Title: The Cohomology of Groups (Johnson-Freyd/Guo) - Lecture 4

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Collection: The Cohomology of Groups (Johnson-Freyd/Guo)

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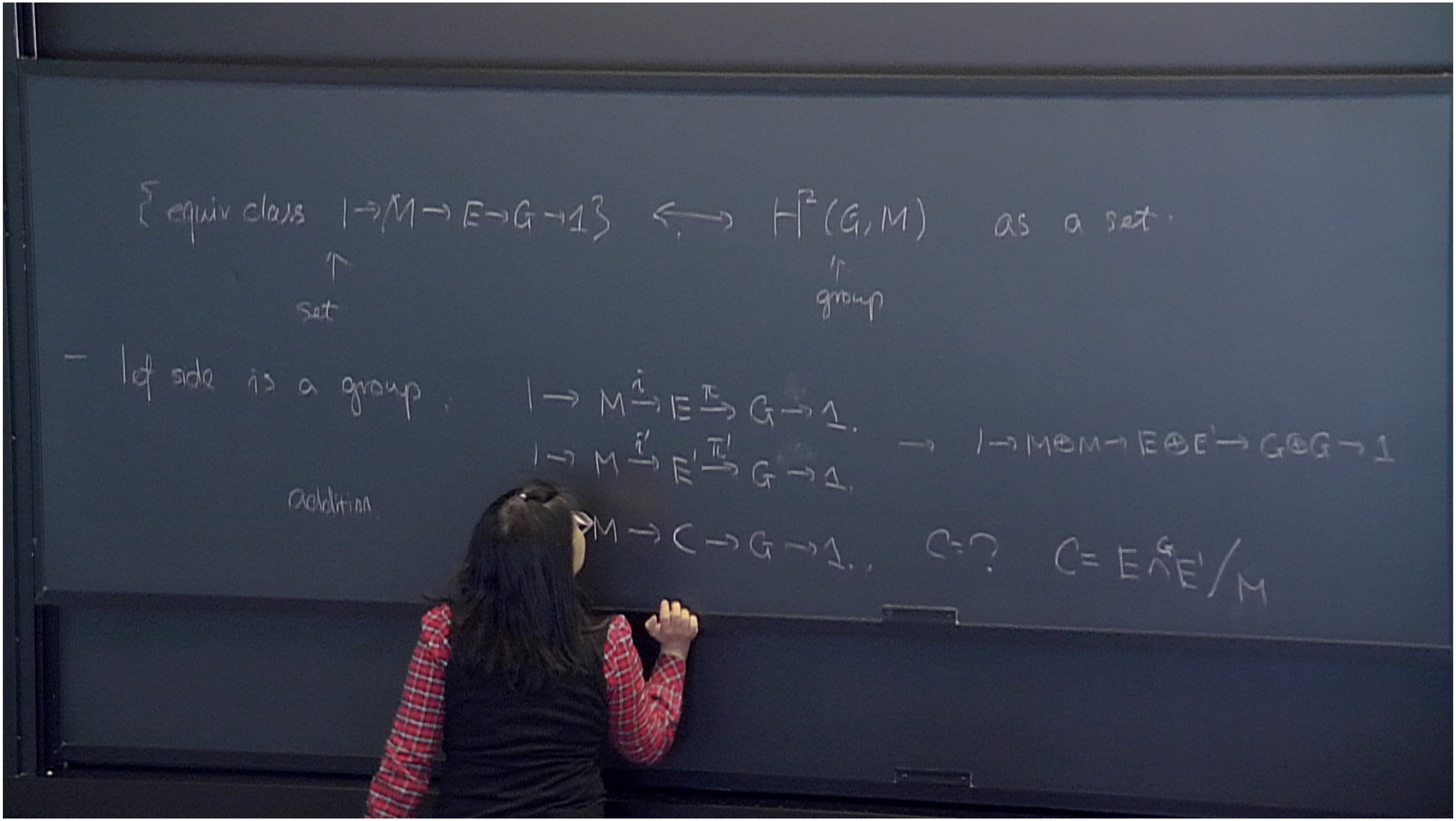

{equiv class 1→M→E→G→1} ⟷ H²(G,M) as a set.
set
group
– lef side is a group.
1→M →i E →π G→1.
1→M →i' E' →π' G→1.
→ 1→M⊕M→E⊕E'→G⊕G→1
addition
M→C→G→1. C=? C= E ∧G E'/M

– left side is a group:

$$1 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to \underline{1}.$$
$$1 \to M \xrightarrow{i'} E' \xrightarrow{\pi'} G \to \underline{1}.$$

$$\to \quad 1 \to M \oplus M \to E \oplus E' \to G \oplus G \to \underline{1}$$

addition

$$1 \to M \to C \to G \to \underline{1}., \qquad C = ? \qquad C = E \wedge^G E' / M$$

$$E \wedge^G E' = \{ (e, e') \in E \times E' \mid \pi(e) = \pi'(e') \in G \}$$

$$\begin{array}{ccc} E \wedge^G E' & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ E' & \xrightarrow[\pi']{} & G \end{array}$$

pullback

$$M \hookrightarrow E \wedge^G E'$$
$$m \longmapsto (i(m), i'(m))$$

identity

$$1 \to M \to M \rtimes G \to G \to \underline{1}$$

inverse

$$1 \to M \xrightarrow{-i} E \to G \to \underline{1}$$

Examples: $\cdot \to \mathbb{Z}/2 \qquad \to \mathbb{Z}/2 \to 1$, only $\mathbb{Z}/2$-action on $\mathbb{Z}/2$ is trivial.

$$\rightsquigarrow H^2(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$$

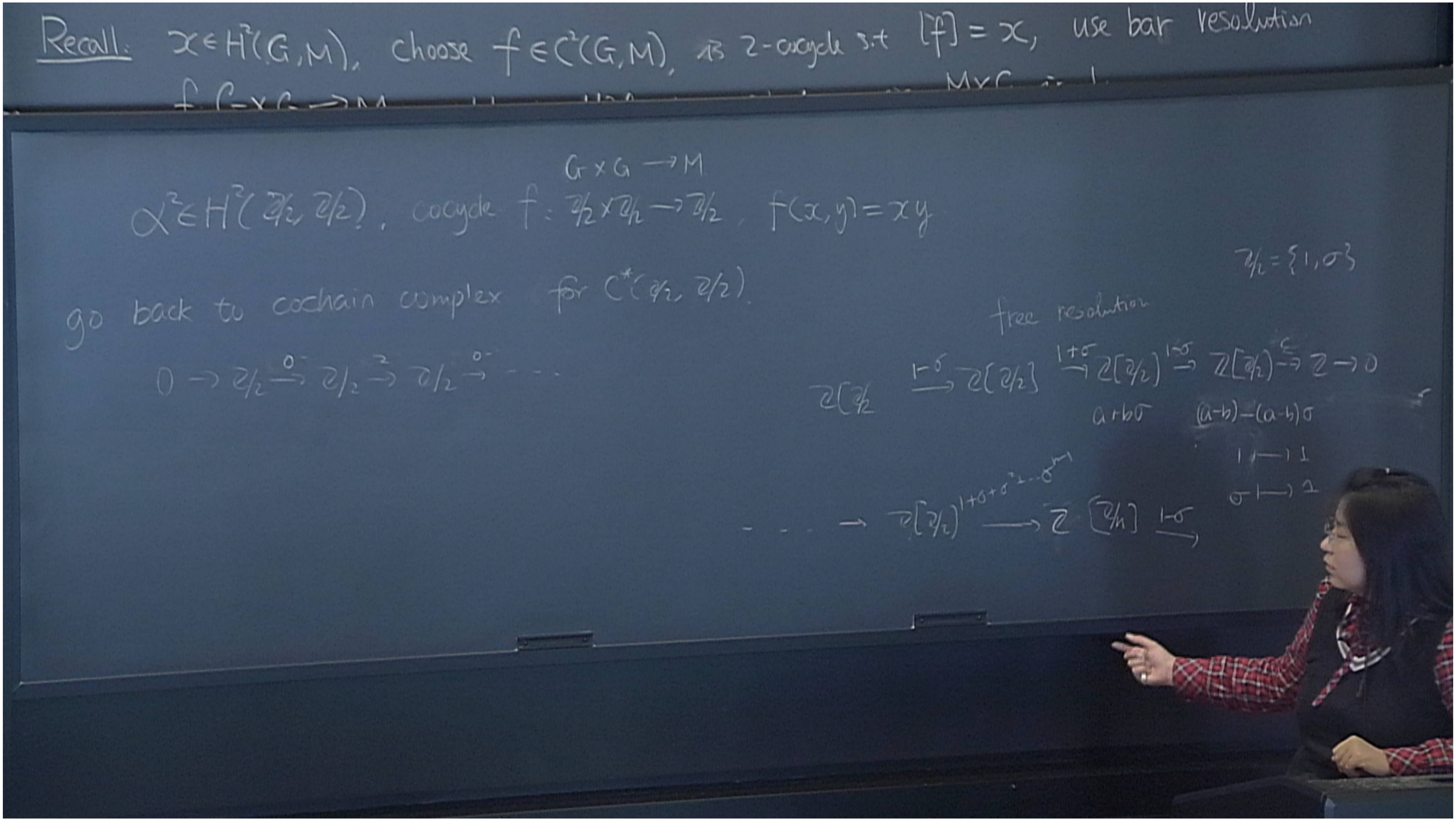
$$\mathbb{Z}/2 \times \mathbb{Z}/2 \longrightarrow 0$$

$$\mathbb{Z}/4 \longrightarrow \alpha^2 \in \mathbb{Z}/2$$

Recall: $x \in H^2(G, M)$, choose $f \in C^2(G, M)$, is 2-cocycle s.t. $[f] = x$, use bar resolution

$f: G \times G \to M$, the multiplication structure on $M \times G$ is by

$$(m_1, g_1) \cdot (m_2, g_2) = (m_1 + g_1 \cdot m_2 + f(g_1, g_2),\ m_1 m_2)$$


Recall: x ∈ H²(G,M), choose f ∈ C²(G,M), is 2-cocycle s.t [f] = x, use bar resolution
G × G → M
α² ∈ H²(Z/2, Z/2), cocycle f: Z/2 × Z/2 → Z/2, f(x,y) = xy
Z/2 = {1, σ}
go back to cochain complex for C*(Z/2, Z/2)
free resolution
0 → Z/2 →0 Z/2 →2 Z/2 →0 . . .
Z[Z/2 →1−σ Z[Z/2] →1+σ Z[Z/2] →1−σ Z[Z/2] →ε Z → 0
a+bσ (a−b)−(a−b)σ
1 ↦ 1
σ ↦ 1
. . . → Z[Z/2] →1+σ+σ²+...σ^{n−1} Z[Z/n] →1−σ

Recall: $x \in H^2(G,M)$, choose $f \in C^2(G,M)$, a 2-cocycle s.t. $[f] = x$, use bar resolution

$f: G\times G \to M$, the multiplication structure on $M \times G$ is by

$$(m_1, g_1)\cdot(m_2, g_2) = (m_1 + g_1\cdot m_2 + f(g_1,g_2),\ g_1 g_2)$$

$G\times G \to M$

$\alpha^2 \in H^2(\mathbb{Z}/2, \mathbb{Z}/2)$, cocycle $f: \mathbb{Z}/2\times\mathbb{Z}/2 \to \mathbb{Z}/2$, $f(x,y) = xy$

$\mathbb{Z}/2 = \{0,1\}$

$(0,0) \mapsto 0$

$(0,1) \mapsto 0$

$(1,0) \mapsto 0$

$(1,1) \mapsto 1$

go back to cochain complex for $C^*(\mathbb{Z}/2, \mathbb{Z}/2)$

$$\cdots\ \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow[\varepsilon]{} \mathbb{Z} \to 0 \quad \leftarrow \text{free resolution}$$

$\longrightarrow$ cochain complex $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(-, \mathbb{Z}/2)$

deg 0

$\sigma \mapsto 1$

$$\cdots \xleftarrow{1-\sigma} \underset{\{0,1\}}{\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z}/2)} \xleftarrow{1+\sigma} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z}/2) \xleftarrow{1-\sigma} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z}/2) \longleftarrow 0$$

$$\xleftarrow{0} \mathbb{Z}/2 \qquad\qquad \xleftarrow{0} \mathbb{Z}/2 \qquad\qquad \xleftarrow{0} \mathbb{Z}/2$$

$\longrightarrow$ Cochain complex $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(-, \mathbb{Z}/2)$

deg 0

$$\cdots \xleftarrow{1-\sigma} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z}/2) \xleftarrow{1+\sigma} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z}/2) \xleftarrow{1-\sigma} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z}/2) \longleftarrow 0$$

$\sigma \mapsto 1$, $\{0,1\}$

$$\xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2$$

on $M \times G = \mathbb{Z}/2 \times \mathbb{Z}/2$

$(x_1, y_1)(x_2, y_2) = (x_1 + x_2 + y_1 y_2, \; y_1 + y_2)$

$(0,0) \longleftarrow$ identity element in extension

$a = (1,0)$

$b = (0,1)$

$c = (1,1)$

$a^2 = (1,0)(1,0) = (1+1+0\cdot 0, 0) = (0,0)$

$b^2 = (0,1)(0,1) = (0+0+1\cdot 1, 1+1) = (1,0) = a \implies b^4 = 1 \implies \mathbb{Z}/4$

$c = b^3, \quad a = b^2, \quad b, \quad 1$

<u>Recall:</u> $x \in H^2(G,M)$. Choose $f \in C^2(G,M)$ as 2-cocycle s.t. $[f] = x$, use bar resolution

$f: G\times G \to M$, the multiplication structure on $M \times G$ is by

$$(m_1, g_1)\cdot(m_2, g_2) = (m_1 + g_1\cdot m_2 + f(g_1, g_2),\ m_1 m_2)$$

$$G \times G \to M$$

$\alpha^2 \in H^2(\mathbb{Z}/2, \mathbb{Z}/2)$, cocycle $f: \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$, $f(x,y) = xy$

$\mathbb{Z}/2 = \{0,1\}$

$(0,0) \mapsto 0$

$(0,1) \mapsto 0$

$(1,0) \mapsto 0$

$(1,1) \mapsto \underline{1}$

go back to cochain complex for $C^*(\mathbb{Z}/2, \mathbb{Z}/2)$.

$$\cdots\ \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-\sigma} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow[\varepsilon]{} \mathbb{Z} \to 0 \quad \leftarrow \text{free resolution}$$

$\to$ cochain complex $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(-, \mathbb{Z}/2)$

$x \in H^i(G,M)$

$y \in H^j(G,M)$

$\rightsquigarrow$

$x \cdot y = x \cup y \in H^{i+j}(G,M)$

$\sigma \mapsto 1$ deg 0

$$\cdots \xleftarrow{1-\sigma} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z}/2) \xleftarrow{1+\sigma} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z}/2) \xleftarrow{1-\sigma} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z}/2) \leftarrow 0$$

$$\{0,1\} \qquad \cong \qquad \cong \qquad \cong$$

$$\xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2$$

Examples

$$1 \to \mathbb{Z}/2 \to \, ? \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1 \quad \rightsquigarrow \quad H^2(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

$$\langle \alpha^2 \rangle \quad \langle \alpha\beta \rangle \quad \langle \beta^2 \rangle$$

$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \quad \longrightarrow 0$

$\mathbb{Z}/4 \times \mathbb{Z}/2$

D_8

Q_8

D_8

Q_8

The extension corresponding to α^2.

$$\alpha^2 : G \times G \longrightarrow M$$

$$(\mathbb{Z}/2 \times \mathbb{Z}/2) \times (\mathbb{Z}/2 \times \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

$$(y_1, z_1), (y_2, z_2) \longmapsto y_1 \cdot y_2$$

$$\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2$$
$$x \longmapsto (x, 0)$$

$$\rightsquigarrow H^2(\mathbb{Z}/2 \times \mathbb{Z}/2 ; \mathbb{Z}/2) \longrightarrow H^2(\mathbb{Z}/2, \mathbb{Z}/2)$$
$$\alpha^2 \longrightarrow \alpha^2$$

On $M \times G = \mathbb{Z}/2 \times (\mathbb{Z}/2 \times \mathbb{Z}/2)$, the multiplication structure

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2 + \alpha^2((y_1, z_1), (y_2, z_2)), y_1 + y_2, z_1 + z_2) = (\overbrace{x_1 + x_2 + y_1 y_2, y_1 + y_2}^{\mathbb{Z}/4}, \overbrace{z_1 + z_2}^{\times \mathbb{Z}/2})$$

$$\alpha^2: \quad 1 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to \underline{1}$$

$$1 \longmapsto (2, 0)$$

$$(1, 0) \longmapsto (1, 0)$$

$$(0, 1) \longmapsto (0, 1)$$

$$\beta^2: \quad 1 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to \underline{1}.$$

$$(1, 0) \longmapsto (0, 1)$$

$$(0, 1) \longmapsto (1, 0)$$

Examples:

$$1 \to \mathbb{Z}/2 \to \,?\, \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1 \quad \rightsquigarrow H^2(\mathbb{Z}/2 \times$$

$$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \longleftrightarrow 0$$

$$\alpha^2,\ \alpha^2+\beta^2 \longleftrightarrow \mathbb{Z}/4 \times \mathbb{Z}/2$$

$$\beta^2$$

$$\alpha\beta \longrightarrow D_8$$

$$Q_8$$

$D_8 = \langle r, s \mid r^4 = s^2 = 1,\ rsrs = 1 \rangle$

On 2-cocycle. $\alpha\beta : G \times G \longrightarrow M$

$(\mathbb{Z}/2 \times \mathbb{Z}/2) \times (\mathbb{Z}/2 \times \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$

$(y_1, z_1)\ (y_2, z_2) \longmapsto y_1 z_2$

Multiplication structure on $M \times G = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$

$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2 + y_1 z_2,$

$0,0)$

$1,0)$ $\quad a^2=1,\ b^2=1,\ c^2=1,\ d^2=1,\ e^2=1$

$0,1)$ $\quad f^2=a$

$1,0)$ $\quad g^2=a \Rightarrow f^4=g^4=1$

$0,1)$

$$bc=(0,1,0)\cdot(0,0,1)=(1,1,1)=g$$

$1,1)$

$$cb=(0,0,1)\cdot(0,1,0)=(0,1,1)=f \rightsquigarrow bc\neq cb$$

$1,1)$

$a = (1,0,0)$

$b = (0,1,0)$ $\quad a^2 = 1,\ b^2 = 1,\ c^2 = 1,\ d^2 = 1,\ e^2 = 1$

$c = (0,0,1)$ $\quad f^2 = a$

$d = (1,1,0)$ $\quad \Rightarrow f^4 = g^4 = 1$

$g^2 = a$

$e = (1,0,1)$

$f = (0,1,1)$ $\quad bc = (0,1,0)\cdot(0,0,1) = (1,1,1) = g$

$cb = (0,0,1)\cdot(0,1,0) = (0,1,1) = f$

$g = (1,1,1)$

$\langle f, b \mid f^4 = b^2 = 1,\ fbfb = 1 \rangle$

$=1,\ b^2=1,\ c^2=1,\ d^2=1,\ e^2=1$

$$\left.\begin{array}{l} f^2 = a \\ g^2 = a \end{array}\right\} \Rightarrow f^4 = g^4 = 1.$$

$$bc = (0,1,0)\cdot(0,0,1) = (1,1,1) = g$$

$$cb = (0,0,1)\cdot(0,1,0) = (0,1,1) = f \quad \rightsquigarrow \quad bc \neq cb$$

$$\langle f, b \mid f^4 = b^2 = 1,\ fbfb = 1 \rangle = D_8$$

Examples:

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow ? \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \longrightarrow 1$$

$$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \longleftrightarrow 0$$

$$\alpha^2, \beta^2, \alpha^2+\beta^2 \longleftrightarrow \mathbb{Z}/4 \times \mathbb{Z}/2$$

$$\alpha\beta, \ \alpha^2+\alpha\beta, \ \alpha\beta+\beta^2 \longrightarrow D_8$$

$$\alpha^2+\alpha\beta+\beta^2 \longrightarrow Q_8$$

$$
\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N \arrow[r] & \mathrm{Aut}(N) \arrow[r] & \mathrm{Out}(N) \arrow[r] & 1
\end{tikzcd}
$$

N nonabelian

$$
G \to \mathrm{Out}(N) = \mathrm{Aut}(N) / \mathrm{Inn}(N)
$$

$$
\mathrm{Inn}(N) = \{ n \mapsto x n x^{-1},\ x \in N \}
$$

$$
\begin{tikzcd}
 & M \arrow[d, phantom, "\cong"] & & & \\
1 \arrow[r] & \mathrm{Inn}(N) \arrow[r] & \mathrm{Aut}(N) \arrow[r] & \mathrm{Out}(N) \arrow[r] & 1
\end{tikzcd}
$$

Choose $\psi : G \to \mathrm{Out}(N)$, if $1 \to N \to E \to G \to 1$ exists

either no such existence, or if exists $= H^2$

Check Brown's book

(G, M). (G', M'),

$$\mathbb{Z}[G\times G'] = \mathbb{Z}[G]\otimes_{\mathbb{Z}} \mathbb{Z}[G']$$

$M\times M'$ is a $\mathbb{Z}[G\times G']$-module

$$(g, g')\cdot(m, m') = (g\cdot m,\ g'\cdot m')$$

(G, M), (G', M'), $\mathbb{Z}[G \times G'] = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G']$

$M \times M'$ is a $\mathbb{Z}[G \times G']$-mo

$(g, g') \cdot (m, m') = (g \cdot m, g' \cdot m')$

Recall: Thm: $F_\bullet \to \mathbb{Z}$ free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$, $F'_\bullet$ over $\mathbb{Z}[G']$, then $F_\bullet \otimes F'_\bullet$ is a free resolut

$\mathbb{Z}[\mathbb{Z}/2]$ $C_\bullet$ free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/2]$

$\mathbb{Z}[G \times G'] = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G']$

$M \times M'$ is a $\mathbb{Z}[G \times G']$-module

$(g, g') \cdot (m, m') = (g \cdot m, g' \cdot m')$

solution of $\mathbb{Z}$ over $\mathbb{Z}[G]$, $F'_\bullet \to \mathbb{Z}$ free resolution of $\mathbb{Z}$

$F_\bullet \otimes F'_\bullet$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G \times G']$

esolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/2]$, $\leadsto$ $C_\bullet \otimes C_\bullet$ is ..... of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]$.

$$\otimes \left(F'_{\bullet} \otimes_{\mathbb{Z}[G']} M' \right) \longrightarrow \left(F_{\bullet} \otimes F'_{\bullet} \right) \otimes_{\mathbb{Z}[G \times G']} \left(M \otimes M' \right)$$

chain complex $C_{\bullet}(G', M')$

Chain complex $C_{\bullet}(G \times G', M \otimes M')$

$$(F_\bullet \otimes_{\mathbb{Z}[G]} M) \otimes (F'_\bullet \otimes_{\mathbb{Z}[G']} M) \longrightarrow (F_\bullet \otimes F'_\bullet) \otimes_{\mathbb{Z}[G\times}$$

$$(x \otimes m) \otimes (y \otimes m') \longmapsto (x \otimes y) \otimes (m \otimes$$

Chain complex $C_\bullet(G, M)$

Chain complex $C_\bullet(G', M')$

Chain comple

$$(x\otimes m)\otimes(y\otimes m') \longmapsto (x\otimes y)\otimes(m\otimes m'$$

Chain complex $C_\cdot(G,M)$

Chain complex $C_\cdot(G',M')$

Chain complex

$$\underset{x}{\underset{\Psi}{H_*(G,M)}}\otimes \underset{y}{\underset{\Psi}{H_*(G',M')}} \longrightarrow \underset{x\times y}{H_*(G\times G',\ M\otimes M')}$$

$a = (1,0)$

$b = (0,1)$

$c = (1,1)$

$a^2 = (1,0)(1,0) = (1+1+0\cdot 0\,;\, 0) = (0,0)$ –

$b^2 = (0,1)(0,1) = (0+0+1\cdot 1,\ 1+1) = (1,0) = ($

mplex

M')

Chain complex $C_\bullet(G \times G', M \otimes M')$

$) \longrightarrow H_*(G \times G', M \otimes M')$ cross product

$x \times y$

$\Rightarrow \mathbb{Z}/4$

$(1,0)(1,0) = (1+1+0\cdot 0, 0) = (0,0)$

$(0,1)(0,1) = (0+0+1\cdot 1, 1+1) = (1,0) = a \Rightarrow b^4 = 1$

Do the same thing

$$\operatorname{Hom}_{\mathbb{Z}[G]}(F_\bullet, M) \otimes \operatorname{Hom}_{\mathbb{Z}[G']}(F'_\bullet, M') \longrightarrow \operatorname{Hom}_{\mathbb{Z}[G\times G']}$$

$$f \qquad\qquad g \longmapsto (x \otimes y \mapsto$$

cohomology $\rightsquigarrow$

$$H^*(G, M) \otimes H^*(G', M') \longrightarrow H^*(G \times G', M \times M')$$

ing

$$M) \otimes \operatorname{Hom}_{\mathbb{Z}[G']}(F'_\bullet, M') \longrightarrow \operatorname{Hom}_{\mathbb{Z}[G\times G']}(F_\bullet \otimes F'_\bullet, M \otimes M')$$

$$\otimes\ g \longmapsto (x \otimes y \longmapsto f(x) \otimes g(y))$$

$$) \otimes H^*(G', M') \longrightarrow H^*(G \times G', M \times M')$$

$G = G'$, $M = M'$; $F_\cdot = F'_\cdot$

$$C^*(G,M) \otimes C^*(G,M) \longrightarrow C^*(G\times G, M\otimes M)$$

$$G \longrightarrow G\times G$$
$$g \longmapsto (g,g)$$

$\rightsquigarrow$ $M\otimes M$ is $\mathbb{Z}[G]$-module

$\rightsquigarrow$ $C^*(G\times G, M\otimes M) \longrightarrow C^*(G, M\otimes M)$

M trivial G-action

$$\to C^*(G\times G, M\otimes M) \longrightarrow C^*(G, M\otimes M) \longrightarrow C^*(G, M)$$

$M\otimes M$ is $\mathbb{Z}[G]$-module

$$\to C^*(G\times G, M\otimes M) \longrightarrow C^*(G, M\otimes M)$$

$G = G'$, $M = M$; $F_\cdot = F_\cdot$

$$C^*(G,M) \otimes C^*(G,M) \longrightarrow C^*(G\times G, M\otimes M) \longrightarrow C^*(G, M$$

$$G \longrightarrow G\times G$$
$$g \longmapsto (g,g)$$

$\rightsquigarrow$ $M\otimes M$ is $\mathbb{Z}[G]$-module

$\rightsquigarrow$ $C^*(G\times G, M\otimes M) \longrightarrow C^*(G, M\otimes M)$

taking H^*: $H^*(G,M)\otimes H^*(G,M) \longrightarrow H^*(G,M)$

$x, \; y \longmapsto x\cup y$ cup product

M trivial G-action

$$\to C^*(G\times G, M\otimes M) \longrightarrow C^*(G, M\otimes M) \xrightarrow[\text{on } M]{\text{multiplication}} C^*(G, M)$$

$M\otimes M$ is $\mathbb{Z}[G]$-module

$$\rightsquigarrow C^*(G\times G, M\otimes M) \longrightarrow C^*(G, M\otimes M)$$

$$\longrightarrow H^*(G, M)$$

$$\longmapsto x \cup y \quad \text{cup product}$$

, $y \in H^j$, $x \cup y \in H^{i+j}$

s identity. $1 \in H^0(G,M) = M^G$ $\quad 1 \cup x = x \cup 1 = x$

tisfies associativity. $(x \cup y) \cup z = x \cup (y \cup z)$

($\leftarrow$ R is a trivial G-module) is a graded ring.

rmk: ① $x \in H^i,\ y \in H^j,\ x \cup y \in H^{i+j}$

② It has identity. $1 \in H^0(G,M) = M^G$ $\quad 1 \cup x$

③ It satisfies associativity. $(x \cup y) \cup z = x$

$\rightsquigarrow H^*(G,R)$ ($\leftarrow R$ is a trivial G-module) is a graded

$$\begin{array}{ccc} x_1 & + & x_2 \\ \cap & & \cap \\ H^i & & H^i \end{array}$$

$(1,0) \longmapsto (0,1)$

$(0,$

② It has identity. $1 \in H^0(G,M) = M^G$ ...

③ It satisfies associativity. $(x \cup y) \cup z = x$ ...

$\rightsquigarrow H^*(G,R)$ ($\leftarrow R$ is a trivial G-module) is a graded ...

$\underbrace{x_1 + x_2}_{\text{abelian gp}}$, $\underbrace{x \cdot y := x \cup y}_{\text{mulplication structure of ring}}$

$x_1 \in H^i$, $x_2 \in H^i$

$(1,0) \longmapsto (0,1)$

$(0,1) \longmapsto (1,0)$

④ graded commutative

$$x \cup y = (-1)^{|x||y|}\, y \cup x$$

e.g. $H^*(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2[\alpha]$. $\alpha \in H^1(\mathbb{Z}/2, \mathbb{Z}/2)$

$$\alpha \cdot \alpha = \alpha \cup \alpha \in H^2(\mathbb{Z}/2, \mathbb{Z}/2)$$