Title: The Cohomology of Groups (Johnson-Freyd/Guo) - Lecture 1

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Collection: The Cohomology of Groups (Johnson-Freyd/Guo)

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Background reading:

- Brown, Cohomology of Groups, 1982
- Gelfand + Manin, Methods in homological algebra, 1997
- Moore, lecture notes on group theory.

  www.physics.rutgers.edu/~gmoore/618Spring2019/GTLect1-AbstractGroupTheory-2019.pdf

Origins of homological algebra

1. Poincare: combinatorial topology

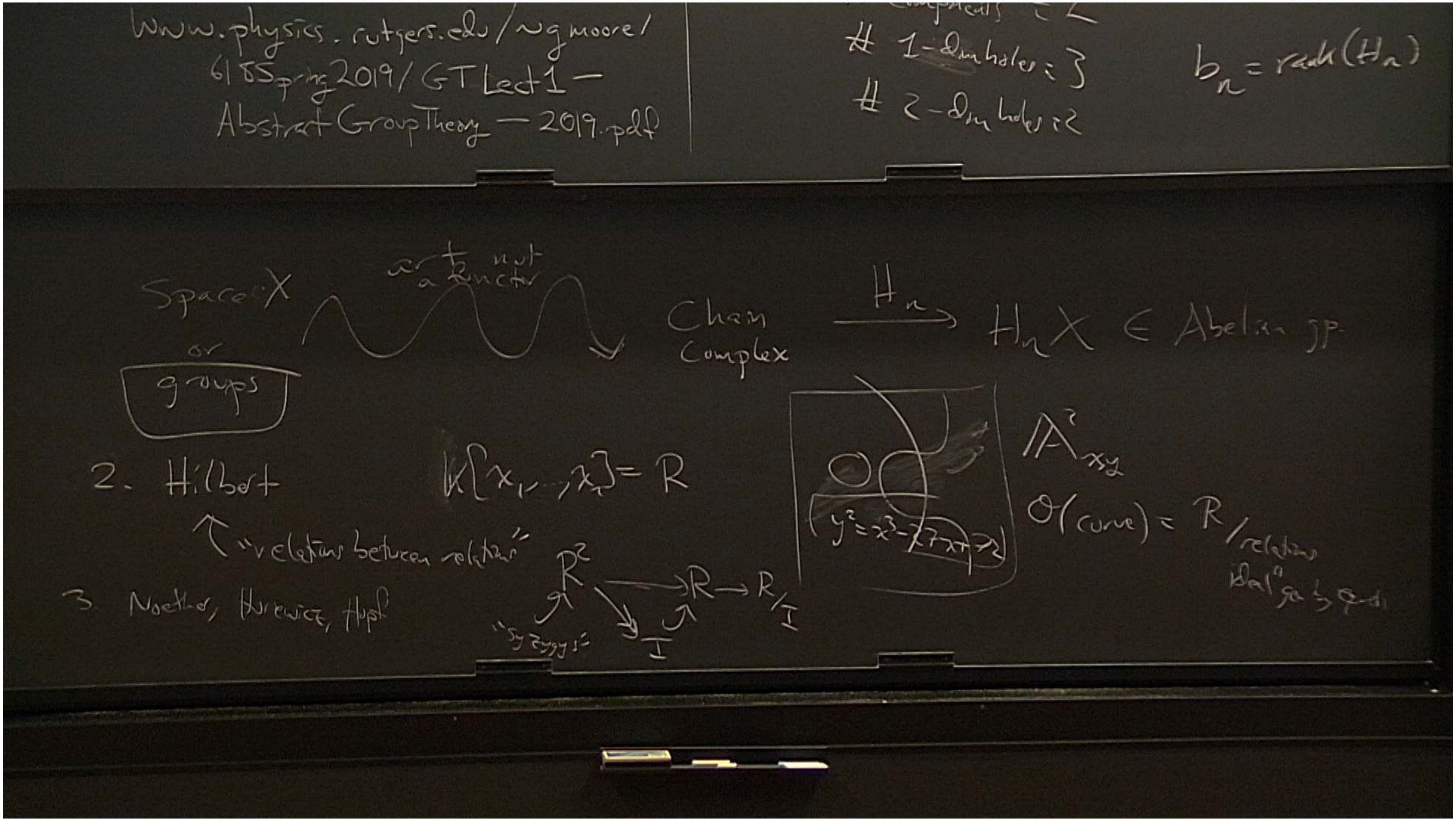
\# components = 2

\# 1-dim holes = 3

\# 2-dim holes = 2

"Betti numbers"

$b_n = \mathrm{rank}(H_n)$


www.physics.rutgers.edu/~gmoore/
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AbstractGroupTheory-2019.pdf
1-Dim holes = ?
2-Dim holes = ?
$b_n = \mathrm{rank}(H_n)$
Space X
or
groups
not a functor
Chain Complex
H_n
$H_n X \in$ Abelian gp.
2. Hilbert
"relations between relations"
3. Noether, Hurewicz, Hopf
$k[x_1, \dots, x_n] = R$
$R^2 \to R \to R/I$
"syzygies"
I
$y^2 = x^3 - 27x + 72$
$\mathbb{A}^2_{xy}$
$\mathcal{O}(\text{curve}) = R/\text{relations}$

Groups will have homology:

$H_0(G;\mathbb{Z}) = H_0 G = \mathbb{Z}$.

$H_1 G = G_{ab} = G/[G,G]$ abelianization of G.

<u>Hopf</u> $H_2 G = \dfrac{I \cap [F,F]}{[I,F]}$

$G = F/I$ (F: free group, I: relations) $I \trianglelefteq F$.

F = free gp on "generators".

$I \trianglelefteq F$ normal subgp gen by some "relations".

If M is a G-module, $\exists$ homology + coh w/ coeffs in M.

$H_n(G;M)$

$H^n(G;M)$

$= \frac{F^{\text{free group}}}{I^{\text{relation}}}$ $I \triangleleft F$ $[I,F]$

generators.

$I \triangleleft F$ normal subgp gen by some "relations".

E.g. $G = \mathbb{Z}/2$ $F = \mathbb{Z}$.

$H_0 G = \mathbb{Z}$

$H_1 G = \mathbb{Z}/2$ $2\mathbb{Z} = I \triangleleft \mathbb{Z}$

$$H_2 G = \frac{2\mathbb{Z} \cap [\mathbb{Z},\mathbb{Z}]}{[2\mathbb{Z},\mathbb{Z}]} = \frac{*}{*} = *$$

Goals: Computational tools

- Mayer-Vietoris $H_\cdot$ (free product)
- Kunneth formula $H_\cdot(G_1 \times G_2)$
- Long exact sequences $H_\cdot(G, M)$ ← interesting module.

$\Rightarrow$ - Leray-Serre spectral sequence $H_\cdot$ (extension)

- cobordism ...

Defn:

A chain complex of abelian gps is a sequence of abelian gps connected by linear maps

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \xrightarrow{\partial_{-2}} \cdots$$

s.t. $\partial_i \circ \partial_{i+1} = 0$. i.e. $Z_i = \mathrm{Ker}(\partial_i) \supseteq \mathrm{im}(\partial_{i+1}) =: B_i$.

∂ "boundary operator"

"cycles"

"boundaries"

elements of C_n = "chains of deg. n"

$$H_n(C_\bullet) := \frac{Z_n}{B_n} = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}.$$

Defn:

of abelian grps

Defn:

A cochain complex (of abelian gps) is a sequence of abelian gps connected by linear maps

" chain complex w/ diff. [illegible]

$$\cdots \xrightarrow{\partial^{-3}} C^{-2} \xrightarrow{\partial^{-2}} C^{-1} \xrightarrow{\partial^{-1}} C^{0} \xrightarrow{\partial^{0}} C^{1} \xrightarrow{\partial^{1}} C^{2} \xrightarrow{\partial^{2}} \cdots$$

s.t. $\partial^i \circ \partial^{i-1} = 0$. i.e. $Z^i = \mathrm{Ker}(\partial^i) \supseteq \mathrm{im}(\partial^{i-1}) =: B^i$

∂ "boundary map"

cohomology $\quad H^n(C_\bullet) := \dfrac{Z^n}{B^n} = \dfrac{\mathrm{Ker}(\partial^n)}{\mathrm{im}(\partial^{n-1})}$.

$\text{s.t. } \partial^i \circ \partial^{i-1} = 0. \quad \text{i.e. } Z^i = \mathrm{Ker}(\partial^i) \supseteq \mathrm{im}(\partial^{i-1}) =: B^i$

∂ "boundary map"

cohomology $\quad H^n(C_\bullet) := \dfrac{Z^n}{B^n} = \dfrac{\mathrm{Ker}(\partial^n)}{\mathrm{im}(\partial^{n-1})}$

<u>Defn:</u> A <u>morphism</u> of chain complexes $C_\bullet \xrightarrow{f_\bullet} C'_\bullet$ is a sequence of linear maps s.t. $\partial' \circ f = f \circ \partial$.

is

$$\begin{tikzcd}
\cdots \arrow[r] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & C_{-1} \arrow[r] \arrow[d, "f_{-1}"] & \cdots \\
\cdots \arrow[r] & C'_1 \arrow[r, "\partial'"] & C'_0 \arrow[r, "\partial'"] & C'_{-1} \arrow[r] & \cdots
\end{tikzcd}$$

$G = \langle \ \mid \ \rangle$ ... $\cong \langle \ \mid \ \rangle$. | $I \triangleleft F$ normal subgp gen by some "relations"

I^{relation} $I \triangleleft F$.

Lemma: H_n is a functor $\forall n$.

chain complex $\to$ ab gps.

Defn: A chain complex is **exact at C_n** if $H_n(C_\cdot) = 0$.

$C_\cdot$ **acyclic** if it is exact everywhere

$\|$

long exact sequence

long exact sequence

A short exact sequence

is

$$* \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} *$$

exact everywhere

trivial ab gp: $1, 0, *$.

$f(A) = \ker(g)$

$\ker f = *$

f is an injection

$g(B) = C$

$C_\bullet \overset{f_\bullet}{\underset{g_\bullet}{\rightrightarrows}} D_\bullet$ chain complexes (not nec. exact)

a cycle
$\partial \alpha = 0$

$$
\begin{tikzcd}
\cdots \arrow[r] & C_1 \arrow[r] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dr, "\eta"] & C_0 \arrow[r, "\partial"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dr, "\eta"] & C_{-1} \arrow[r] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \cdots \\
& D_1 \arrow[r] & D_0 \arrow[r] & D_{-1} \arrow[r] & \cdots
\end{tikzcd}
$$

$\partial(\eta(\alpha)) = f(\alpha) - g(\alpha)$

a homotopy equivalence of morphisms

$f \simeq g$ is

a map $C_n \xrightarrow{\eta} D_{n+1}$ $\forall\, n$

s.t. $\partial \circ \eta + \eta \partial = f - g$.

Defn: A <u>homotopy equivalence</u> of chain complexes

is $C_\bullet \xrightarrow{f} D_\bullet$

s.t. $\exists$ "f^{-1}" $: D_\bullet \to C_\bullet$ s.t. $f \circ f^{-1} \simeq id.$

$$f^{-1} \circ f \simeq id.$$

Defn: A *weak equivalence* $C_\cdot \xrightarrow{f} D_\cdot$

is a morphism of chain complexes

s.t. $H_n(f): H_n(C_\cdot) \to H_n(D_\cdot)$

is an iso $\forall n$.

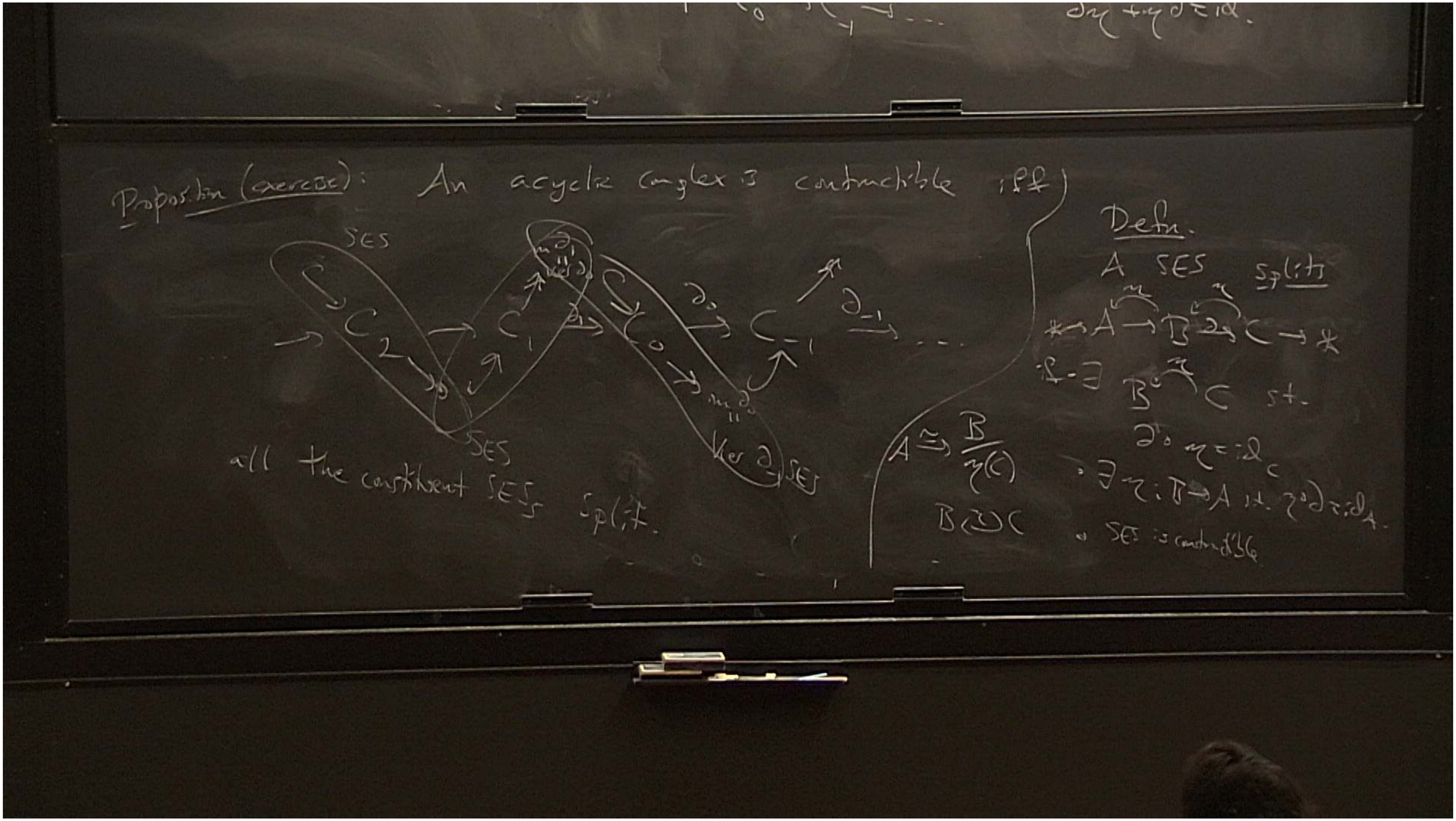
Exercise Homotopy equiv $\Rightarrow$ weak equivalence.

Examples.

- acyclic = weakly equivalent to the 0 complex
- contractible = homotopy equiv to the 0 complex

$$= \quad \cdots \to C_1 \to C_0 \xrightarrow{\partial} C_{-1} \to \cdots \qquad \exists\, \eta \leftarrow \text{contracting homotopy s.t. } \partial\eta + \eta\partial = \mathrm{id}.$$

LESs are stitched out of SESs:


Proposition (exercise): An acyclic complex is contractible iff
SES
SES
all the constituent SESs split.
Defn.
A SES splits
st.
SES is contractible

Recollection: A category is some collection of objects + homomorphisms w/ associative composition.

A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$

sends · objects to objects.

· morphisms to morphisms compatibly.

E.g.

$\left\{ \begin{array}{l} \text{· ab. gps.} \\ \text{· linear maps} \end{array} \right\}$

$\left\{ \begin{array}{l} \text{· chain complexes} \\ \text{· morphisms} \end{array} \right\}$

ie $X, Y \in \text{objects}(\mathcal{C})$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] & Y \arrow[r, "g"] & Z \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[rr, bend left, "F(g \circ f)"] \arrow[rr, bend right, "F(g) \circ F(f)"'] & F(Y) \arrow[r, "F(g)"] & F(Z) \end{tikzcd}$$

respecting composition

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(g) \circ F(f)"'] & F(Y) \arrow[r, "F(g)"] & F(Z)
\end{tikzcd}
$$

respects composition

E.g. $\forall n$ $\quad H_n : \mathrm{Ch} \longrightarrow \mathrm{Ab}$

Ch = chain complexes

Ab = abelian gps.

$$
\begin{tikzcd}
\beta \arrow[d, mapsto] & \partial_{n+1}(\beta) + \alpha \arrow[r, mapsto] & 0 \\
 \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
f(\beta) \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & \mathrm{im} \\
 & & 0
\end{tikzcd}
$$

$f(\alpha) \in$ cycle

$f(\alpha + \partial\beta) = f(\alpha) + \partial f(\beta)$

$$
\begin{aligned}
C_\bullet &\mapsto \frac{\ker(\partial_n) \subseteq C_n}{\mathrm{im}(\partial_{n+1}) \subseteq C_n} = H_n(C) \ni [\alpha] \\
\Big\downarrow f \quad & \qquad \Big\downarrow H_n(f) \\
D_\bullet &\mapsto \frac{\ker(\partial'_n)}{\mathrm{im}(\partial'_{n+1})} = H_n(D)
\end{aligned}
$$

Let G a gp. M is a G-module

i.e. M is an abelian gp

and G acts on M by (

group ring

$$\mathbb{Z}[G] = \bigoplus_{g \in G} \mathbb{Z}g.$$

M is a $\mathbb{Z}[G]$-

$$(7g + 8h)(g) = 7(g^2) + 8(hg).$$

$\ldots = \ldots x(g) + g(h_s).$

Build a "free resolution" of M

"relations" $= \ker(\partial_0)$

pick some generating relations i.e. gens of

generators of M.

$\ker(\partial_1)$ $=$ "syzygies"

$$\cdots \longrightarrow \overset{2}{R^{k_2}} \longrightarrow \overset{1}{R^{k_1}} \xrightarrow{\partial_1} \overset{0}{R^{k_0}} \xrightarrow{\partial_0} \overset{-1}{M}$$

basis here $\longmapsto$ generators of $\ker(\partial_0)$

$Re_1 \oplus Re_2 \oplus \cdots$

spanned by $m_1, \ldots, m_k$

$e_i \longmapsto m_i$

$re_i \longmapsto r \cdot m_i$, $r \in R$

by construction, this whole complex is acyclic, and it's free except in degree -1

A free resolution of M as an R-module

is a nonnegative degree "chain complex" $F_\bullet$ of free R-modules

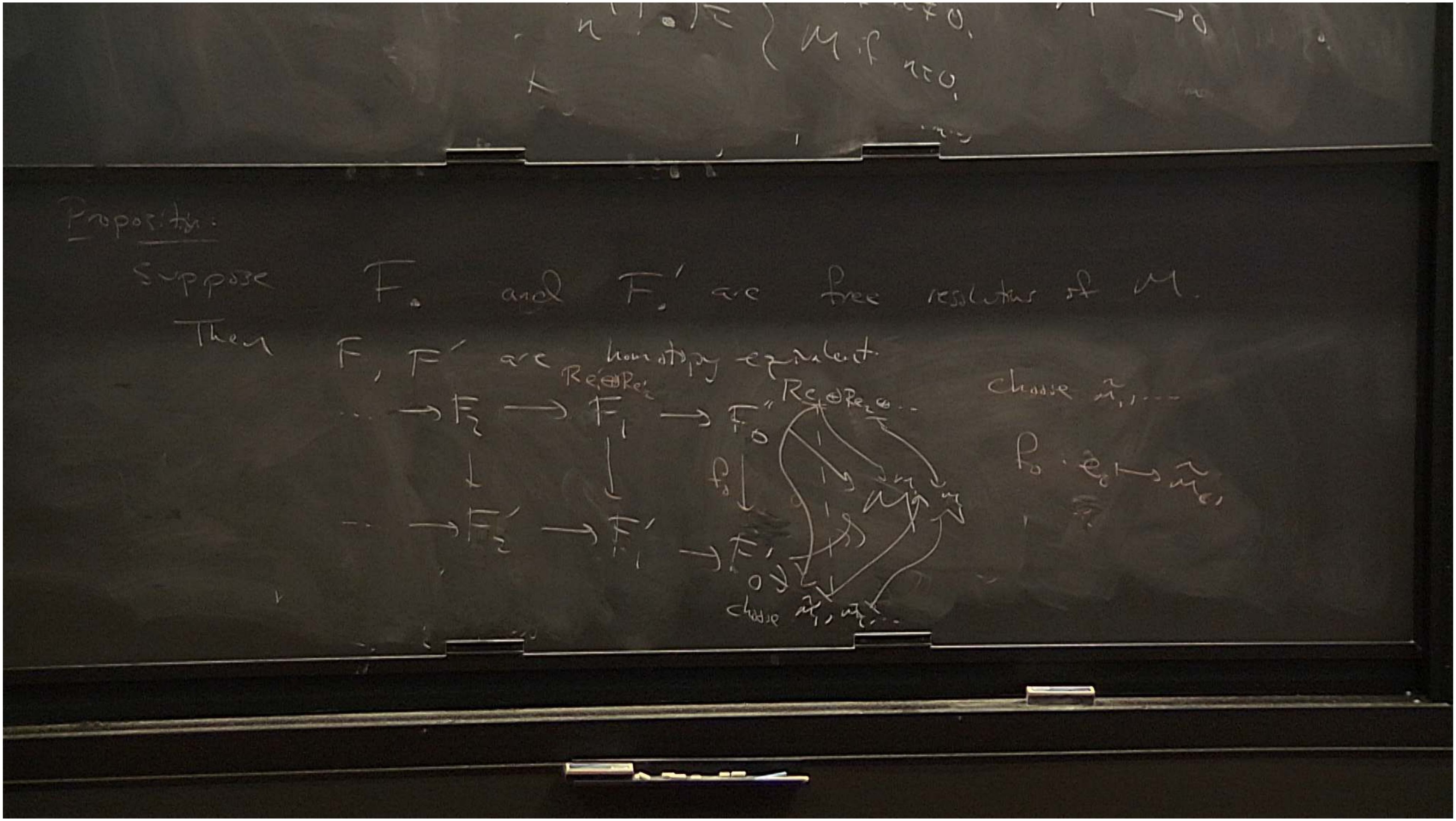
$\Uparrow$ "connective"

$$\begin{tikzcd}
& 3 & 2 & 1 & 0 & \\
& \cdots \arrow[r] & R^{k_2} \arrow[r] \arrow[d] & R^{k_1} \arrow[r] \arrow[d] & R^{k_0} \arrow[r] \arrow[d] & 0 \\
& & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

weak equiv of complexes

s.t. $$H_n(F_\bullet) = \begin{cases} 0 & \text{if } n \neq 0. \\ M & \text{if } n = 0. \end{cases}$$

A category is some collection of objects + homomorphism



Proposition:
Suppose F. and F.′ are free resolutions of M.
Then F, F′ are homotopy equivalent.
choose

except in degree $= -1$

$r e_i \longmapsto r \cdot m_i$
$r \in R$

Strategy: $M \rightsquigarrow F_M$

Choose a functor $\mathcal{F}$ $\left\{ \begin{array}{c} \text{chain complexes} \\ \text{of } R\text{-modules} \end{array} \right\} \xrightarrow{\mathcal{F}} \left\{ \begin{array}{c} \text{chain complexes} \\ \text{of} \\ \text{abelian gps} \end{array} \right\} \xrightarrow{H_n}$ abelian gps

$\{R\text{-modules}\} \xrightarrow[\text{free resolution}]{}$

because homotopy equivalence is diagrammatic - it's preserved by functors. $\therefore$ Doesn't depend on choice

If $R = \mathbb{Z}[G] \curvearrowright M$,

the coinvariants $M_G := M/_{m = gm}$ $\forall m \in M,\ g \in G$.

e.g. $\mathbb{Z}[G]_G = \mathbb{Z}$.

$\mathbb{Z}[G]^k_G = \mathbb{Z}^k$.

$H_n(G; M) :=$

$H_n\left((\cancel{\text{any}}\ \text{free res. of } M)_G \right)$

E.g. $M = \mathbb{Z} \circlearrowleft G$ trivially

$R = \mathbb{Z}[G]$

$\langle g|h\rangle \longmapsto g\langle h\rangle - \langle gh\rangle + \langle g\rangle$

basis vector is "$\langle\rangle$"

$\langle g\rangle \longmapsto g\langle\rangle - \langle\rangle$

$$\bigoplus_{g,h,j\in G} R\langle g|h|j\rangle \xrightarrow{\partial_3} \bigoplus_{g,h\in G} R\langle g|h\rangle \xrightarrow{\partial_2} \bigoplus_{g\in G} R\langle g\rangle \xrightarrow{\partial_1} R\langle\rangle \xrightarrow{\partial_0} \mathbb{Z}$$

$\langle\rangle \longmapsto 1$

$\partial_3 : \langle g|h|j\rangle \mapsto g\langle h|j\rangle - \langle gh|j\rangle + \langle g|hj\rangle - \langle g|h\rangle$

$\langle g\rangle \mapsto g\langle\rangle - \langle\rangle$

$\langle gh\rangle \mapsto gh\langle\rangle - \langle\rangle$

$g\langle h\rangle \mapsto g(h\langle\rangle - \langle\rangle) = gh\langle\rangle - g\langle\rangle$

So, the kernel: $g\langle\rangle - \langle\rangle \longmapsto g\cdot 1 = 1 \in \mathbb{Z}$, $g\cdot 1 - 1 = 0$

$\in \ker(\partial_0)$

$- \langle gh | j \rangle$
$+ \langle g | hj \rangle$
$- \langle g | h \rangle$

$\langle gh \rangle \mapsto gh\langle\rangle - \langle\rangle$

$g\langle h \rangle \mapsto g(h\langle\rangle - \langle\rangle) = gh\langle\rangle - g\langle\rangle$

s.t. the kernel

$g\langle\rangle - \langle\rangle \mapsto g\cdot 1 = 1 \in \mathbb{Z}$

$\in \ker(\partial_1)$

$g\cdot 1 - 1 = 0.$

$\downarrow$ Coinvariants

"Bar Complex": $\cdots \longrightarrow \bigoplus_{g,h\in G} \mathbb{Z}\langle g|h \rangle \longrightarrow \bigoplus_{g\in G} \mathbb{Z}\langle g \rangle \xrightarrow{\partial_1} \mathbb{Z}\langle\rangle$

$\partial_2 : \langle g|h \rangle \mapsto \langle h \rangle - \langle gh \rangle + \langle g \rangle$

$\partial_1 : \langle g \rangle \mapsto \langle\rangle - \langle\rangle$

$\partial_3 : \langle g|h|j \rangle$
$\mapsto \langle h|j \rangle - \langle gh|j \rangle + \langle g|hj \rangle - \langle g|h \rangle$

Defn. $H_n G = H_n$ of the complex.

normalized bar complex.

$$\bigoplus_{g,h\neq e} \longrightarrow \bigoplus_{g\neq e} R\langle g\rangle \longrightarrow R\langle\,\rangle \longrightarrow \mathbb{Z}$$

$$\langle e\rangle \longmapsto e\cdot\langle\,\rangle - \langle\,\rangle = 0$$

Check.

$H_1 G = G_{ab}$

↓ coinvariants

$$H_1 = \frac{\bigoplus_{g\in G} \mathbb{Z}\langle g\rangle}{\langle gh\rangle = \langle g\rangle + \langle h\rangle = \langle h\rangle + \langle g\rangle} = G_{ab}$$

e.g. $\langle g^2\rangle = 2\cdot\langle g\rangle$

$0 = \partial_1$

"bar complex": $\cdots \longrightarrow \bigoplus_{g,h\in G} \mathbb{Z}\langle g|h\rangle \longrightarrow \bigoplus_{g\in G} \mathbb{Z}\langle g\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle\rangle$

$H_0 = \mathbb{Z}$

$\partial_2 : \langle g|h\rangle \mapsto \langle h\rangle - \langle gh\rangle + \langle g\rangle$

$\partial_1 : \langle g\rangle \mapsto \langle\rangle - \langle\rangle$

$\partial_3 : \langle g|h|j\rangle \mapsto \langle h|j\rangle - \langle gh|j\rangle + \langle g|hj\rangle - \langle g|h\rangle$

<u>Defn.</u> $H_n G = H_n$ of the complex

E.g. $M = \mathbb{Z} \circlearrowleft G$ trivially

$R = \mathbb{Z}[G]$

$\ldots - \langle gh|\ \rangle + \langle g|h\ \rangle - \langle g|h\rangle$ ... complex.

$G = \mathbb{Z}/2 = \{e, \tau\}$

normalized bar complex

$$\xrightarrow{\times 2} \mathbb{Z}\langle\tau|\tau|\tau\rangle \xrightarrow{0} \mathbb{Z}\langle\tau|\tau\rangle \xrightarrow{\times 2} \mathbb{Z}\langle\tau\rangle \xrightarrow{0} \mathbb{Z}\langle\rangle$$

$$\langle\tau\rangle \longmapsto \langle\rangle - \langle\rangle$$

$$\langle\tau|\tau\rangle \mapsto \langle\tau\rangle - \langle\underbrace{\tau^2}_{=e}\rangle + \langle\tau\rangle = 2\langle\tau\rangle$$

$$H_n(\mathbb{Z}/2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2 & n=1 \\ 0 & n=2 \\ \mathbb{Z}/2 & n=3 \\ 0 & \\ \mathbb{Z}/2 & \ldots \end{cases}$$