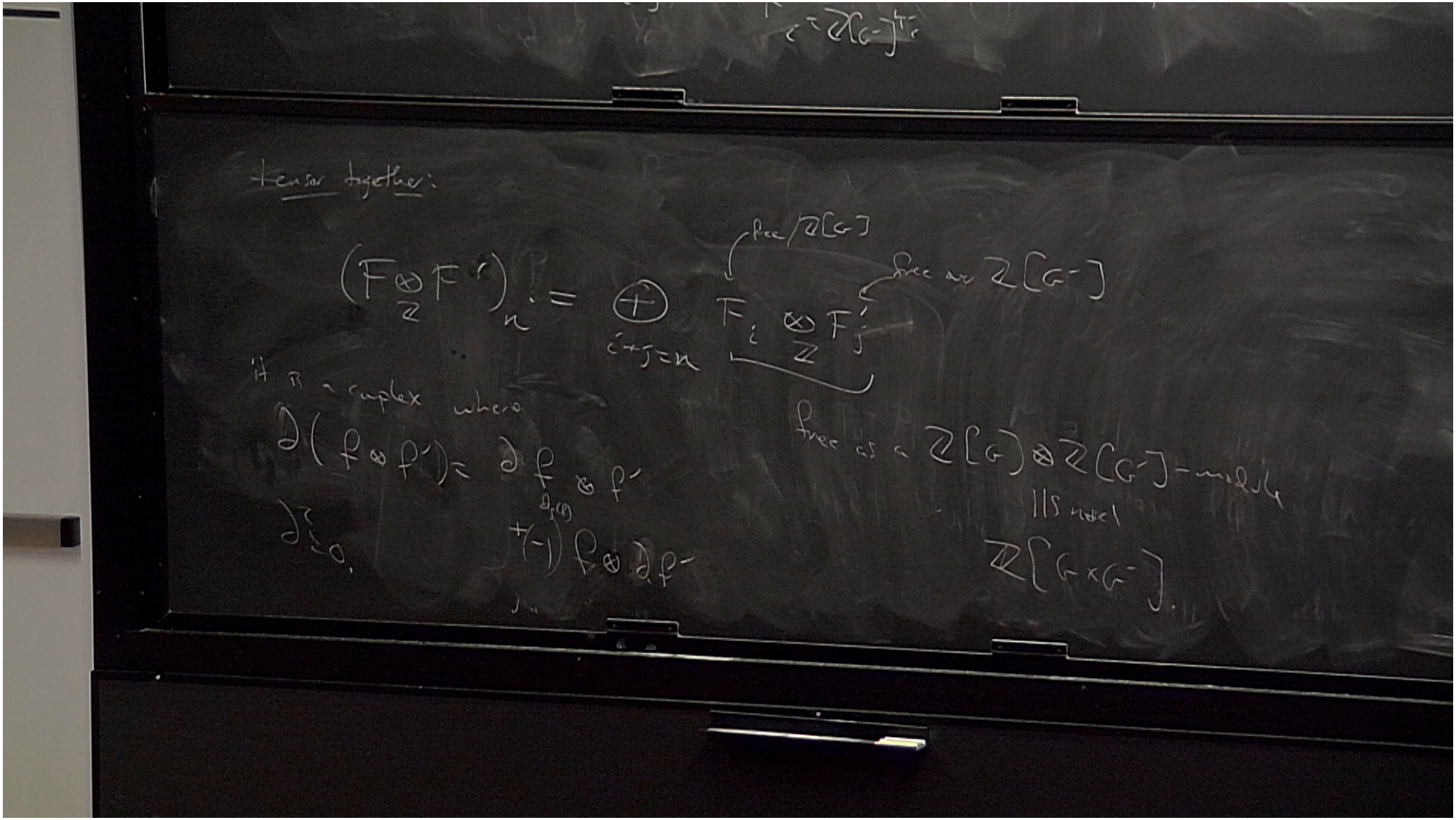
choose $f(p) \in M$ arbitrarily

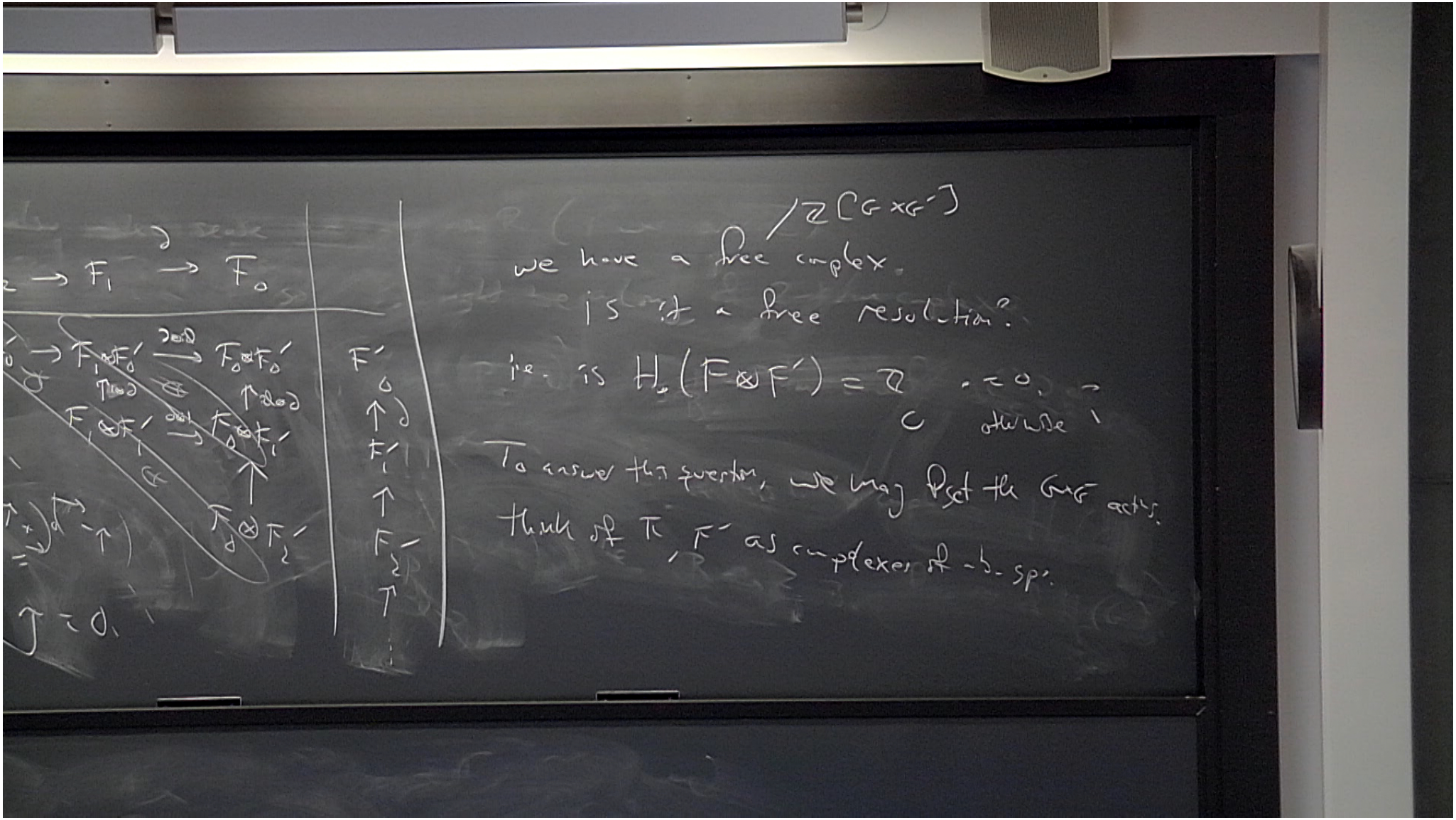

e.g. U(1) ℝ, ℚ, ℚ/ℤ ~ ℂ, ℂ* ≅ ℂ/ℤ
ℝ/ℤ
$H^i(G; \mathbb{C}^\times) = \mathrm{Hom}(H_i G; \mathbb{C}^\times)$


Künneth theorem: about $H_*(G \times G')$ and $H^*(G \times G')$.
Pick free resolutions
$\to F_1 \to F_0 \to \mathbb{Z}$
of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module.
$F'_* \to \mathbb{Z}$
as $\mathbb{Z}[G']$-module.
$F_i = \mathbb{Z}(G)^{r_i}$
$F'_i = \mathbb{Z}(G')^{r'_i}$


tensor together:
free/Z[G]
free on Z[G^-]
free as Z[G] ⊗ Z[G^-]-module
113 model
Z[G × G^-].




tensor together:
free/ℤ[G]
free as ℤ[G⁻]
$(F \otimes_{\mathbb{Z}} F')_n := \bigoplus_{i+j=n} F_i \otimes_{\mathbb{Z}} F'_j$
if F, F' complex where
$\partial(f \otimes f') = \partial f \otimes f' + (-1)^{\deg f} f \otimes \partial f'$
$\partial^2 = 0$.
free as $\mathbb{Z}[G] \otimes \mathbb{Z}[G^-]$-module
$\mathbb{Z}[G \times G^-]$.


$\to F_1 \to F_0$
$F_1 \otimes F_0' \to F_0 \otimes F_0'$
$F_1 \otimes F_1' \to F_0 \otimes F_1'$
$F_0 \otimes F_2'$
F'
F_1'
F_2'
$\partial = 0$,
we have a free complex over $\mathbb{Z}[G \times G']$
is it a free resolution?
i.e. is $H_*(F \otimes F') = \mathbb{Z}$ if $* = 0$, 0 otherwise?
To answer this question, we may first set the $G \times G'$ actions.
think of F, F' as complexes of G-sp.

free over $\mathbb{Z}[G] \Rightarrow$ free over $\mathbb{Z}$

so $F_\cdot, F'_\cdot$ are free res of $\underbrace{\mathbb{Z}}_{\text{already free}}$ as an ab gp.

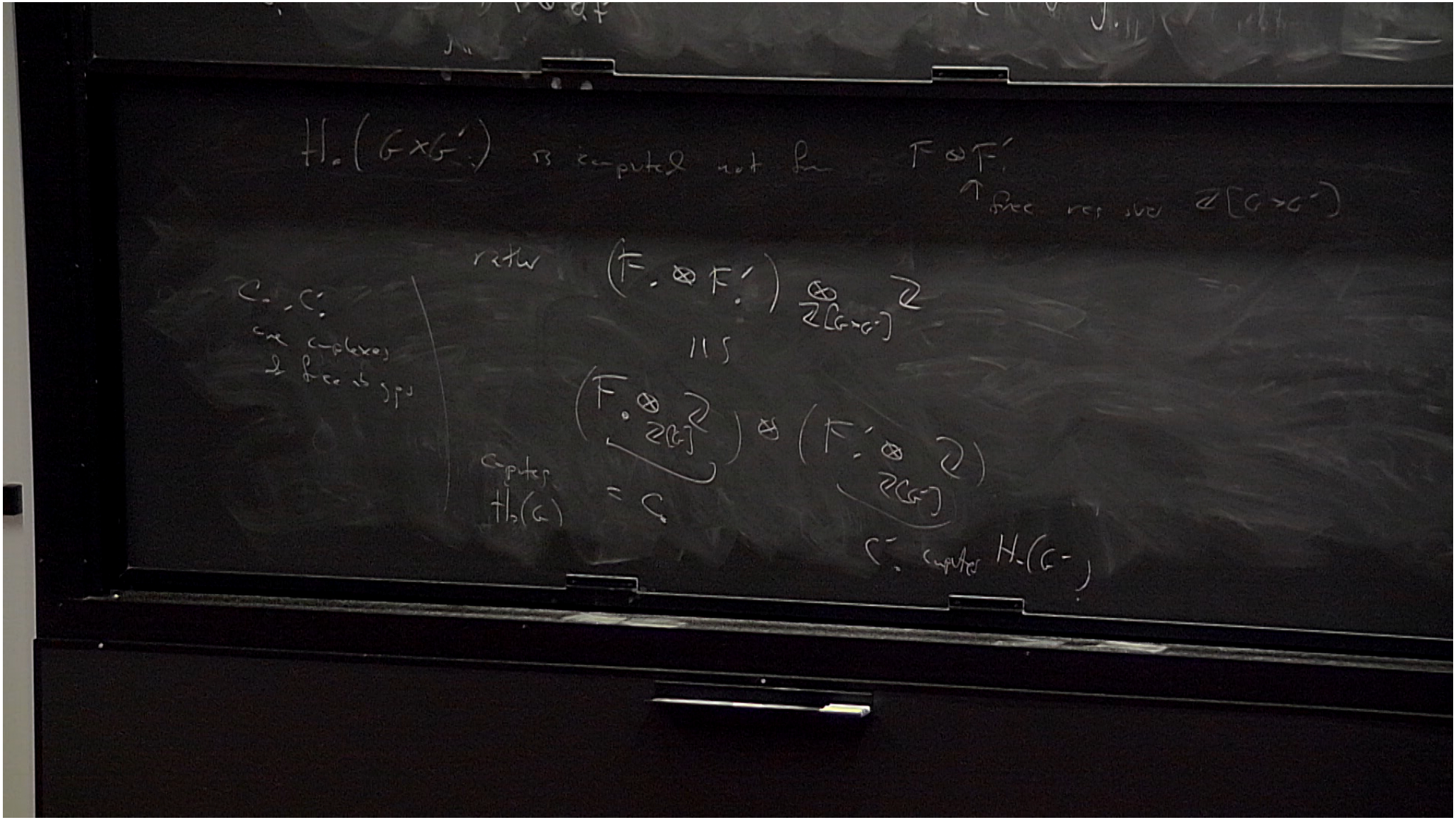
as complexes of ab. gps (not as G, G' (merely)

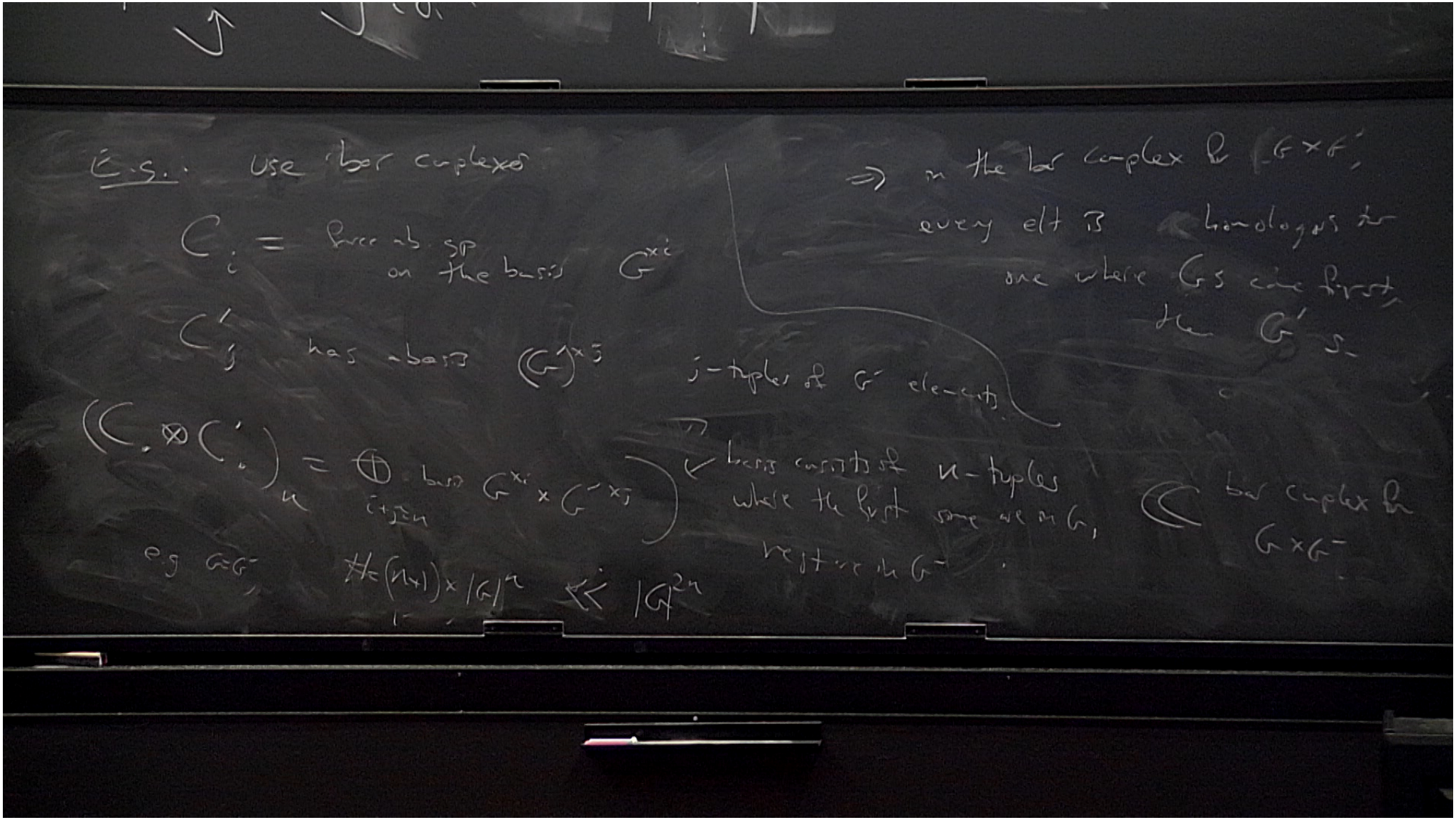
there are homotopy equivs

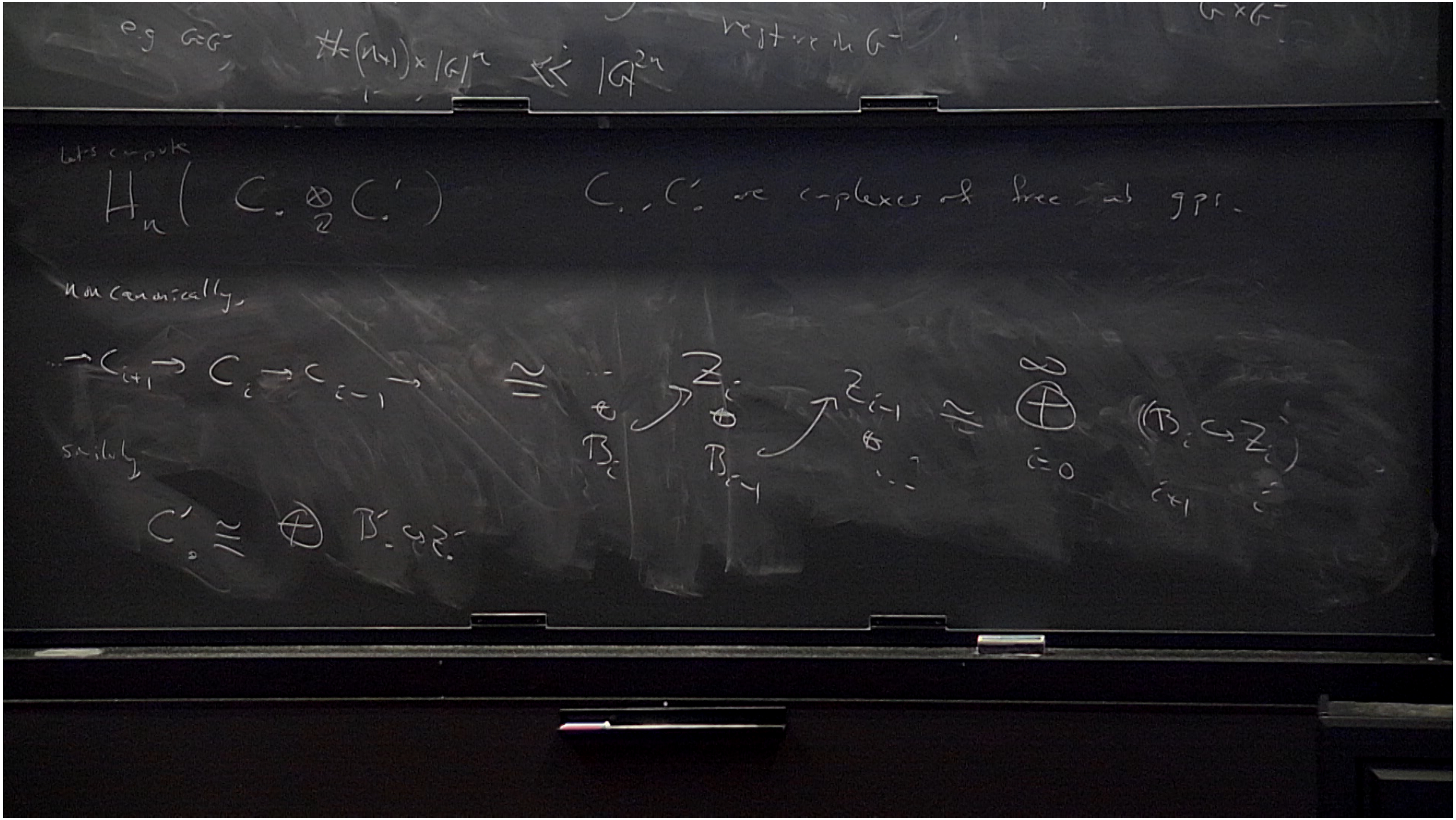
$$F_\cdot \simeq \mathbb{Z} \qquad F'_\cdot \simeq \mathbb{Z}.$$

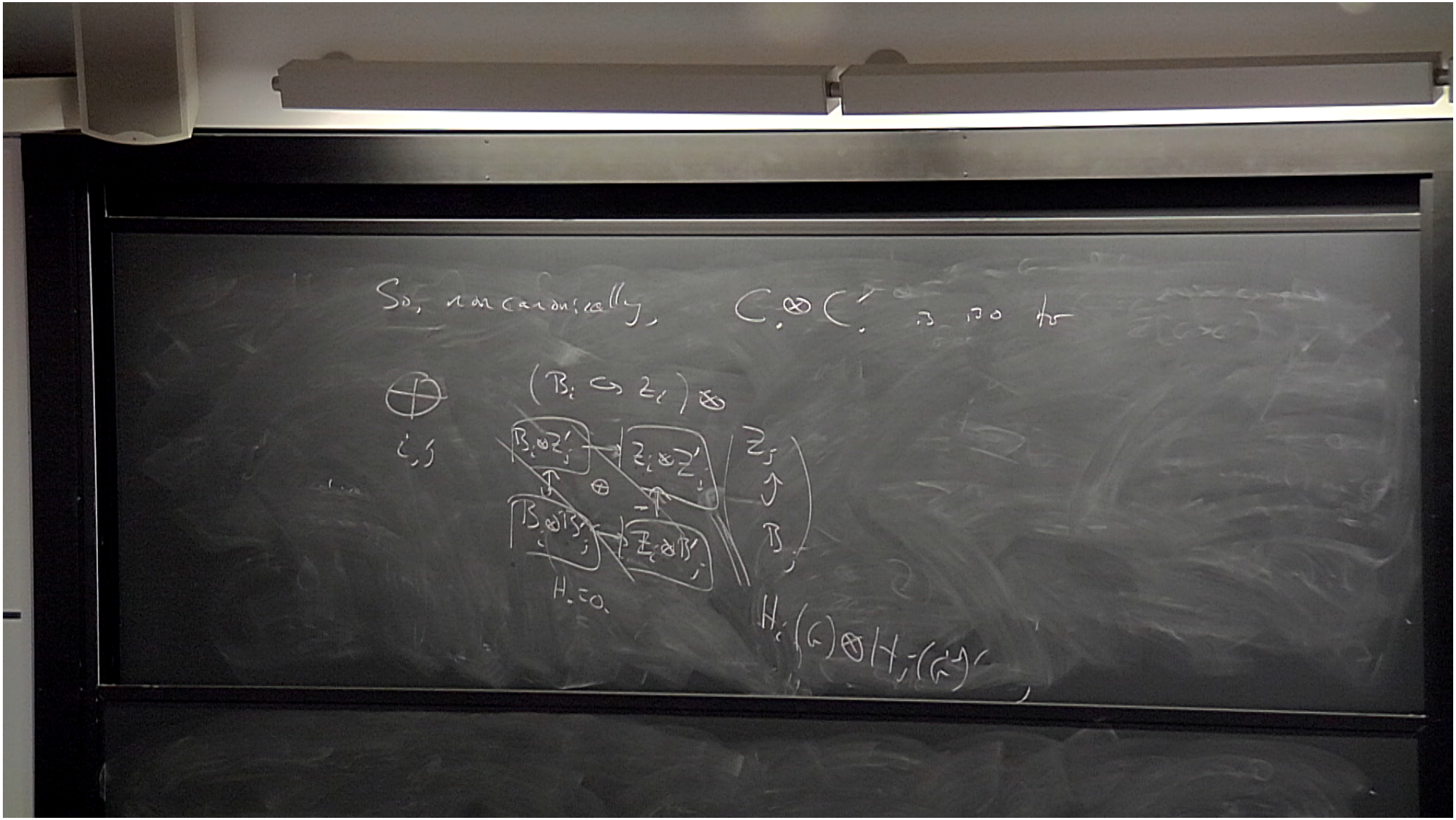
$$\Rightarrow F_\cdot \otimes F'_\cdot \simeq \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}.$$

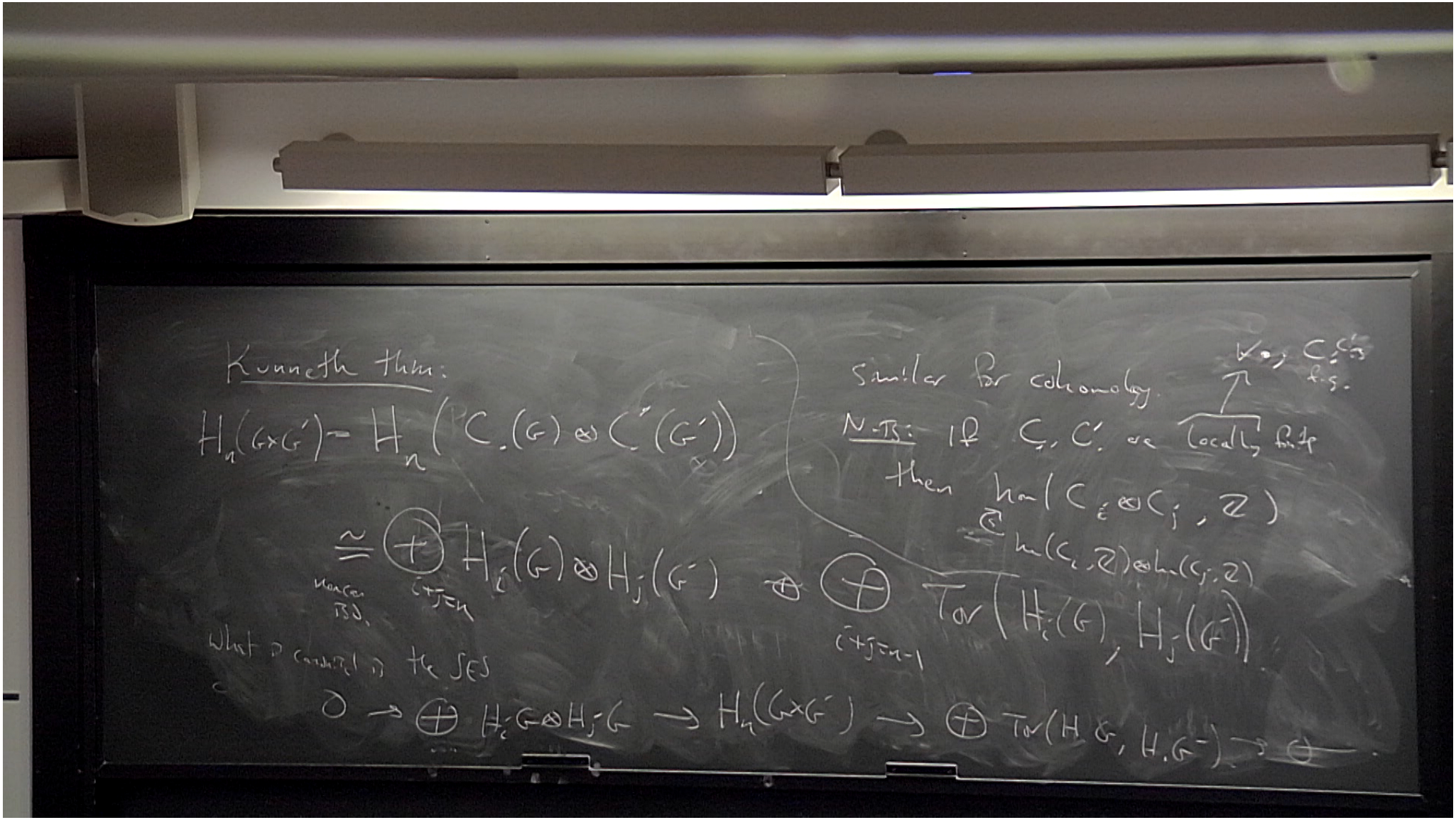
as complexes of ab. gps

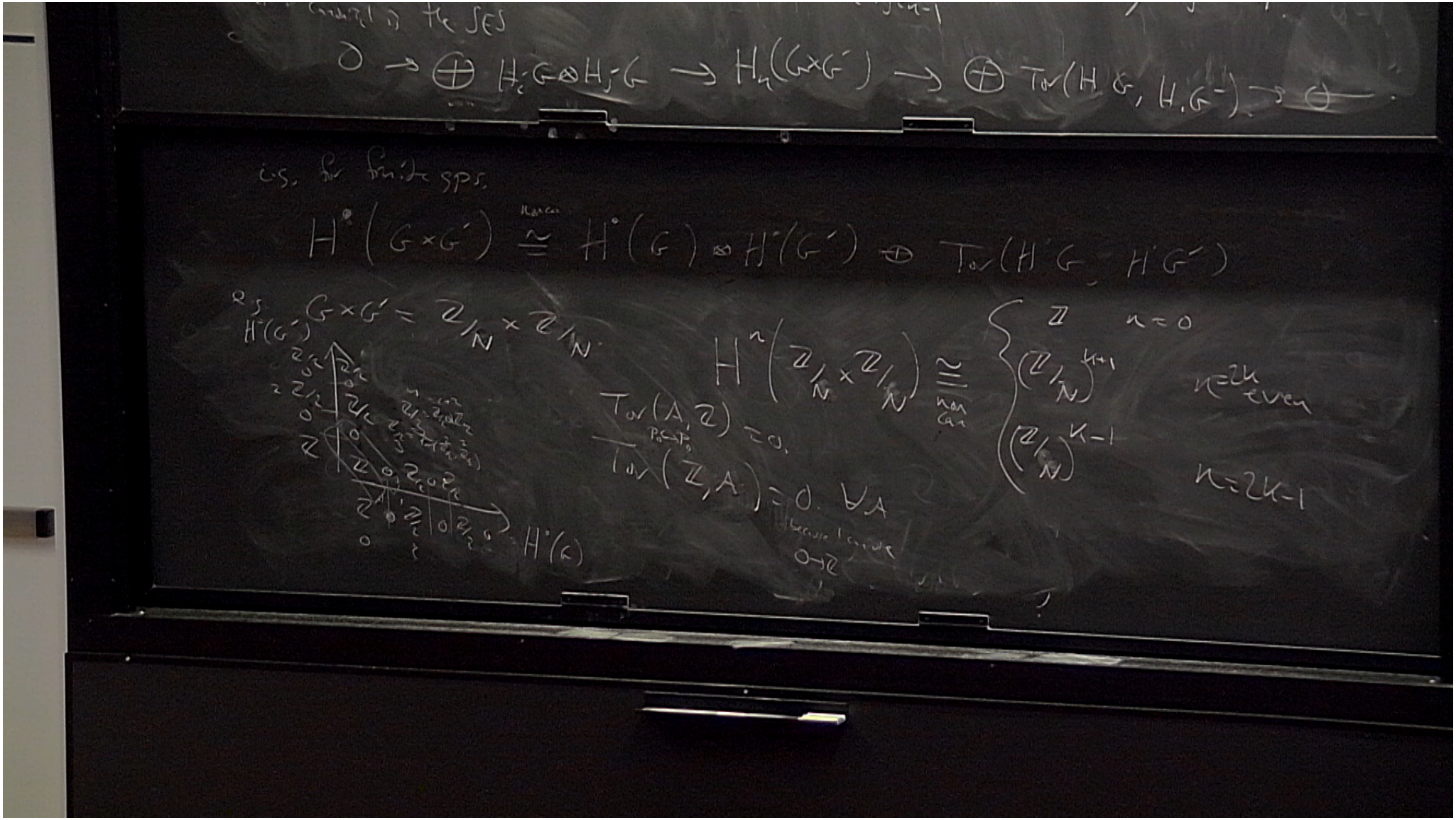

$H_*(G \times G')$ is computed not from $F \otimes F'$
free resolution of $\mathbb{Z}[G \times G']$
retain $(F_* \otimes F'_*) \otimes_{\mathbb{Z}[G \times G']} \mathbb{Z}$
$\cong$
$(F_* \otimes_{\mathbb{Z}[G]} \mathbb{Z}) \otimes (F'_* \otimes_{\mathbb{Z}[G']} \mathbb{Z})$
$C_* \cdot C'_*$
computes $H_*(G)$
$= C_*$
C'_* computes $H_*(G')$

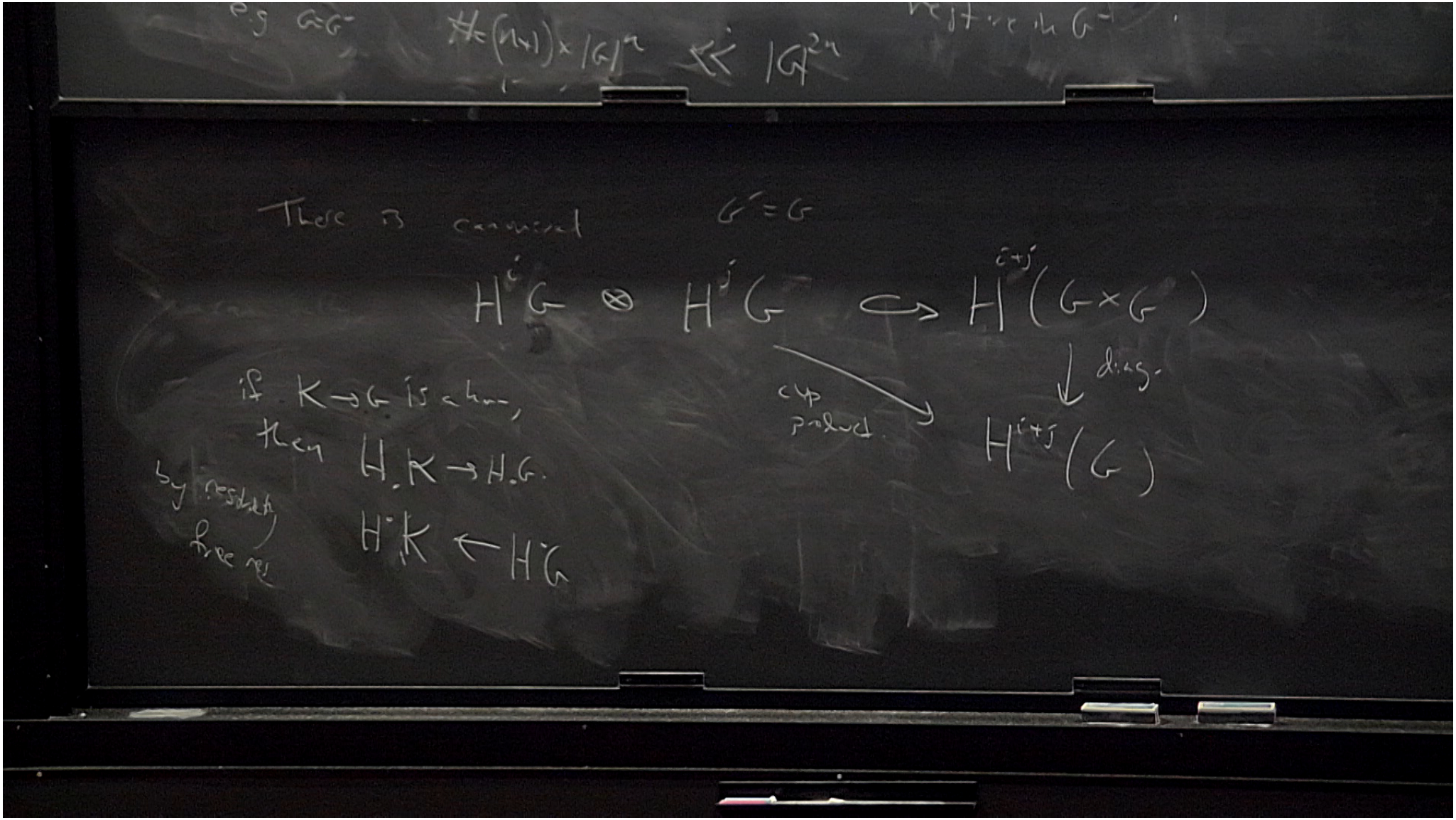

E.S.: use bar complexes
C_i = free ab. sp. on the basis G^{xi}
C'_j has basis $(G^{-})^{xj}$
j-tuples of G^- elements
$(C_\cdot \otimes C'_\cdot)_n = \bigoplus_{i+j=n}$ basis $G^{xi} \times G^{-xj}$
basis consists of n-tuples where the first some # in G, rest in G^-
$\Rightarrow$ in the bar complex for $G \times G^-$, every elt is homologous to one where G's come first, then G^-'s
bar complex for $G \times G^-$
e.g. $G = G^-$, $\#(n+1) \times |G|^n \ll |G|^{2n}$


e.g. $G=G^-$
$\#=(n+1)\times|G|^n \ll |G^{2n}$
Let's compute
$H_n(C_\cdot \otimes_{\mathbb{Z}} C'_\cdot)$
$C_\cdot, C'_\cdot$ are complexes of free ab gps.
Now canonically,
$\cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots \cong \cdots \to B_i \oplus Z_i \to B_{i-1} \oplus Z_{i-1} \to \cdots \cong \bigoplus_{i=0}^{\infty} (B_i \hookrightarrow Z_i)[i+1]$
Similarly
$C'_\cdot \cong \bigoplus B'_\cdot [-i]$


So, naturally, C. ⊗ C'.
(B_i ↪ Z_i) ⊗
i,j
B_i ⊗ Z'_j
Z_i ⊗ Z'_j
B_i ⊗ B'_j
Z_i ⊗ B'_j
Z_j
B_j
H.=0.
H_i(C) ⊗ H_j(C')


Kunneth thm:
$H_n(G\times G') \cong H_n(C_\cdot(G)\otimes C'_\cdot(G'))$
$\cong \bigoplus_{i+j=n} H_i(G)\otimes H_j(G') \oplus \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(G), H_j(G'))$
Monoidal Eq.
Similar for cohomology
N.B.: if $C_\cdot, C'_\cdot$ are locally fin.t
then $\mathrm{Hom}(C_i\otimes C_j, \mathbb{Z}) \cong \mathrm{Hom}(C_i,\mathbb{Z})\otimes \mathrm{Hom}(C_j,\mathbb{Z})$
What's connecting in the SES
$0 \to \bigoplus H_iG\otimes H_jG \to H_n(G\times G') \to \bigoplus \mathrm{Tor}(H_iG, H_jG) \to$


$0 \to \bigoplus_{i+j=n} H_i G \otimes H_j G' \to H_n(G \times G') \to \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i G, H_j G') \to 0$
c.s. for finite gps.
$H^*(G \times G') \overset{\text{Künneth}}{\cong} H^*(G) \otimes H^*(G') \oplus \mathrm{Tor}(H^* G, H^* G')$
$G \times G' = \mathbb{Z}/N \times \mathbb{Z}/N$
$H^n(\mathbb{Z}/N \times \mathbb{Z}/N) \overset{\text{non can}}{\cong} \begin{cases} \mathbb{Z} & n=0 \\ (\mathbb{Z}/N)^{k+1} & n=2k \text{ even} \\ (\mathbb{Z}/N)^{k-1} & n=2k-1 \end{cases}$
$\mathrm{Tor}(A, \mathbb{Z}) = 0$
$\mathrm{Tor}(\mathbb{Z}, A) = 0 \ \forall A$
because I can use $0 \to \mathbb{Z}$
$H^*(G)$


There is canonical
$G' = G$
$H^i G \otimes H^j G \hookrightarrow H^{i+j}(G \times G)$
cup product
diag.
$H^{i+j}(G)$
if $K \to G$ is chain,
then $H_\cdot K \to H_\cdot G$.
by restriction
free res
$H^\cdot K \leftarrow H^\cdot G$