Title: The Cohomology of Groups (Johnson-Freyd/Guo) - Lecture 2

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Collection: The Cohomology of Groups (Johnson-Freyd/Guo)

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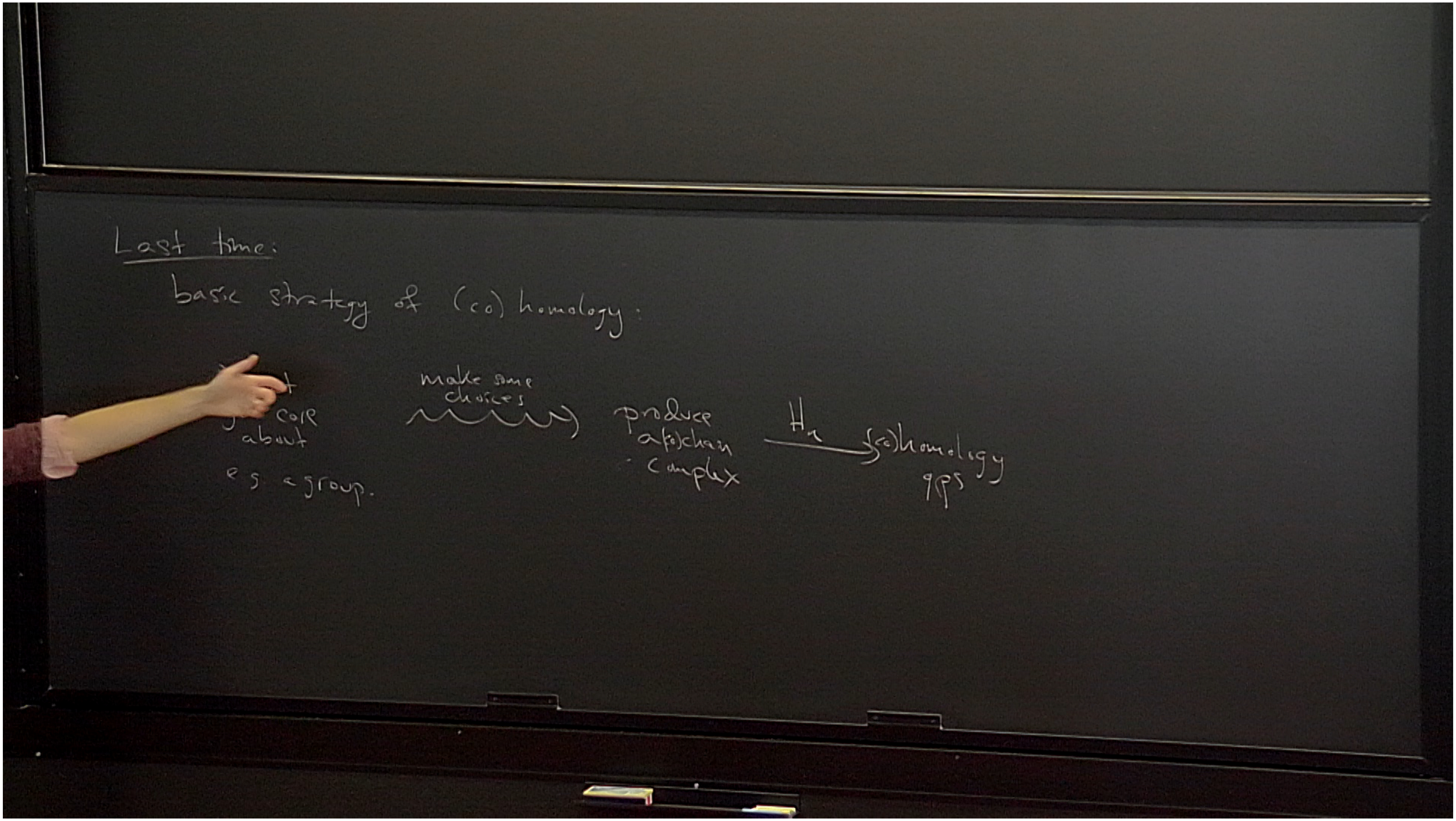
Last time:
basic strategy of (co)homology:
care
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e.g. a group.
make some
choices
produce
a (co)chain
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H_n
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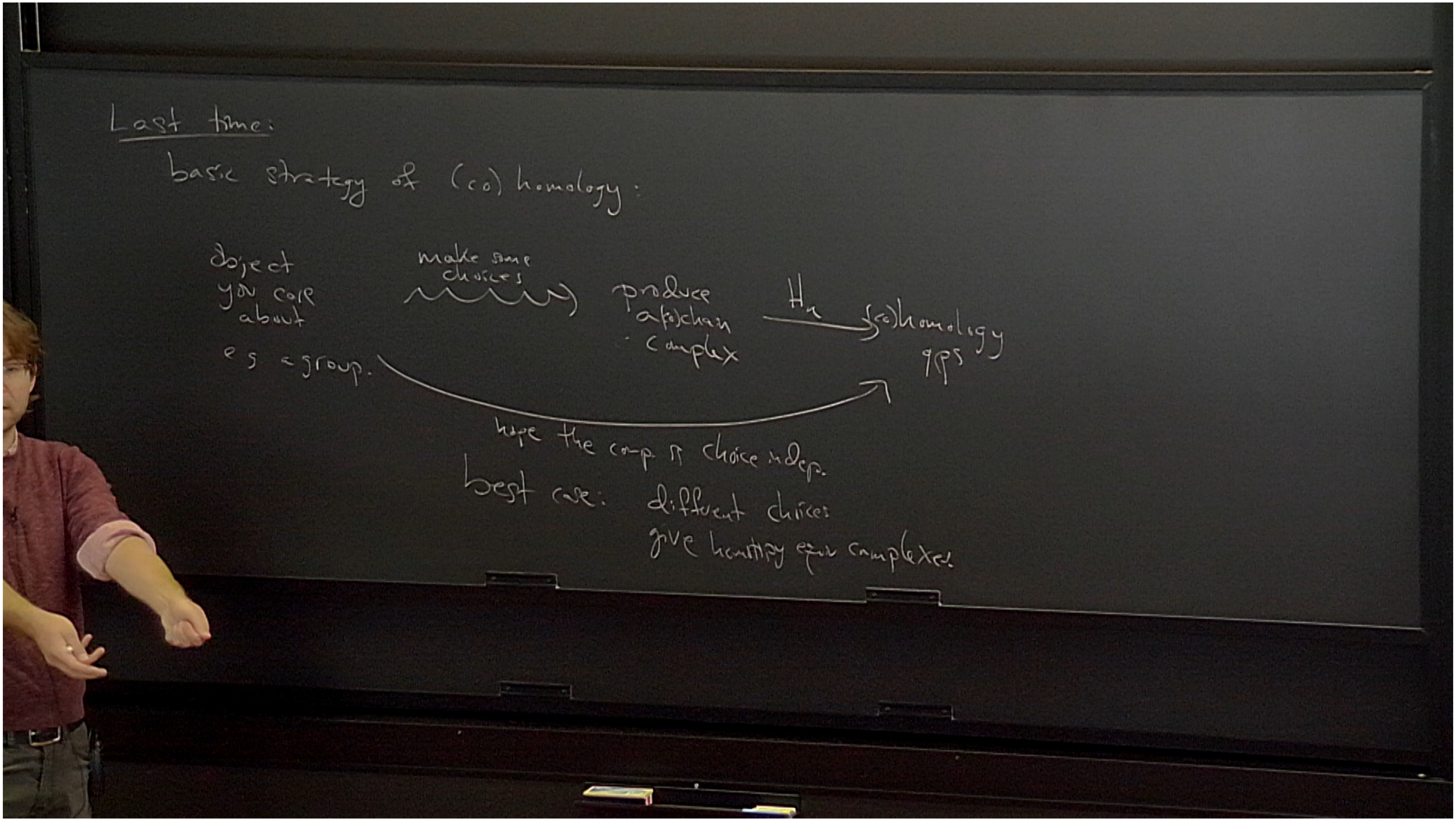
gps


Last time:
basic strategy of (co)homology:
object you care about
e.g. a group.
make some choices
produce a (co)chain complex
Hn
(co)homology gps
hope the comp. is choice indep.
best case: different choices give homotopy equiv complexes

Last time:

basic strategy of (co)homology:

functor

$$\begin{tikzcd} \text{object you care about} \arrow[r, squiggly, "\text{make some choices}"] \arrow[rr, bend right, "\text{hope the comp. is choice indep.}"'] & \text{produce a (co)chain complex} \arrow[r, "H_n"] & \text{(co)homology grps} \end{tikzcd}$$

e.g. a group.

best case: different choices give homotopy equiv complexes

Specifically:

G a group.

complex

e.g. a group. gps

hope the comp. is choice indep.

best case: different choices give homotopy equiv complexes

Specifically:

G a group $\circlearrowleft$ M some G-module

$\left(R = \mathbb{Z}[G] \right.$
its group ring $\left. \right)$

$\rightsquigarrow$ find a free resolution

$F_* \twoheadrightarrow M$ i.e. exact complex

$\cdots \to F_1 \to F_0 \twoheadrightarrow M \to 0$

F_i = free modules = R^{n_i}

apply some functor to F_*

e.g. functor of G-coinvariants.

Remarks:

(0) "free module" makes sense over any ring.

(1) Given $M \xrightarrow{\varphi} N$ map of R-modules

choose free resolution

$$\begin{array}{ccccc} \cdots \to & F_1 & \xrightarrow{\partial} & F_0 & \twoheadrightarrow M \\ & \downarrow \tilde{\varphi} & & \downarrow \tilde{\varphi} & \downarrow \varphi \\ \cdots \to & E_1 & \xrightarrow{\partial} & E_0 & \twoheadrightarrow N \end{array}$$

$f' \mapsto f \mapsto m$, $\tilde{\varphi}(f) = e$, $e \mapsto \varphi(m)$

Prop: Then $\exists$ lift of $\tilde{\varphi}$ of φ

$f' \in F_1$, look at $\tilde{\varphi}(\partial f') \in E_0$.

it lifts to E_1 iff $\partial \tilde{\varphi} \partial f' = 0$

by exactness.

$\tilde{\varphi} \partial^2 f' = 0$. ✓

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(0) "free module" makes sense over any ring.

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$$\begin{array}{ccccccc} \cdots \to & F_1 & \xrightarrow{\partial} & F_0 & \twoheadrightarrow & M & \ni m \\ & \downarrow \tilde{\varphi} & & \downarrow \tilde{\varphi} & & \downarrow \varphi & \\ \cdots \to & E_1 & \xrightarrow{\partial} & E_0 & \twoheadrightarrow & N & \ni \varphi(m) \end{array}$$

$f' \in F_1$, $f \in F_0$, $\tilde{\varphi}(f) = e$, $\tilde{\varphi}(f) \in E_0$

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$\tilde{\varphi} \partial^2 f' = 0$. ✓

N.B. $\tilde{\varphi}$ is not canonical,

but different choices

are related by a chain homotopy.

choose free resolutions

$$\cdots \to F_1 \xrightarrow{\partial} F_0 \to M \ni m$$

$$\tilde{\varphi}(f) = e \qquad \varphi$$

$$\cdots \to E_1 \xrightarrow{\partial} E_0 \to N \ni \varphi(m)$$

e

N.B.: $\tilde{\varphi}$ is not canonical,

but different choices

are related by a chain homotopy.

Cor: Different free resolutions are htpy equivalent.

$f' \in F_1$. Look at $\tilde{\varphi}(\partial f') \in E_0$.

it lifts to E_1 iff $\partial \tilde{\varphi} \partial f' = 0$.

by exactness.

$\tilde{\varphi} \partial^2 f' = 0.$ ✓

choose free resolution $\cdots \to F_1 \xrightarrow{\partial} F_0 \twoheadrightarrow M$

N.B. $\tilde{\varphi}$ is not canonical,

but different choices

are related by a chain homotopy.

Cor: different free resolutions are htpy equivalent.

$\cdots \quad \tilde{\varphi}(e) = e \quad \varphi$

$\cdots \to E_1 \to E_0 \twoheadrightarrow N \ni \varphi(m)$

$f' \in F_1$, look at $\tilde{\varphi}(\partial f) \in E_0$.

it lifts to E_1 iff $\partial \tilde{\varphi} \partial f = 0$

by exactness.

$\tilde{\varphi} \partial^2 f = 0.$ ✓

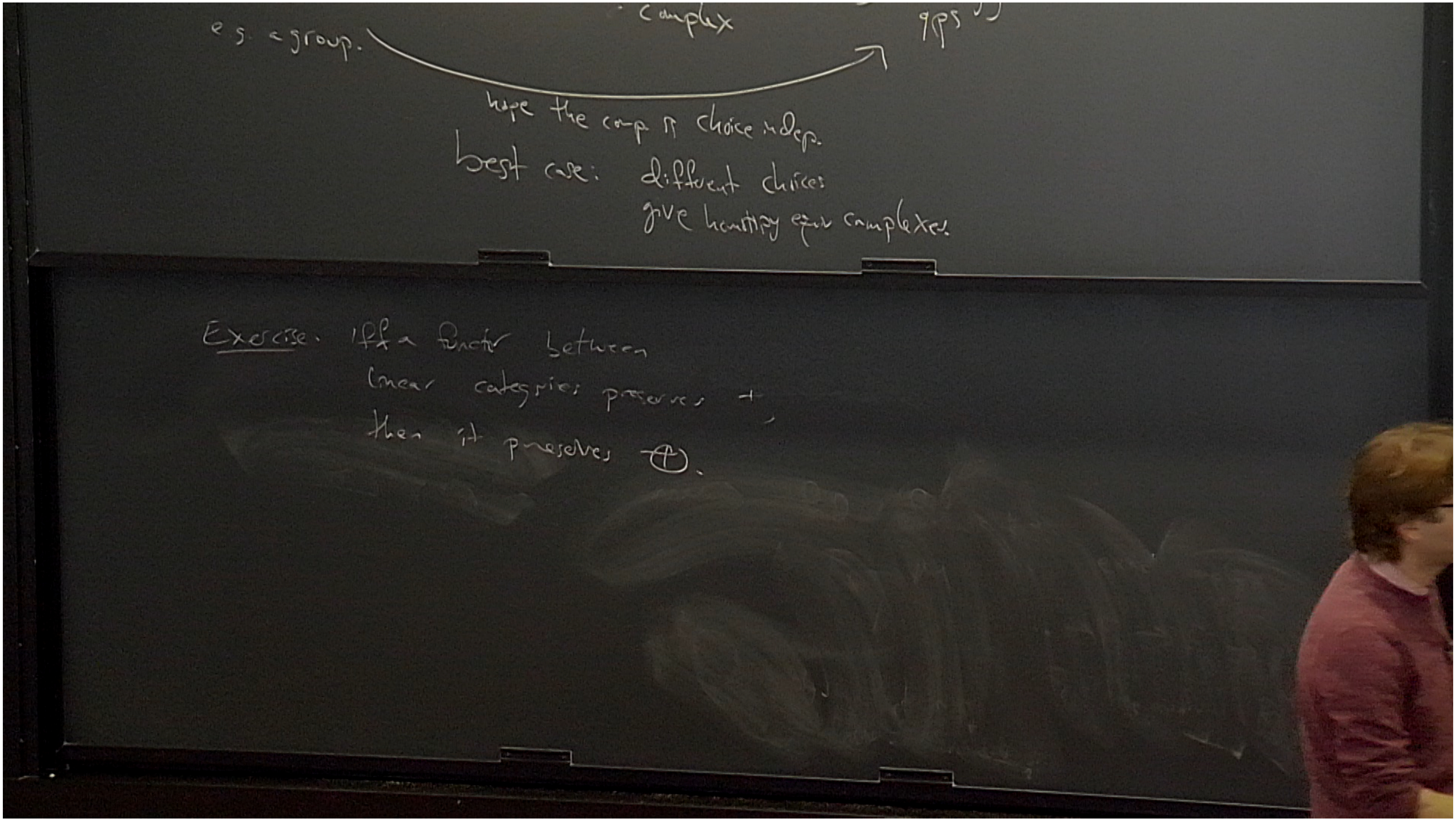
Defn. Suppose $\mathcal{F}$: R-modules $\longrightarrow$ Ab Gps or whatever

preserves addition. ($\Rightarrow$ takes chain complexes to chain complexes)

$$M \rightsquigarrow F_* \to M \mapsto \mathcal{F}(F_*) \in \text{chain complex of abelian gps} \overset{H_n}{\longmapsto} H_n(\mathcal{F}(F_*)) = \mathbb{L}_n\mathcal{F}(M)$$

Does not depend on F_*. It is a functor in M-variable

$M \mapsto \mathbb{L}_n\mathcal{F}(M)$ is the nth left-derived functor of $\mathcal{F}$.


e.g. a group.
complex
gps
hope the comp is choice indep.
best case: different choices
give homotopy equiv complexes
Exercise. If a functor between
linear categories preserves +,
then it preserves ⊕.

choose free resolution $\cdots \to F_1 \xrightarrow{\partial} F_0 \to M$

N.B.: $\tilde{\varphi}$ is not canonical,
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$$\begin{array}{ccccc} \cdots \to & F_1 & \xrightarrow{\partial} & F_0 & \to M \\ & \downarrow \tilde{\varphi} & & \downarrow \tilde{\varphi} & \downarrow \varphi \\ \cdots \to & E_1 & \xrightarrow{\partial} & E_0 & \to N \ni \varphi(m) \end{array}$$

$\tilde{\varphi}(f) = e$

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Defn. Suppose $\mathcal{F}$: R-modules $\longrightarrow$ Ab Gps or whatever
preserves addition ($\Rightarrow$ takes chain complexes to chain complexes) $\quad \mathcal{F}^{\mathbb{L}}_n$

$M \rightsquigarrow F_* \to M \longmapsto \mathcal{F}(F_*) \in$ chain complex of abelian gps $\overset{H_n}{\longmapsto} H_n(\mathcal{F}(F_*)) = (\mathbb{L}_n \mathcal{F})(M)$

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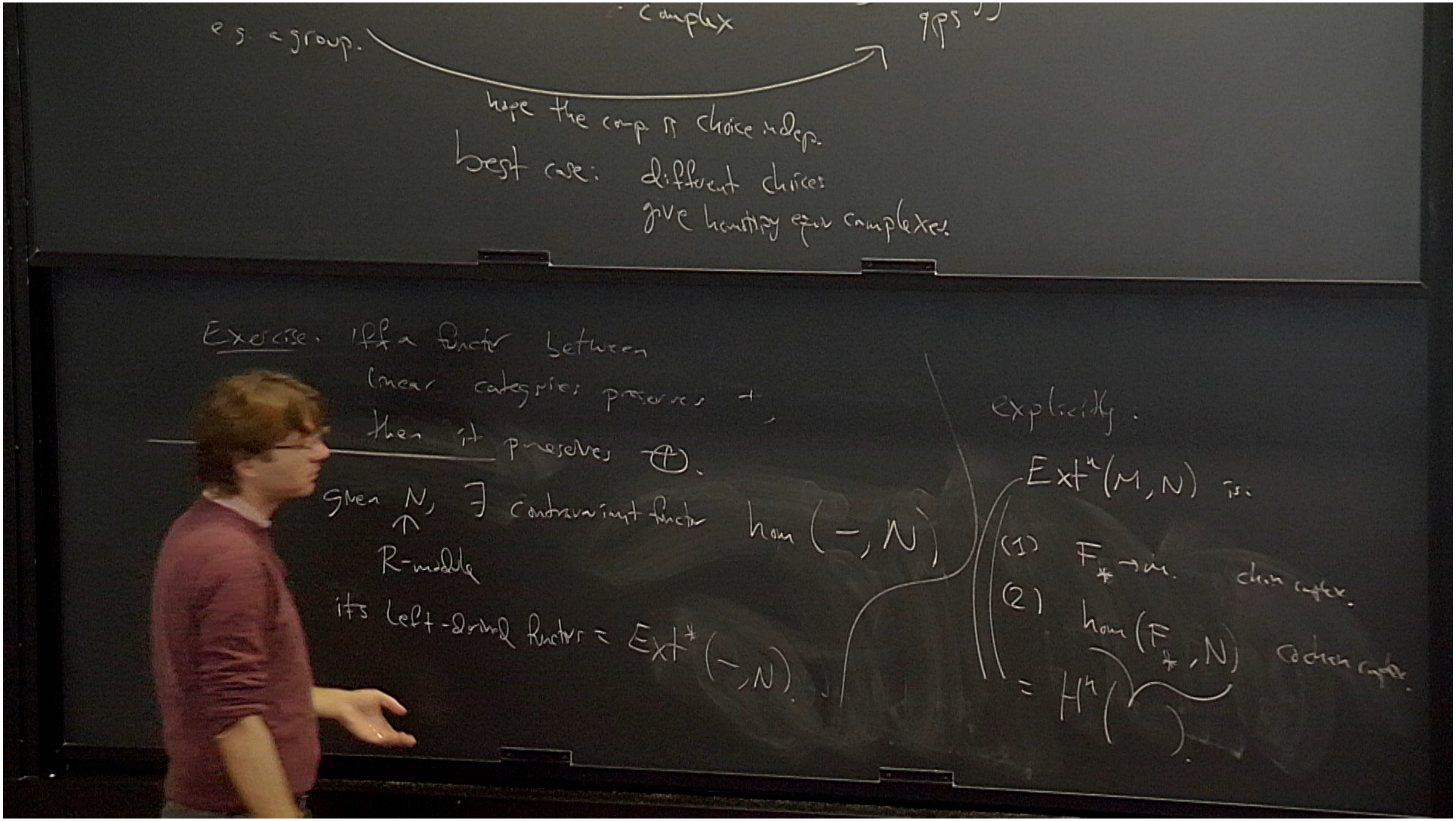
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Exercise. If a functor between
linear categories preserves +,
then it preserves ⊕.
given N, ∃ contravariant functor hom(−, N)
R-module
its left-derived functors = Ext*(−, N).
explicitly:
Ext^n(M, N) is:
(1) F_* → M. chain complex.
(2) hom(F_*, N) cochain complex.
= H^n( )

R-module

its left-derived functor $= \mathrm{Ext}^*(-, N)$.

$\hom(-, N)$

(1) $F_* \to M$. chain complex.

(2) $\hom(F_*, N)$ cochain complex.

$= H^n(\quad)$.

G is a (discrete) gp.

$R = \mathbb{Z}[G]$

M is a G-module

<u>Defn.</u> $H^n_{gp}(G, M) := \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$

trivial G-module

<u>Point is.</u> Given any free resolution $\cdots \to R^{n_1} \to R^{n_0} \to \mathbb{Z}$, you get a complex computing ext gps.

e.g. the bar resolution

$\uparrow$

R-module

its left-derived functor = $\mathrm{Ext}^*(-, N)$.

$\mathrm{hom}(-, N)$

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If free resolution consists of finite-rank free modules,

$$F_i = R^{m_i} \quad m_i < \infty.$$

then $\mathrm{hom}_R(F_i, M) = \mathrm{hom}(R^{m_i}, M)$

$$= M^{m_i}$$

Exercise: If a functor between linear categories preserves $+$, then it preserves $\oplus$.

E.g. given N, $\exists$ contravariant functor $\hom(-, N)$
(N an R-module)

its left-derived functors $= \mathrm{Ext}^*(-, N)$.

explicitly: $\mathrm{Ext}^n(M, N)$ is:

(1) $F_* \to M$. chain complex.

(2) $\hom(F_*, N)$ cochain complex.

$= H^n(\hom(F_*, N))$.

G is a (discrete) grp.

$R = \mathbb{Z}[G]$

M ...

If free resolution consists of finite-rank free modules,

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$f_0 \ni f_e$

N.B. $\tilde\varphi$ is not canonical, but different choices are related by a chain homotopy.

Cor: different free resolutions are htpy equivalent.

$f' \in F_1$, look at $\tilde\varphi(\partial f) \in E_0$. it lifts to E_1 iff $\partial \tilde\varphi \partial f = 0$. by exactness. $\tilde\varphi \partial \partial f = 0.$ ✓

$H^*(G;M)$

$H^0 = G$-fixed elts of M.

$$\bigoplus_{g \in G} \mathbb{Z}[G]\langle g \rangle \longrightarrow \mathbb{Z}[G] \to \mathbb{Z}$$

basis $\langle\rangle \mapsto 1$

$\langle g \rangle \mapsto g\langle\rangle - \langle\rangle$.

$\mathrm{hom}_G(-,M)$ functions$(G \to M)$ M

$(g \mapsto gm - m) \longleftarrow m$

choose free resolution $\cdots \to F_1 \xrightarrow{\partial} F_0 \to M$

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$\mathrm{hom}_G(-, M)$

functions$(G \to M)$ $\quad M$

$(g \mapsto gm - m) \longleftarrow m$

H^1 = conj. classes of group homs φ

$M \rtimes G \leftarrow G$

if $G \curvearrowright M$, s.t. $\pi \varphi = \mathrm{id}_G$

build semidirect product

$M \rtimes G \xrightarrow[\pi]{\text{grp hom}} G.$

as a set, $M \times G$.

but the mult. is $\quad (\varphi(g), g) \leftarrow g$

$(m_1, g_1)(m_2, g_2)$

$= (m_1 + g_1 m_2, g_1 g_2)$

$(m_1, g_1)\cdot(m_2, g_2)$

$= (m_1 + g_1 \triangleright m_2,\ g_1 \cdot g_2)$

$\mathrm{Hom}_G(-, M)$ $\quad$ $\mathrm{Functions}(G \to M)$ $\quad$ M

$(g \mapsto gm)$ $\longleftarrow\!\shortmid$ m

E.g.: $G = \mathbb{Z}/n$ $\quad$ $\mathbb{Z}[G] = \mathbb{Z}[t]/(t^n - 1) = R$

$\{1, t, t^2, \dots, t^{n-1}\}$ $\quad$ $(t^n - 1) = 0$

$(t-1)(\underbrace{t^{n-1} + t^{n-2} + \dots + t + 1}_{N})$

$$R \xrightarrow{\cdot N} R \xrightarrow{\cdot (t-1)} R \longrightarrow \mathbb{Z}$$

$1 \longmapsto 1$

$7t^2 \longmapsto 7$

$t - 1 \longmapsto 0$

$N \in R$.

if $|G| < \infty$, then

$N = \sum_{g \in G} g \in \mathbb{Z}[G]$

"norm element"

$N^2 = (\# G) \cdot N$

$\mathrm{Hom}_G(-, M)$ $\qquad$ $\mathrm{Functions}(G \to M)$ $\qquad$ M

$(g \mapsto gm - m) \longleftarrow\!\shortmid\ m$

$(\varphi(g), g) \longleftarrow\!\shortmid\ g$

$(m_1, g_1)\cdot(m_2, g_2) = (m_1 + g_1 \triangleright m_2,\ g_1 \cdot g_2)$

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$\cdots \xrightarrow{\cdot N} R \xrightarrow{\cdot (t-1)} R \xrightarrow{\cdot N} R \xrightarrow{\cdot (t-1)} R \longrightarrow \mathbb{Z}$

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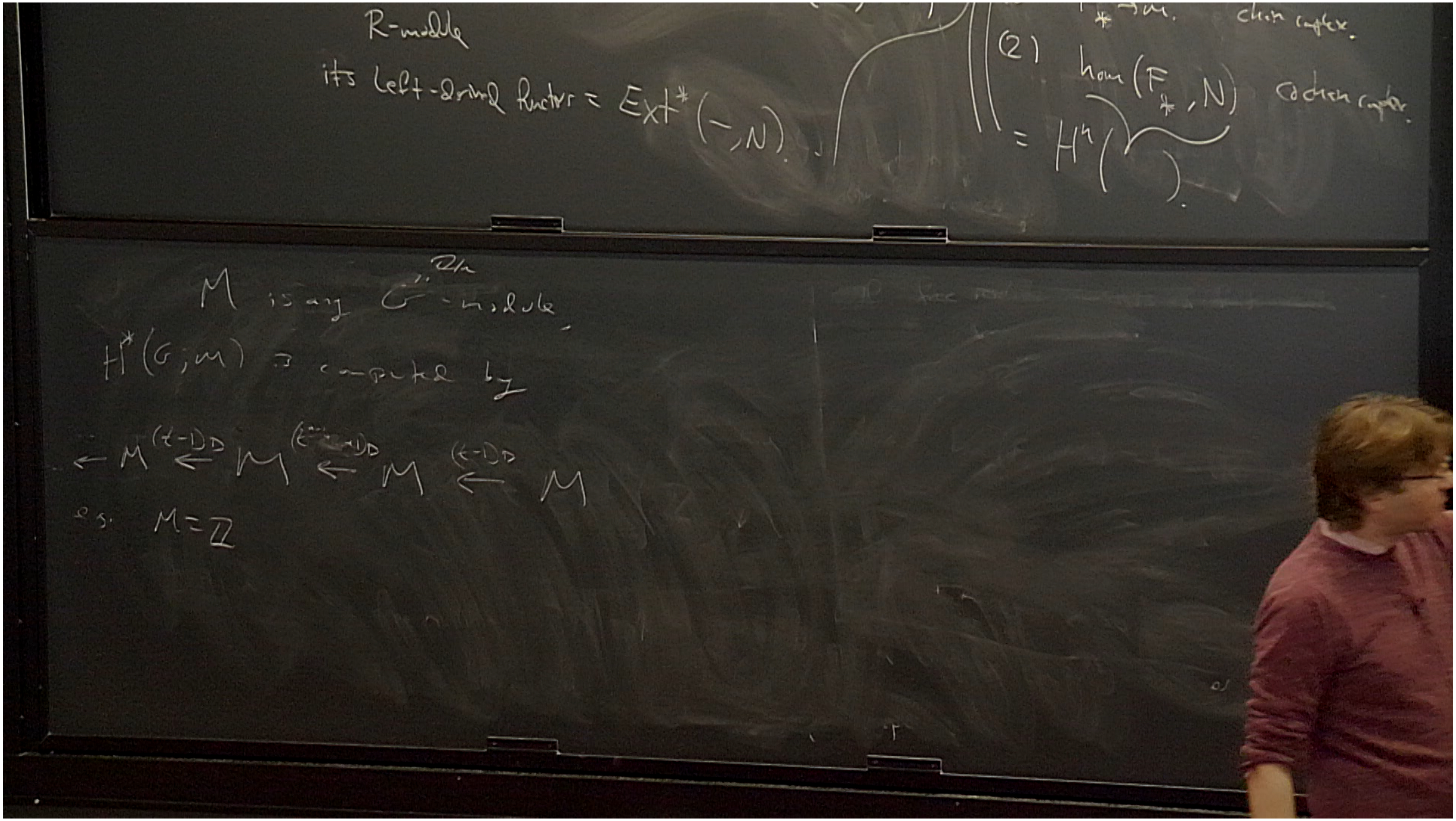

R-module
its left-derived functor = Ext*(−, N).
chain complex.
(2) hom(F*, N) cochain complex
= Hⁿ( ).
M is any Z/n G-module.
H*(G; M) is computed by
← M ←(t−1)D M ←(t−1)D M ←(t−1)D M
e.g. M = Z

$\mathrm{Hom}_G(-, M)$ Functions$(G \to M)$ M

$(g \mapsto gm \ldots m) \longleftarrow\!\shortmid\; m$

$(\varphi(g), g) \leftarrow\!\shortmid\; g$

$(m_1, g_1)\cdot(m_2, g_2)$

$= (m_1 + g_1 \triangleright m_2,\ g_1 g_2)$

E.g.: $G = \mathbb{Z}/n = \{1, t, t^2, \ldots, t^{n-1}\}$.

$\mathbb{Z}[G] = \mathbb{Z}[t]/(t^n - 1) = R$

$(t^n - 1) = 0$

$= (t-1)(\underbrace{t^{n-1} + t^{n-2} + \cdots + t + 1}_{N})$

$N \in R$.

$$\cdots \xrightarrow{\cdot N} R \xrightarrow{\cdot(t-1)} R \xrightarrow{\cdot N} R \xrightarrow{\cdot(t-1)} R \longrightarrow \mathbb{Z}$$

$1 \longmapsto 1$

$7t^2 \longmapsto 7$.

$t-1 \longmapsto 0$

$1 \mapsto Nm$ $1 \mapsto m$

$\mathrm{Hom}(R, M) \longleftarrow \mathrm{Hom}(R, M)$

precompose w/ $\cdot N$.

if $|G| < \infty$, then

$N = \sum_{g \in G} g \in \mathbb{Z}[G]$

"norm element"

$N^2 = (\#G) \cdot N$

R-module

its left-derived functor $= \mathrm{Ext}^*(-, N)$.

(2) $\hom(F_*, N)$ cochain complex.

$= H^n(\ \ )$.

M is any $G = \mathbb{Z}/n$-module.

$H^*(G; M)$ is computed by

$$\cdots \leftarrow \overset{3}{M} \xleftarrow{(t-1)D} \overset{2}{M} \xleftarrow{(t-1)D} \overset{1}{M} \xleftarrow{(t-1)D} \overset{0}{M}$$

e.g. $M = \mathbb{Z}$ trivial

$$\xleftarrow{\cdot n} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot n} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$$

$\mathbb{Z}/n$, 0, $\mathbb{Z}/n$, 0, $\mathbb{Z}$

M is any $G^{\mathbb{Z}/n}$-module.

$H^*(G;M)$ is computed by

$$\begin{array}{ccccccccc} & 3 & & 2 & & 1 & & 0 \\ \leftarrow & M & \xleftarrow{(t-1)D} & M & \xleftarrow{(t^{n-1}+\dots+1)D} & M & \xleftarrow{(t-1)D} & M \end{array}$$

e.g. $M=\mathbb{Z}$ trivial

$$\xleftarrow{\cdot n} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot n} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$$

$\mathbb{Z}/n,\ 0,\ \mathbb{Z}/n,\ 0,\ \mathbb{Z}$

$M=\mathbb{R}/\mathbb{Z}$ w/ trivial action.

$$\xleftarrow{n} \mathbb{R}/\mathbb{Z} \xleftarrow{0} \mathbb{R}/\mathbb{Z} \xleftarrow{\cdot n} \mathbb{R}/\mathbb{Z} \xleftarrow{0} \mathbb{R}/\mathbb{Z}$$

$H^3=\mathbb{Z}/n \quad H^2=0 \quad H^1=\mathbb{Z}/n \quad H^0(\mathbb{Z}/n;\mathbb{R}/\mathbb{Z})=\mathbb{R}/\mathbb{Z}.$

$$\xleftarrow{\cdot n} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot n} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$$

$\mathbb{Z}/n, \quad 0, \quad \mathbb{Z}/n, \quad 0, \quad \mathbb{Z}$

$H^1 = \mathbb{Z}/n \qquad H^2 = 0. \qquad H^1 \cong \mathbb{Z}/n \qquad H^0(\mathbb{Z}/n, \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}.$

$G = \mathbb{Z}/n$, n = even $= 2k$

$\boxed{\pi = \tfrac{1}{2}}$

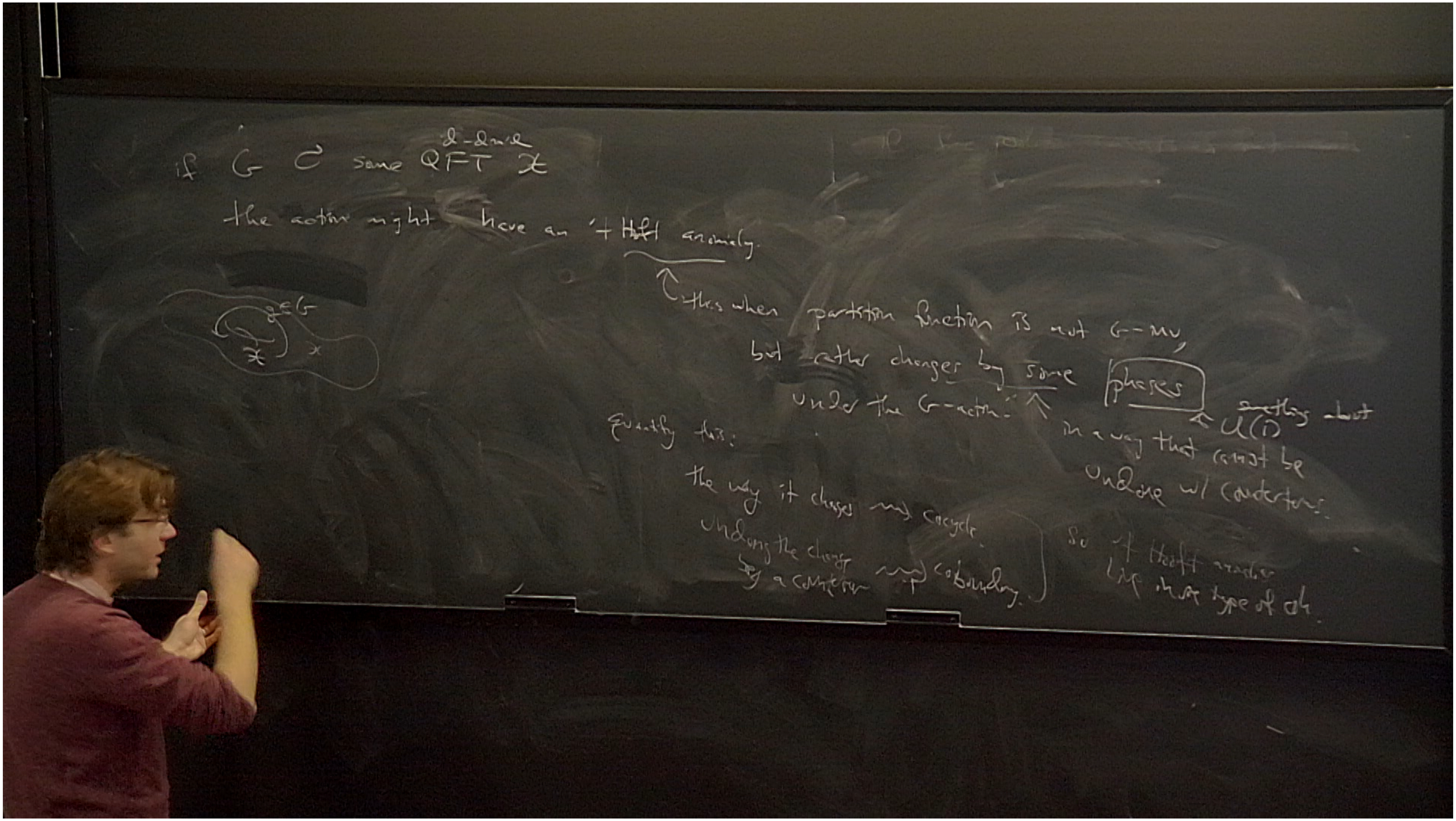
$G \circlearrowright \mathbb{R}/2\pi\mathbb{Z}$ (c.c. on $U(1)$)

t acts by -1

$M = (\mathbb{R}/\mathbb{Z})^T$ "time reversal"

$$\begin{array}{ccccccc} 3 & & 2 & & 1 & & 0 \\ \cdots \xleftarrow[\times(-2)]{t-1} & & \mathbb{R}/\mathbb{Z} \xleftarrow[\times 0]{t^{n-1}+t^{n-2}+\cdots+t+1} & & \mathbb{R}/\mathbb{Z} \xleftarrow[\times(-2)]{(t-1)} & & \mathbb{R}/\mathbb{Z} \end{array}$$

$H^2 = \mathbb{Z}/2 \qquad H^1 = 0. \qquad H^0 = \mathbb{Z}/2 = \{0, \pi\}.$

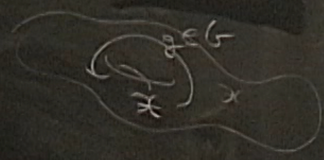

d-dim'l
if G ↻ some QFT X
the action might have an 't Hooft anomaly.
this when partition function is not G-inv,
but rather changes by some phases
under the G-action.
something about U(1)
in a way that cannot be undone w/ counterterms.
g∈G
Quantify this:
the way it changes ~ cocycle.
Undoing the change by a counterterm ~ coboundary.
So 't Hooft anomalies live in this type of coh.

if $G \circlearrowright$ some $\overset{d\text{-dim'l}}{\text{QFT}}$ $\mathfrak{X}$

the action might have an 't Hooft anomaly

this when partition function is not G-inv, but rather changes by some $\boxed{\text{phases}}$ under the G-action." in a way that cannot be undone w/ counterterms.

something about $U(1)$



Quantify this:

the way it changes $\leadsto$ cocycle.

undoing the change by a counterterm $\leadsto$ coboundary.

Some (many) 't Hooft anomalies live specifically in

$$H^{d+1}_{\text{ɡr}}(G; U(1))$$

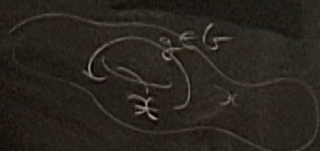
"no mixed gauge-gravity anomaly"

So 't Hooft anomalies live in some type of coh.

if $G \circlearrowright$ some d-dim'l QFT $\mathfrak{X}$

the action might have an 't Hooft anomaly

← this when partition function is not G-inv, but rather changes by some $\boxed{\text{phases}}$ under the G-action.



Some (many) 't Hooft anomalies live specifically in $H^{d+1}_{gp}(G; U(1))$

"no mixed gauge-gravity anomaly"

Quantify this:

the way it changes ⇝ cocycle

Undoing the change by a counterterm ⇝ coboundary.

I could allow symmetries in G that reverse Time.

$G \to \mathbb{Z}/2$

which records whether $g \in G$ reverses Time.

$\mathbb{Z}/2 \subset U(1)$ by c.c.

phases ← something about $U(1)$ in a way that cannot be undone w/ counterterms.

So 't Hooft anomalies live in a type of coh.

Some (many) 't Hooft anomalies live specifically in $H^{d+1}_{gp}(G; U(1)^T)$

"no mixed gauge-gravity anomaly"

Quantify this:

$U(1)$ w/ action of $G \to \mathbb{Z}/2 \subset \mathrm{Aut}(U(1))$

the way it changes $\rightsquigarrow$ cocycle

undoing the change by a counterterm $\rightsquigarrow$ coboundary.

in a way that cannot be undone w/ counterterms.

So 't Hooft anomalies live in this type of coh.

From SES to LES

G gp. $R = \mathbb{Z}[G]$

Fix a free resolution $F_* \to \mathbb{Z}$ as a G-module.

then for any G-module M,

$F_n = R^{k_n}$

$\rightsquigarrow H^*(G, M) = H^*(\overbrace{\mathrm{hom}_G(F_*, M)})$

each term in this complex is M^{k_n}

Study case where I have a short exact sequence of G-modules

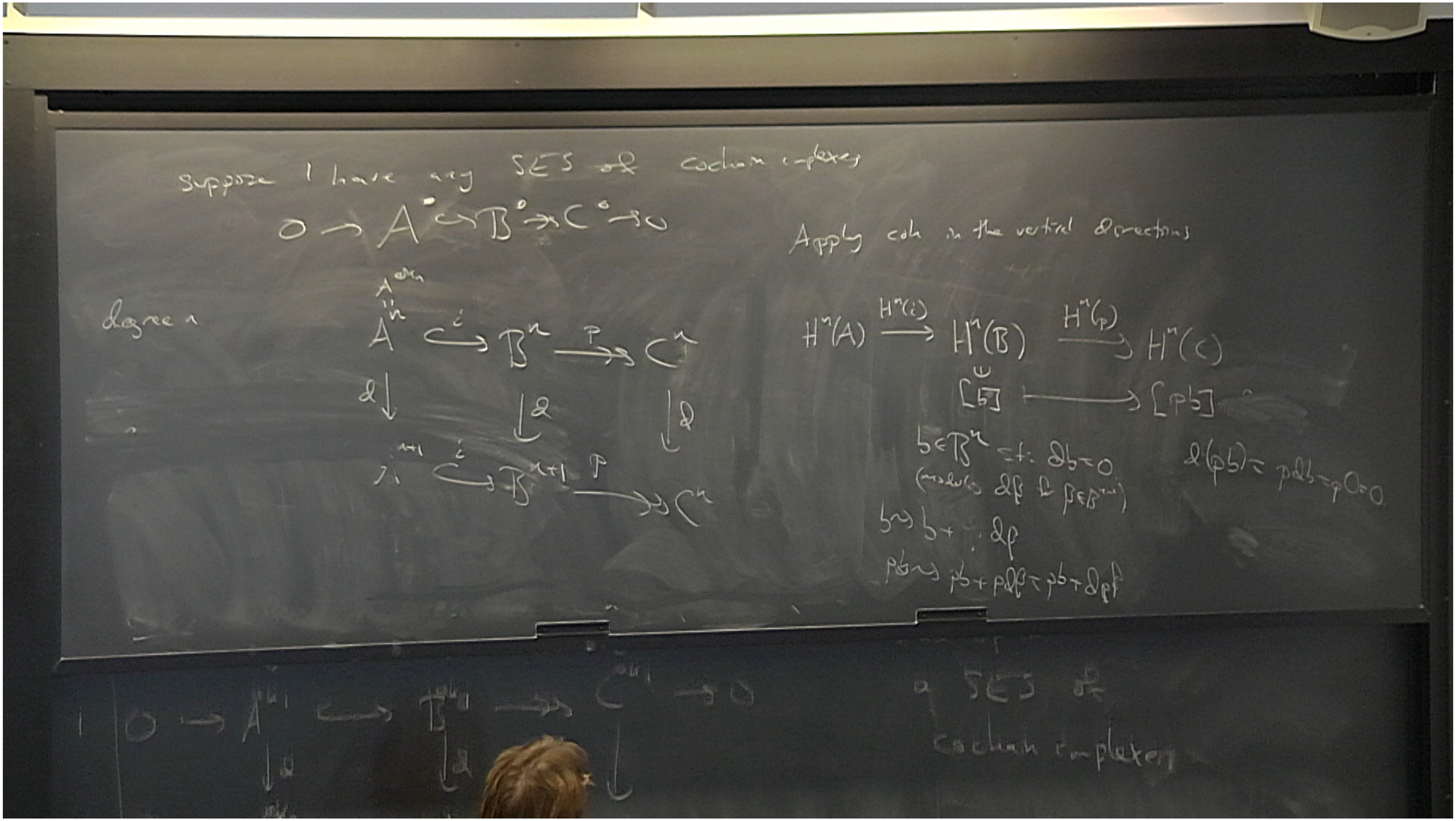
$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$.

$\mathrm{im}(i) = \ker(p)$

$$\begin{tikzcd}
A^{k_0} \arrow[d, "\ell"] & B^{k_0} \arrow[d, "\ell"] & C^{k_0} \arrow[d] \\
A^{k_1} \arrow[d, "\ell"] & B^{k_1} \arrow[d, "\ell"] & C^{k_1} \arrow[d] \\
A^{k_2} \arrow[d] & B^{k_2} \arrow[d] & C^{k_2} \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

$$
\begin{tikzcd}
\text{coh deg} & & & & & \\
0 & 0 \arrow[r] & A^{\oplus k_0} \arrow[r, hook, "i^{\oplus k_0}"] \arrow[d, "\partial"] & B^{\oplus k_0} \arrow[r, two heads, "p^{k_0}"] \arrow[d, "\partial"] & C^{k_0} \arrow[r] \arrow[d] & 0 \\
1 & 0 \arrow[r] & A^{\oplus k_1} \arrow[r, hook] \arrow[d, "\partial"] & B^{\oplus k_1} \arrow[r, two heads] \arrow[d, "\partial"] & C^{\oplus k_1} \arrow[r] \arrow[d] & 0 \\
2 & 0 \arrow[r] & A^{\oplus k_2} \arrow[r, hook] \arrow[d] & B^{\oplus k_2} \arrow[r, two heads] \arrow[d] & C^{\oplus k_2} \arrow[r] \arrow[d] & 0 \\
3 & & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

end up w/ a SES of cochain complexes


suppose I have any SES of cochain complexes
$0 \to A^\bullet \hookrightarrow B^\bullet \twoheadrightarrow C^\bullet \to 0$
Apply coh in the vertical directions
degree n
$H^n(A) \xrightarrow{H^n(i)} H^n(B) \xrightarrow{H^n(p)} H^n(C)$
$[b] \longmapsto [pb]$
$b \in B^n$ s.t. $\partial b = 0$
(modulo $\partial\beta$ for $\beta \in B^{n-1}$)
$b \mapsto b + \partial\beta$
$pb \mapsto pb + p\partial\beta = pb + \partial p\beta$
$\partial(pb) = p\partial b = p0 = 0$
a SES of cochain complexes

$$\begin{array}{ccccc} \partial\downarrow & & \downarrow\partial & & \downarrow\partial \\ A^{n+1} & \overset{i}{\hookrightarrow} & B^{n+1} & \overset{p}{\twoheadrightarrow} & C^{n} \end{array}$$

$[a] \mapsto [ia]$ $\quad [b] \longmapsto [pb]$

$b \in B^n$ s.t. $\partial b = 0$ $\quad \partial(pb) = p\partial b = p0 = 0$

(modulo $\partial\beta$ for $\beta \in B^{n-1}$)

$b \rightsquigarrow b + \partial\beta$

$pb \rightsquigarrow pb + p\partial\beta = pb + \partial p\beta$

Claim: the sequence

$$H^n(A) \longrightarrow H^n(B) \longrightarrow H^n(C)$$

is exact at $H^n(B)$.

WTS: if $[bp] = 0$ in $H^n(C)$

then $[b] = [ia]$ in $H^n(B)$

for some $[a]$ in $H^n(A)$

Some (many) 't Hooft anomalies live specifically in

$$H^{d+1}_{gp}(G; U(1)^T)$$

"no mixed gauge-gravity anomaly"

Quantify this: the way it changes my cocycle under the G-action

U(1) w/ action of G

undoing the change by a counterterm = coboundary

something about U(1) in a way that cannot be undone w/ counterterms.

So 't Hooft anomalies live in this type of coh.

Know: • $b \in B^n$ s.t. $\partial b = 0$.

• $pb = \partial \gamma$ for $\gamma \in C^{n-1}$

Know: $\cdot$ $b \in B^n$ s.t. $\partial b = 0$.

$\cdot$ $pb = \partial \gamma$ for $\gamma \in C^{n-1}$.

exactness of a-ly row — can choose $\beta \in B^{n-1}$ s.t. $p\beta = \gamma$

Look at $[b - \partial\beta]$

$p(b-\partial\beta) = pb - p\partial\beta = pb - \partial_p \beta$

$= pb - \partial\gamma = 0$

$[ia] = [b]$

so exactness of B^n

$\Rightarrow b - \partial\beta = ia$ for some $a \in A^n$.

$\checkmark$.

I could allow symmetries in G that reverse Time.

$G \to \mathbb{Z}/2$

which records whether $g \in G$ reverses Time.

$\mathbb{Z}/2 \subset U(1)$ by c.c.

might have an 't Hooft anomaly

this when partition function is not G-inv,

but rather changes by some

Rhetorical question:

is $0 \to H^n(A) \to H^n(B) \xrightarrow{H^n(p)} H^n(C) \to 0$

exact at C? i.e. is $H^n(p)$ surjective?

Given $[c] \in H^n(C)$

Want to construct $\underbrace{[b]}_{\text{this requires } \partial b = 0}$ s.t. $[pb] = [c]$

I can certainly find $b \in B^n$ s.t. $pb = c$.

So I win if $\partial b = 0$.

I know $\partial c = 0$. $p \partial b = \partial p b = \partial c = 0$

$ia = \partial b \in B^{n+1} \to 0 \in C^{n+1}$

$a \in A^{n+1}$

... construct $\underbrace{[b]}_{\text{this requires } db=0.}$ s.t. $[pb]=[c]$

So I win if $db=0$.

I know $dc=0$. $\quad p\,db = dpb = dc = 0$

$\qquad\qquad\qquad\qquad\quad \Downarrow$
$\qquad\qquad\qquad\qquad\quad 0$

$ia = db \in B^{n+1} \mapsto 0 \in C^{n+1}$

$a \in A^{n+1}$

<u>Note.</u> $da = 0$ since i is an injection

and $i\,da = dia = d^2 b = 0$.

<u>Claim.</u> $\underset{H^n(C)}{\underset{\Uparrow}{[c]}} \rightsquigarrow \underset{H^{n+1}(A)}{\underset{\Uparrow}{[a]}}$ is well-defined ✓

$c \rightsquigarrow c + d\gamma$ then choose β s.t. $p\beta = \gamma$.

$b \rightsquigarrow b + d\beta$. $ia = db \rightsquigarrow db + d^2\beta = db$. $a \rightsquigarrow a$.

to prove the claim, I need to show that $[a]$ does not change if

- $b \rightsquigarrow b + i\alpha$ $\qquad [a] \rightsquigarrow [a + d\alpha] = [a]$ ✓.
- $c \rightsquigarrow c + d\gamma$

$db \rightsquigarrow db + di\alpha = db + id\alpha$
$\qquad = ia + id\alpha$

$$[a] \mapsto [ia] \qquad [b] \longmapsto [pb]$$

$b \in B^n$ s.t. $\partial b = 0$

(modulo $\partial\beta$ for $\beta \in B^{n-1}$)

$b \rightsquigarrow b + \partial\beta$

$pb \rightsquigarrow pb + p\partial\beta = pb + \partial p\beta$

$\partial(pb) = p\partial b = p0 = 0$

$A^{n+1} \xhookrightarrow{i} B^{n+1} \xrightarrow{p} C^n$

Thm: Given a SES $0 \to A \to B \to C \to 0$ of G-modules get a LES

$$H^n(G,A) \xrightarrow{i} H^n(G,B) \xrightarrow{p} H^n(G,C) \xrightarrow{\delta} H^{n+1}(G,A) \xrightarrow{i} H^{n+1}(G,B) \xrightarrow{p} H^{n+1}(G,C) \to$$

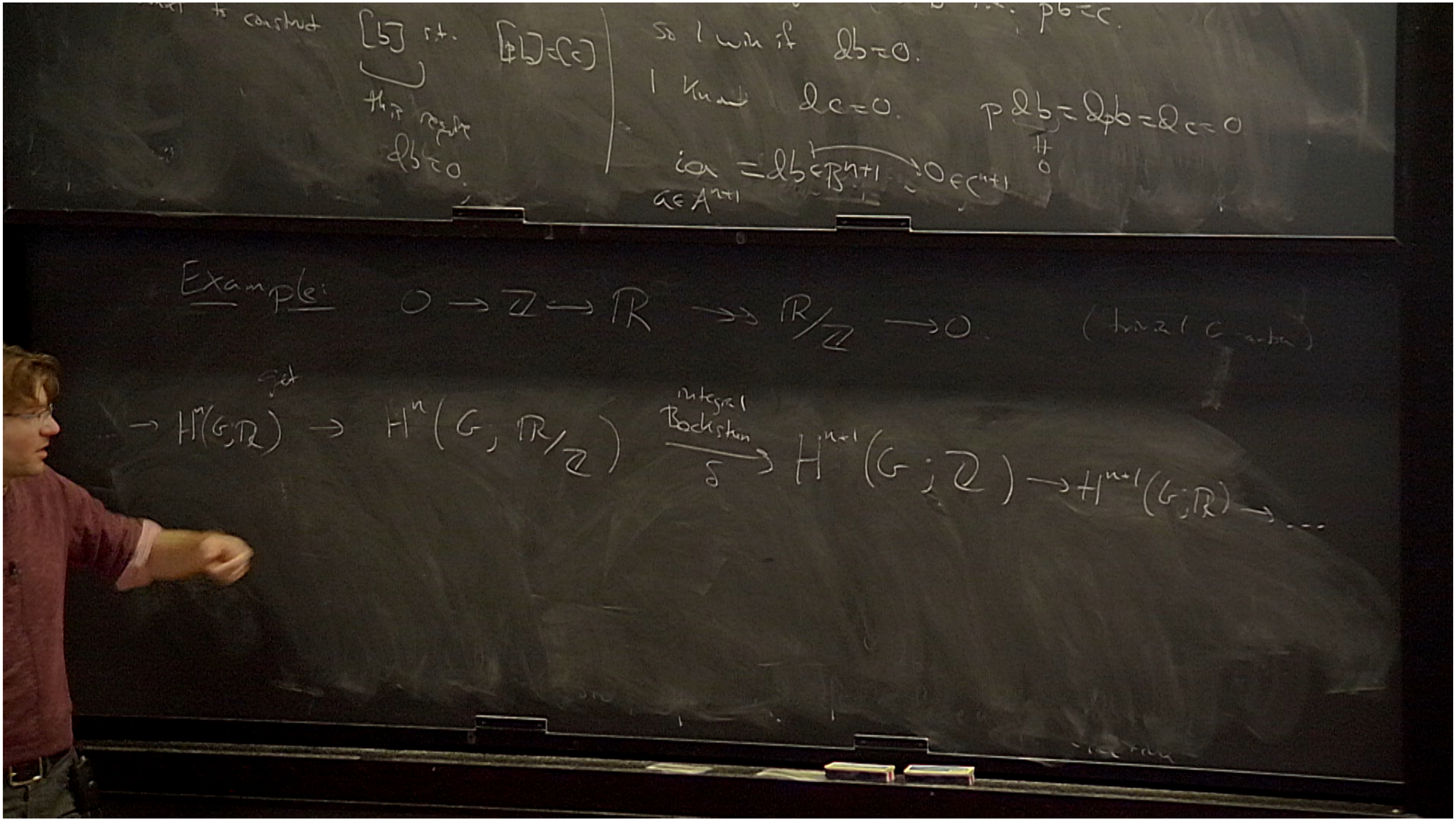
by construction, this 4-term complex is exact at $H^n(C)$.

Claim $i \circ \delta = 0$ ✓.

$\delta c = a \quad ia = \partial b$

Claim.

this long sequence is exact at $H^{n+1}(A)$


to construct [b] s.t. [pb]=[c]
this requires ∂b=0.
So I win if ∂b=0.
I know ∂c=0. p∂b=∂pb=∂c=0
ia = ∂b ∈ B^{n+1} → 0 ∈ C^{n+1}
a ∈ A^{n+1}
Example: 0 → Z → R → R/Z → 0. (trivial G-action)
get
→ H^n(G;R) → H^n(G, R/Z) → H^{n+1}(G;Z) → H^{n+1}(G;R) → ...
integral Bockstein
δ

... any sol'n.

Example: $0 \to \mathbb{Z} \to \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z} \to 0$. (trivial G action)

get

$$\cdots \to H^n(G,\mathbb{R}) \to H^n(G,\mathbb{R}/\mathbb{Z}) \xrightarrow[\delta]{\text{integral Bockstein}} H^{n+1}(G;\mathbb{Z}) \to H^{n+1}(G;\mathbb{R}) \to \cdots$$

$\circlearrowleft$ this is a $\mathbb{R}$-vector space.

$\mathbb{R}$

if $|G|<\infty$, it must be the zero v.s.

any G-action.

Example: $0 \to \mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z} \to 0$. (trivial G-action)

get

$$\cdots \to H^n(G,\mathbb{R}) \to H^n(G;\mathbb{R}/\mathbb{Z}) \xrightarrow[\delta]{\text{integral Bockstein}} H^{n+1}(G;\mathbb{Z}) \to H^{n+1}(G;\mathbb{R}) \to \cdots$$

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if $|G|<\infty$, it must be the zero v.s.

Cor: if $|G|<\infty$, then

$$H^n(G,\mathbb{R}/\mathbb{Z}) \underset{\sim}{\xrightarrow{\text{bockstein}}} H^{n+1}(G;\mathbb{Z}) \text{ if } n>0$$

Lemma: $\times|G|$ acts by zero on $H^n(G;M)$ $n>0$.
w/ trivial coeffs

pf: • works w/ bar complex.

• write explicitly a coboundary for $|G|x$, any cochain.

$H^1(\mathbb{Z};\mathbb{R}) = \mathbb{R}^1$

$H^4(SU(2) ;\mathbb{R}) = \mathbb{R}$.
& discrete gp

$\cdots \xrightarrow{\delta} H^{n+1}(G;\mathbb{Z}) \to H^{n+1}(G;\mathbb{R}) \to \cdots$

this is a $\mathbb{R}$-vector space.

$\mathbb{R}$

if $|G|<\infty$, it must be the zero v.s.

Cor: if $|G|<\infty$, then

$$H^n(G,\mathbb{R}/\mathbb{Z}) \underset{\sim}{\xrightarrow{\text{bockstein}}} H^{n+1}(G;\mathbb{Z}), \text{ if } n>0$$