Title: The Cohomology of Groups (Johnson-Freyd/Guo) - Lecture 3

Speakers: Meng Guo, Theo Johnson-Freyd

Collection: The Cohomology of Groups (Johnson-Freyd/Guo)

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resolution of $\mathbb{Z}$ as $\mathbb{Z}[\mathbb{Z}/n]$-resolution. $\mathbb{Z}/n = \{1, t, t^2, \dots, t^{n-1}\}$

$$\cdots \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{N} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$t^i \longmapsto 1$$

$\{$ M-conjugacy classes $1 \to M \to M \rtimes \overset{s}{\overset{\circ}{G}} \to G \to 1 \} \longleftrightarrow H^1(G, M)$

- more examples $\mathbb{Z}/2 \times \mathbb{Z}/2$, $D_{2n} = \langle s, t \mid s^n = t^2 = 1, (st)^2 = 1 \rangle$
- $H^2(G, M) \longleftrightarrow$ extensions $1 \to M \to E \to G \to 1$

$$C_* = \cdots \to \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1+t} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{1-t} \mathbb{Z}[\mathbb{Z}/2] \to 0]$$

$\mathbb{Z}/2 = \{1, t\}$

If two chain complex C_*, D_*

tensor product of $C_* \otimes D_* = E_*$

$$E_n := (C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

$$\partial_n^E (\underset{\deg p}{c} \otimes \underset{\deg q}{d}) = (\partial_p^C c) \otimes d + (-1)^p (c \otimes \partial_q^D d)$$

Claim: (E_*, ∂^E) is a chain complex

$$\partial_n^E(\underset{\deg p}{c} \otimes \underset{\deg q}{d}) = (\partial_p^C c) \otimes d + (-1)^p (c \otimes \partial_q^D d)$$

$p+q=n$

Claim: (E_*, ∂^E) is a chain complex

C_* is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$

D_* is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[H]$

<u>Claim</u>. $C_* \otimes D_*$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G \times H]$

Pf. Exercise

$C_* \otimes C_*$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]$

chain complex

$$\cdots \mathbb{Z}[\mathbb{Z}/k]\otimes\mathbb{Z}[\mathbb{Z}/k]\oplus\mathbb{Z}[\mathbb{Z}/k]\otimes\mathbb{Z}[\mathbb{Z}/k]\oplus\mathbb{Z}[\mathbb{Z}/k]\otimes\mathbb{Z}[\mathbb{Z}/k] \xrightarrow{\partial_2} \mathbb{Z}[\mathbb{Z}/k]\otimes\mathbb{Z}[\mathbb{Z}/k]\oplus\mathbb{Z}[\mathbb{Z}/k]\otimes\mathbb{Z}[\mathbb{Z}/k] \xrightarrow{\partial_1} \mathbb{Z}[\mathbb{Z}/k]\otimes\mathbb{Z}[\mathbb{Z}/k] \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$\partial_n^c = \begin{cases} 1-t & n \text{ odd} \\ 1+t & n \text{ even} \end{cases}$$

$$C_0\otimes C_1\oplus C_1\otimes C_1\oplus C_2\otimes C_0 \qquad \uparrow \qquad C_0\otimes C_1\oplus C_1\otimes C_0 \qquad \uparrow \qquad C_0\otimes C_0$$

$$a_0\otimes b_1 \longmapsto \partial(a_0)\otimes b_1 + a_0\otimes\partial(b_1) = a_0\otimes(1-t)b_1$$

$$a_1\otimes b_0 \longmapsto \partial(a_1)\otimes b_0 - a_1\otimes\partial(b_0) = (1-t)a_1\otimes b_0$$

Apply $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/k\times\mathbb{Z}/k]}(-,\mathbb{Z})$

$$\cdots \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \leftarrow \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \leftarrow \mathbb{Z}\oplus\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0$$

$$\mathbb{Z} \cong \mathbb{Z}$$

$$\cdots \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \leftarrow \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \leftarrow \mathbb{Z}\oplus\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0$$

$$0 \longleftarrow f$$

$$(2f, 0, 2g) \longleftarrow (f, g)$$

$$f \in \mathrm{Hom}_{\underbrace{\mathbb{Z}[\mathbb{Z}/2\times\mathbb{Z}/2]}_{\mathbb{Z}[\mathbb{Z}/2]\otimes\mathbb{Z}[\mathbb{Z}/2]}}(\mathbb{Z}[\mathbb{Z}/2]\otimes\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z})$$

$$H^n(\mathbb{Z}/2\times\mathbb{Z}/2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \vdots & \end{cases}$$

$$E_1 \xrightarrow{\partial} E_0$$

$$\mathrm{Hom}(-, M) \qquad \mathrm{Hom}(E_1, M) \xleftarrow{f\circ\partial} \mathrm{Hom}(E_0, M)$$

$$\mathrm{Hom}(-, \mathbb{Z}/2)$$

$$(2f, 0, 2g) \longleftarrow\!\shortmid (f, g)$$

$$f \in \underbrace{\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2\times\mathbb{Z}/2]}}_{\mathbb{Z}[\mathbb{Z}/2]\otimes\mathbb{Z}[\mathbb{Z}/2]}(\mathbb{Z}[\mathbb{Z}/2]\otimes\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z})$$

$$H^n(\mathbb{Z}/2\times\mathbb{Z}/2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n=1 \end{cases}$$

$$E_1 \xrightarrow{\partial} E_0$$

$$\mathrm{Hom}(-, M) \quad \mathrm{Hom}(E_1, M) \longleftarrow \mathrm{Hom}(E_0, M)$$

$$f \circ \partial \qquad f$$

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2\times\mathbb{Z}/2]}(-, \mathbb{Z}/2)$$

$$\to H^n(\mathbb{Z}/2\times\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2^{\oplus(n+1)}$$

$$D_{2n} = \langle s, t \mid s^n = t^2 = 1, \quad stst = 1 \rangle$$

Wall, Hamada

Handel, "On products in the cohomology of the dihedral group."

$$\cdots \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \leftarrow \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \leftarrow \mathbb{Z}\oplus\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow 0$$

$$\mathbb{Z}[\mathbb{Z}/2]\otimes\mathbb{Z}[\mathbb{Z}/2]$$

$$0 \longleftarrow 1 : \mathbb{Z}[\mathbb{Z}/2\times\mathbb{Z}/2] \longrightarrow$$

$$1\otimes t,\ t\otimes 1,\ t\otimes t \longmapsto 1$$

$$(2f, 0, 2g) \longleftarrow\!\shortmid (f, g) \qquad f \in \mathrm{Hom}_{\underbrace{\mathbb{Z}[\mathbb{Z}/2\times\mathbb{Z}/2]}_{\mathbb{Z}[\mathbb{Z}/2]\otimes\mathbb{Z}[\mathbb{Z}/2]}}(\mathbb{Z}[\mathbb{Z}/2]\otimes\mathbb{Z}[\mathbb{Z}/2], \mathbb{Z})$$

$$f \longmapsto \underline{f\circ\partial}$$

$$H^n(\mathbb{Z}/2\times\mathbb{Z}/2, \mathbb{Z}) =$$

$$f\circ\partial_1 : a_1\otimes b_0 \xmapsto{\partial_1} (1-t)a_1\otimes b_0 \xmapsto{f} (1-t)\cdot f(a_1\otimes b_1) = 0$$

$$a_0\otimes b_1 \longmapsto a_0\otimes(1-t)b_0 \qquad (1-t)f(a_0\otimes b_1) = 0$$

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2\times\mathbb{Z}/2]}(-, \mathbb{Z}/2)$$

$$\to H^n(\mathbb{Z}/2\times\mathbb{Z}/2, \mathbb{Z}/2) = \quad \oplus(n+1)$$

$$D_{2n} = \langle s, t \mid s^n = t^2 = 1 \quad stst = 1 \rangle$$

Well

$$E_n := (C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

$$\partial_n^E (\underset{\substack{\uparrow \\ \deg p}}{c} \otimes \underset{\substack{\uparrow \\ \deg q}}{d}) = (\partial_p^C c) \otimes d + (-1)^p (c \otimes \partial_q^D d)$$

Claim: (E_*, ∂^E) is a chain complex

free resolutions of $\mathbb{Z}$ over $\mathbb{Z}[D_{2n}]$

$$\cdots \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

$C_p = \mathbb{Z}[D_{2n}] \langle c_p^1, c_p^2, \cdots, c_p^{p+1} \rangle$, c_p^i is basis of C_p as $\mathbb{Z}[D_{2n}]$-module

$$= \underbrace{\underbrace{\mathbb{Z}[D_{2n}]}_{c_p^1} \oplus \underbrace{\mathbb{Z}[D_{2n}]}_{c_p^2} \oplus \cdots \oplus \underbrace{\mathbb{Z}[D_{2n}]}_{c_p^{p+1}}}_{(p+1)\text{-copies}}$$

$\boxed{n=2}$

$D_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$

$$= \underbrace{\mathbb{Z}[D_{2n}]}_{C_p^1} \underbrace{\oplus \mathbb{Z}[D_{2n}] \oplus \cdots \oplus \mathbb{Z}[D_{2n}]}_{(p+1)\text{-copies}}$$

C_p^2 $\qquad$ C_p^{p+1}

$$N = \sum_{i=0}^{n-1} s^i \in \mathbb{Z}[D_{2n}]$$

$$C_0 \to \mathbb{Z}$$

$$C_p \xrightarrow{\partial_p} C_{p-1}$$

$$\varepsilon_i = \begin{cases} -1 & i \equiv 0 \text{ or } 3 \pmod 4 \\ 1 & i \equiv 1 \text{ or } 2 \pmod 4 \end{cases}$$

p even

$$\partial_p(C_p^i) = \begin{cases} C_{p-1}^i(ts + \varepsilon_i\varepsilon_p) + C_{p-1}^i(s-1) & i \text{ even} \\ C_{p-1}^{i-1}(t - \varepsilon_i\varepsilon_p) + C_{p-1}^i N & i \text{ odd} \end{cases}$$

p odd

$$\partial_p(C_p^i) = \begin{cases} C_{p-1}^{i-1}(t - \varepsilon_i\varepsilon_p) - \varepsilon_{p-1}^i N & i \text{ even} \\ C_{p-1}^{i-1}(ts - \varepsilon_i\varepsilon_p) + C_{p-1}^i N & i \text{ odd} \end{cases}$$

$a_0 \otimes b_2 \longmapsto a_0 \otimes (1+t) b_2$

$a_1 \otimes b_1 \longmapsto (1-t) a_1 \otimes b_1 - a_1 \otimes (1-t) b_1 = 0$

$a_2 \otimes b_1 \longmapsto (1+t) a_2 \otimes b_0$

$a_0 \otimes b_1 \longmapsto \partial(a_0) \otimes b_1 + a_0 \otimes \partial(b_1) = a_0 \otimes (1-t) b_1$

$a_1 \otimes b_0 \longmapsto \partial(a_1) \otimes b_0 - a_1 \otimes \partial(b_0) = (1-t) a_1 \otimes b_0$

$$\mathbb{Z}[D_{2n}] \oplus \mathbb{Z}[D_{2n}] \oplus \mathbb{Z}[D_{2n}] \longrightarrow \mathbb{Z}[D_{2n}]\langle c_1^1 \rangle \oplus \mathbb{Z}[D_{2n}]\langle c_1^2 \rangle \longrightarrow \mathbb{Z}[D_{2n}]\langle c_0^1 \rangle \longrightarrow \mathbb{Z} \to 0$$

$c_2^1 \longmapsto c_1^1 N$

$c \longmapsto c_1^1(ts+1) + c_1^2(s-1)$

$c_2^2 \longmapsto c_1^2(t+1)$

$c_1^1 \longmapsto c_0^1(s-1)$

$c_1^2 \longmapsto c_0^1(t-1)$

$c_1^1 N, \quad c_1^2(t+1)$

$c_1^1(ts+1) + c_1^2(s-1) \longleftarrow xyxy = 1$

$\underline{c_0^1} \longmapsto 1$

$st \longmapsto 1$

$\ker = \underline{c_0^1(s-1)}, \quad \underline{c_0^1(t-1)}$

$\to C_1(ts+1)+C_1(s-1)$ $\quad C_1^1 \mapsto C_0^1(t-1)$ $\quad \ker = \underline{C_0^1(s-1)}, \ \underline{C_0^1(t-1)}$

$C_2^3 \mapsto C_1^2(t+1)$

$C_1^1 N, \quad C_1^2(t+1)$

$C_1^1(ts+1)+C_1^2(s-1) \leftarrow xyxy=1$

An extension of G by N is a short exact seq of groups

$$\begin{array}{ccccccccc} 1 & \to & N & \to & E & \to & G & \to & 1 \\ & & \| & & \downarrow \cong & & \| & & \\ 1 & \to & N & \to & E' & \to & G & \to & 1 \end{array}$$

equiv

$(\underline{N \triangleleft E}, \quad E/N \cong G)$

(N is a normal subgroup of E, i.e. $gNg^{-1}=N \ \forall g \in E$)

eg $1 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to 1$

$1 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 1$

Restrict: N to abelian.

s.e.s. gives a G-module structure of N

$g \in G$. Choose some lifting $\tilde{g}$ of g. $\tilde{g} \in E$ w/ $\pi(\tilde{g}) = g$.

$a \in N$, $\quad g \cdot a = \tilde{g}\, a (\tilde{g})^{-1}$, $\quad a = \iota(a) \in E$

Claim: well-defined. Pf: Claim $\in N$. $\Leftrightarrow \pi(\tilde{g} a \tilde{g}^{-1}) = 1$.

Pf: If choose another lifting $\tilde{g}'$,

$\tilde{g}\, a = \tilde{g}\, a(\tilde{g})$, $a = i(a) \in E$

Claim: well-defined

Pf. if choose another lifting $\tilde{g}'$,

Claim $\in N$.

Pf. $\Leftrightarrow \pi(\tilde{g} a \tilde{g}^{-1}) = 1$

prove $\tilde{g}' a (\tilde{g}')^{-1} = \tilde{g} a (\tilde{g})^{-1}$

$$\begin{tikzcd} \tilde{g}' \arrow[dr, mapsto] & \\ \tilde{g} \arrow[r, mapsto] & g \end{tikzcd}$$

$n \in N \longmapsto \tilde{g}^{-1}\tilde{g}' \xmapsto{\pi} 1$

$\Rightarrow \tilde{g}' = \tilde{g} n$

$\rightsquigarrow$ N is G-module.

If N is in center of E, central extension

$\Updownarrow$

G-action of N is trivial

Fix some G-module M

Classify equiv class $1 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to 1$ (s.t. the induced G-module structure by conjugacy agrees w/ given G-module structure)

s

Choose set-theoretic section s

$\pi \circ s = \mathrm{id}$ as a set map,

Odd condition: $s(1) = 1$ ($\Leftarrow$ normalization)

$$M \times G \longrightarrow E$$
$$(m, g) \longmapsto i(m)\, s(g)$$
as a set map

odd condition: $s(1)=1$ ($\Leftarrow$ normalization)

$$M \times G \longrightarrow E$$
$$(m, g) \longmapsto i(m)\, s(g)$$

as a set map

the difference of s from being a group homomorphism

$$g \cdot h \longmapsto s(gh)$$
$$\neq$$
$$s(g) \cdot s(h)$$

$$1 \longrightarrow M \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1$$
$$m \in M$$
$$s(gh) \longmapsto gh$$
$$s(g)s(h) \longmapsto gh$$

$s(g)s(h) = m\, s(gh)$ $\quad m \in M$

$(g,h) \longmapsto m$

$\rightsquigarrow$ a function $G \times G \longrightarrow M$

$$M \times G \longrightarrow E$$
$$(m, g) \longmapsto i(m)\, s(g)$$

$$(m_1, g_1)(m_2, g_2) \longmapsto i(m_1)\, \underbrace{s(g_1)\, i(m_2)}\, s(g_2) = i(m_1)\, i(g_1 m_2)\, \underline{s(g_1)\, s(g_2)}$$
$$= i(m_1)\, i(g_1 m_2)\, i(\underbrace{f(g_1, g_2)}_{\in M})\, s(g_1 g_2)$$

$$i(g_1 \cdot m_2) = s(g_1)\, i(m_2)\, s(g_1)^{-1}$$
$$s(g_1)\, i(m_2) = i(g_1 \cdot m)\, s(g_1)$$

$$= i(m_1 + g_1 m_2 + f(g_1, g_2))\, s(g_1 g_2)$$

potential group structure on $M \times G$ (agrees w/ E) $\quad (m_1, g_1)(m_2, g_2) = (m_1 + g \cdot m_2 + f(g_1, g_2),\ g_1 g_2)$

An extension of G by N is a short exact seq of groups

$$(m_1, g_1)(m_2, g_2) \longmapsto i(m_1)\, s(g_1)\, i(m_2)\, s(g_2) = i(m_1)\, i(g_1 m_2)\, s(g_1)\, s(g_2)$$

$$= i(m_1)\, i(g_1 m_2)\, i(f(g_1, g_2))\, s(g_1 g_2)$$

$$f(g_1, g_2) \in M$$

$$i(g_1 \cdot m_2) = s(g_1)\, i(m_2)\, s(g_1)^{-1}$$

$$= i(m_1 + g_1 m_2 + f(g_1, g_2))\, s(g_1 g_2)$$

$$s(g_1)\, i(m_2) = i(g_1 \cdot m_2)\, s(g_1)$$

potential group structure on $M \times G$ (agrees w/ E) $\quad (m_1, g_1)(m_2, g_2) = (m_1 + g_1 \cdot m_2 + f(g_1, g_2),\ g_1 g_2)$

$f: G \times G \to M$ (normalized $s(1) = 1 \Rightarrow f(1, h) = f(g, 1) = 1$)

① if we want $M \times G$ to be a group, it forces the group multiplication to be the multiplication above.

Question: Is this a group?

Answer: Not yet.

We need associativity, identity, inverse.

identity: $(0, 1)$

associativity: $\big((m_1, g_1)(m_2, g_2)\big)(m_3, g_3) = (m_1, g_1)\big((m_2, g_2)(m_3, g_3)\big)$

$$\big(m_1 + g_1 \cdot m_2 + f(g_1, g_2),\ g_1 g_2\big) \cdot (m_3, g_3) = (m_1, g_1)\big(m_2 + g_2 \cdot m_3 + f(g_2, g_3),\ g_2 g_3\big)$$

$$\big(m_1 + g_1 \cdot m_2 + f(g_1, g_2) + (g_1 g_2) \cdot m_3 + f(g_1 g_2, g_3),\ g_1 g_2 g_3\big) = \big(m_1 + g_1 \cdot m_2 + g_1 g_2 \cdot m_3 + g_1 \cdot f(g_2, g_3) + f(g_1, g_2 g_3),\ g_1 g_2 g_3\big)$$

$$f(g_1, g_2) + f(g_1 g_2, g_3) = g_1 f(g_2, g_3) + f(g_1, g_2 g_3)$$

$$f(g_1, g_2) + f(g_1 g_2, g_3) = g_1 f(g_2, g_3) + f(g_1, g_2 g_3)$$

$$f(g_1, g_2) - f(g_1, g_2 g_3) + f(g_1 g_2, g_3) - g_1 f(g_2, g_3) = 0$$

$$\partial f(g_1, g_2, g_3)$$

$$f : \mathrm{Hom}_{\mathbb{Z}(G)}(\underline{\underline{B_2}}, M)$$

bar resolution $B_\bullet$

$$f(g_1, g_2) - f(g_1, g_2 g_3) + f(g_1 g_2, g_3) - g_1 f(g_2, g_3) = 0$$

$$\partial f(g_1, g_2, g_3)$$

$f : \mathrm{Hom}_{\mathbb{Z}[G]}(\underline{\underline{B_2}}, M)$

bar resolution $B_\bullet$

$\Longrightarrow$ f is a 2-cocycle in $C^2(G, M)$

Conclusion: $\{1 \to M \to E \to G \to 1$ w/ a normalized section $s\} \longleftrightarrow$ 2-cocycle $\in C^2(G, M)$

Claim: choose another section $\tilde{s}$ $\longleftrightarrow$ differs by a 2-coboundary

$\Longrightarrow f$ is a 2-cocycle in $C^2(G, M)$ bar resolution $B_\bullet$

Conclusion: $\{1 \to M \to E \to G \to 1$ w/ a normalized section $s\} \longleftrightarrow$ 2-cocycle $\in C^2(G, M)$

Claim: choose another section $\tilde{s} \longleftrightarrow$ differs by a 2-coboundary

Theorem: $\{$equiv class of $1 \to M \to E \to G \to 1\} \longleftrightarrow H^2(G, M)$

as a set

$$(m_1, g_1)(m_2, g_2) \longmapsto i(m_1)\, s(g_1)\, i(m_2)\, s(g_2) = i(m_1)\, i(g_1 m_2)\, s(g_1 g_2)$$

$$= i(m_1)\, i(g_1 m_2)\, i(f(g_1, g_2))\, s(g_1 g_2)$$

$$f(g_1, g_2) \in M$$

$$i(g_1 \cdot m_2) = s(g_1)\, i(m_2)\, s(g_1)^{-1}$$

$$= i(m_1 + g_1 m_2 + f(g_1, g_2))\, s(g_1 g_2)$$

$$s(g_1)\, i(m_2) = i(g_1 \cdot m)\, s(g_1)$$

potential group structure on $M \times G$ (agrees w/ E) $\quad (m_1, g_1)(m_2, g_2) = (m_1 + g_1 \cdot m_2 + f(g_1, g_2),\ g_1 g_2)$

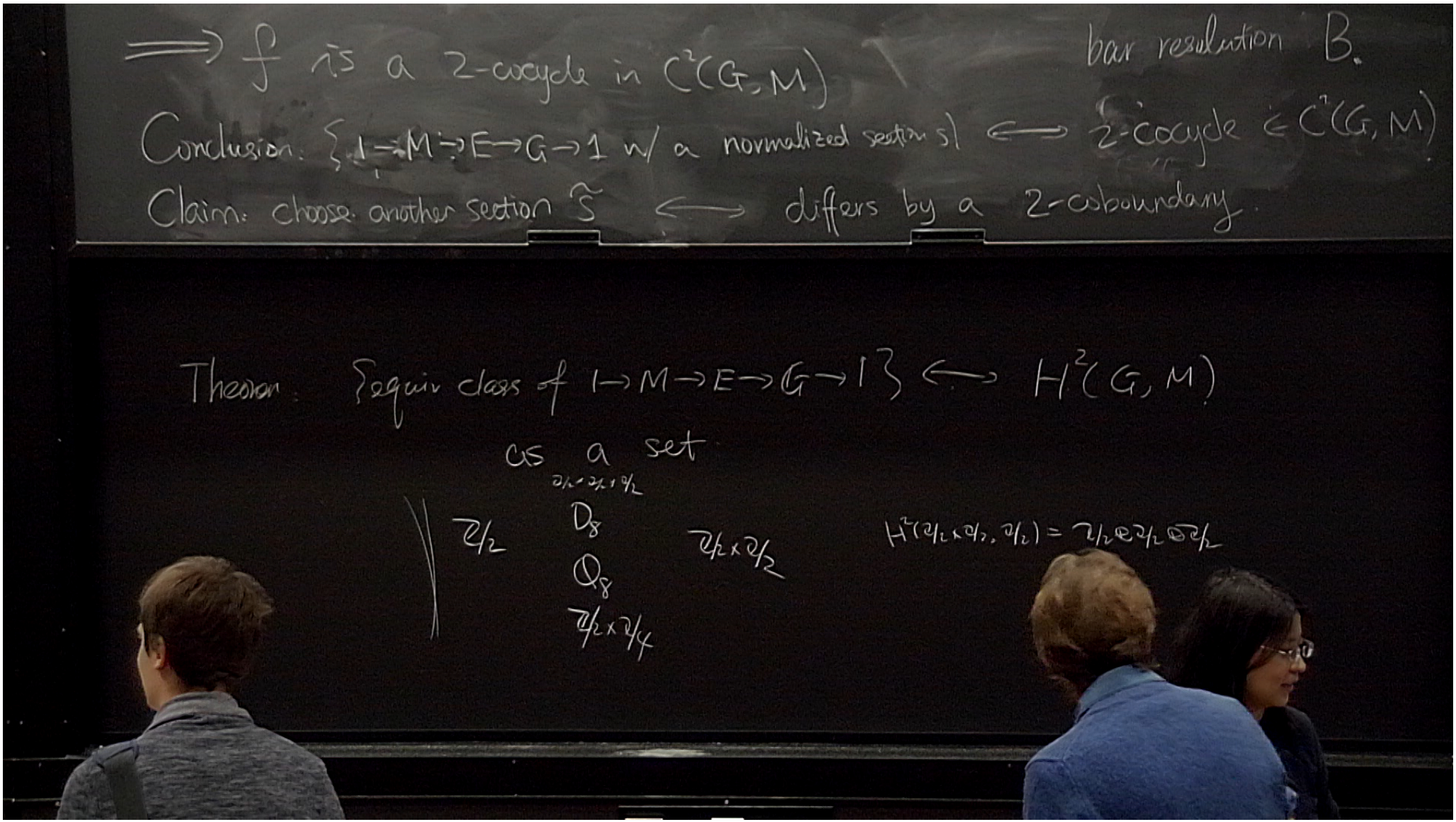
$f: G \times G \to M$ (normalized $s(1) = 1 \Rightarrow f(1, h) = f(g, 1) = 1$)

① if we want $M \times G$ to be a group, it forces the group multiplication to be the multiplication above.

Question: is this a group? $\quad E \longrightarrow M \times G$

Answer: Not yet

We need associativity, identity, inverse.


$\Longrightarrow f$ is a 2-cocycle in $C^2(G, M)$
bar resolution $B_\bullet$
Conclusion: $\{1 \to M \to E \to G \to 1$ w/ a normalized section $s\} \longleftrightarrow$ 2-cocycle $\in C^2(G, M)$
Claim: choose another section $\tilde{s} \longleftrightarrow$ differs by a 2-coboundary.
Theorem: $\{$equiv classes of $1 \to M \to E \to G \to 1\} \longleftrightarrow H^2(G, M)$
as a set
$\mathbb{Z}/2$
D_8
Q_8
$\mathbb{Z}/2 \times \mathbb{Z}/4$
$\mathbb{Z}/2 \times \mathbb{Z}/2$
$H^2(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$