Title: PSI 2019/2020 - Statistical Mechanics (Vieira) - Lecture 9

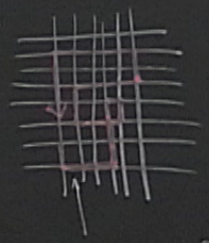
Speakers: Pedro Vieira

Collection: PSI 2019/2020 - Statistical Mechanics (Vieira)

Date: October 18, 2019 - 10:45 AM

URL: http://pirsa.org/19100033

$$= \mathcal{N} \sum_{\text{Stick configs with no open ends and 0 or 1 sticks per link}} v^{\Sigma} \cdot 1 = \mathcal{N} \exp\left[\sum \frac{v^{l}}{2} (-1)^{2w(\gamma)+1} \right]$$



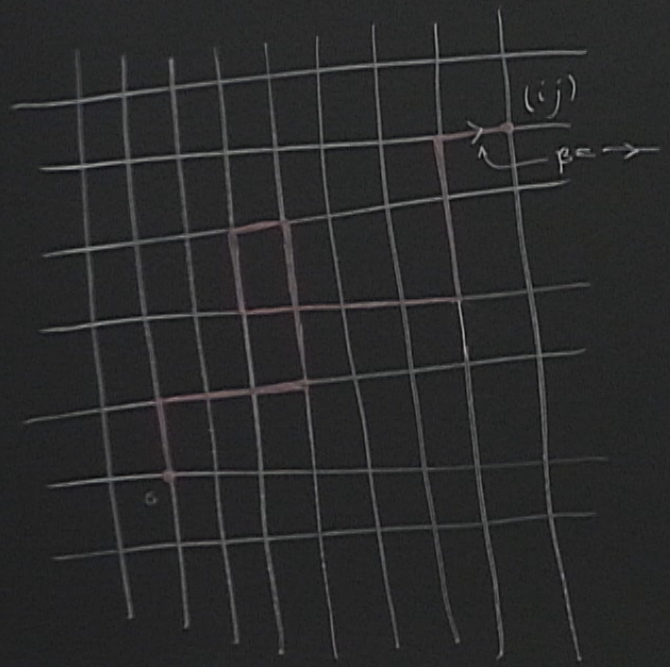
Closed path γ

$$w(\gamma) = \frac{N_L - N_R}{4}$$

$(-1)^{w} \rightarrow \pi$ steps $\{ e^{i\pi/4} \uparrow \; 1 \rightarrow \; e^{-i\pi/4} \downarrow \}$

$$Z_{\text{Ising 2D}} = \underbrace{(\cosh J)^{L^2}}_{\mathcal{N}} \sum_{\{S_i = \pm 1\}} \prod_{\langle ij \rangle} \left(1 + \underbrace{\tanh J}_{v} S_i S_j\right)$$

$$= \mathcal{N} \exp\left[-\frac{1}{2} \sum_{l=0}^{\infty} v^l N_l\right]$$



Strategy to find N_ℓ :

1) N_ℓ ⟶ open path more refined measure $N_\ell(i,j,\beta)$

2) Rec Rel $N_\ell(i,j,\beta) \leftarrow N_{\ell-1}(i\pm1, j\pm1, \beta')$

3) Fourier ⇝ diagonalize & solve Rec Rel.

4) Close the path, sum over β_0 ⇝ N_ℓ

5) Sum over ℓ

$N_l(i,j,\beta)$ = # of l steps paths (Weighted by Winding)
from origin to position (i,j) ending there
from direction β
(and at the origin $\beta_0 = \uparrow$ say $(P_{ren}\Sigma_{\beta_0})$)

1) $N_l(i,j,\rightarrow) = N_{l-1}(i-1,j,\rightarrow) + e^{-i\pi/4} N_{l-1}(i-1,j,$

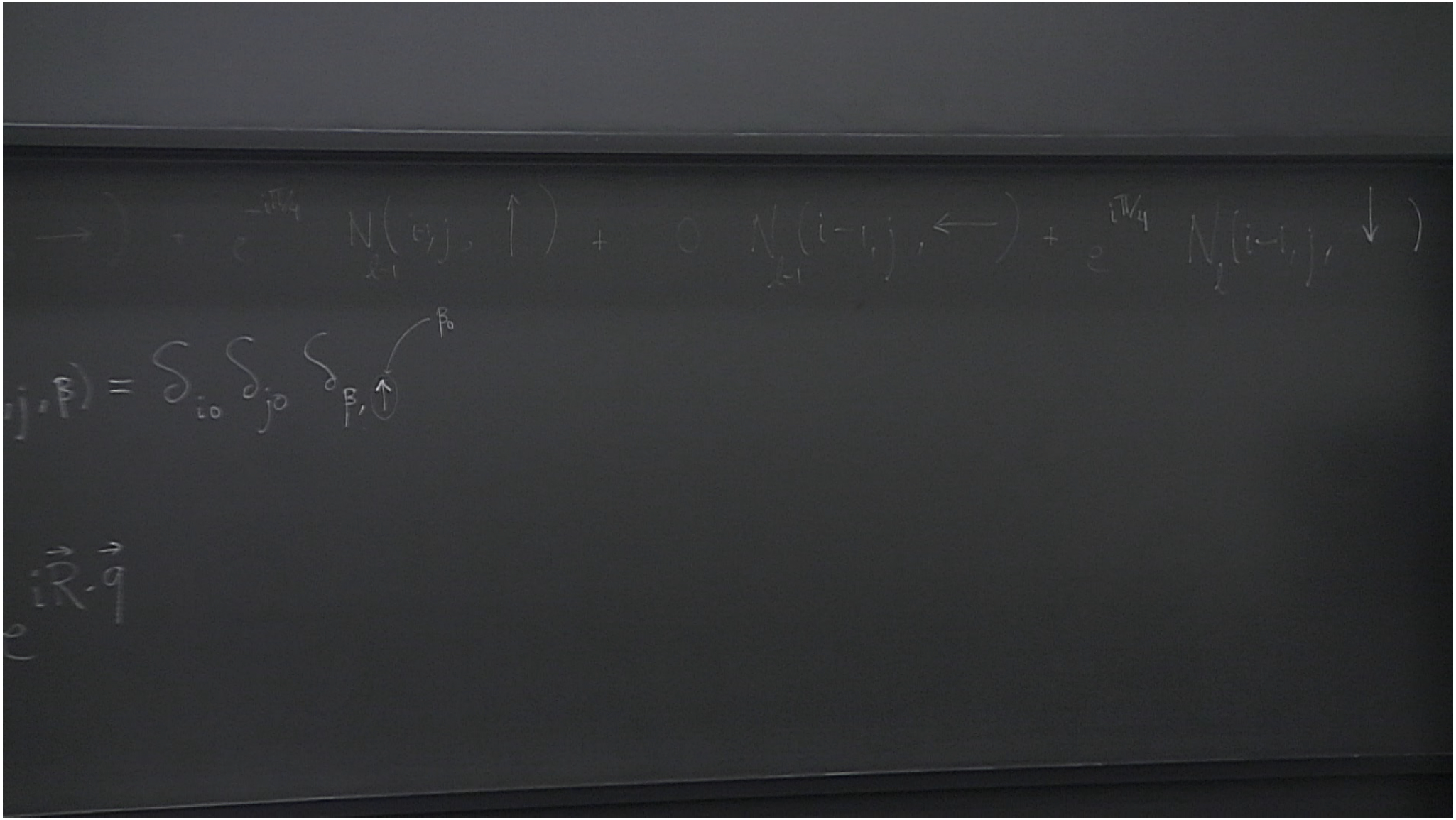
eq $(i+1)$, $\leftarrow$

eq $(j+1)$, $\downarrow$

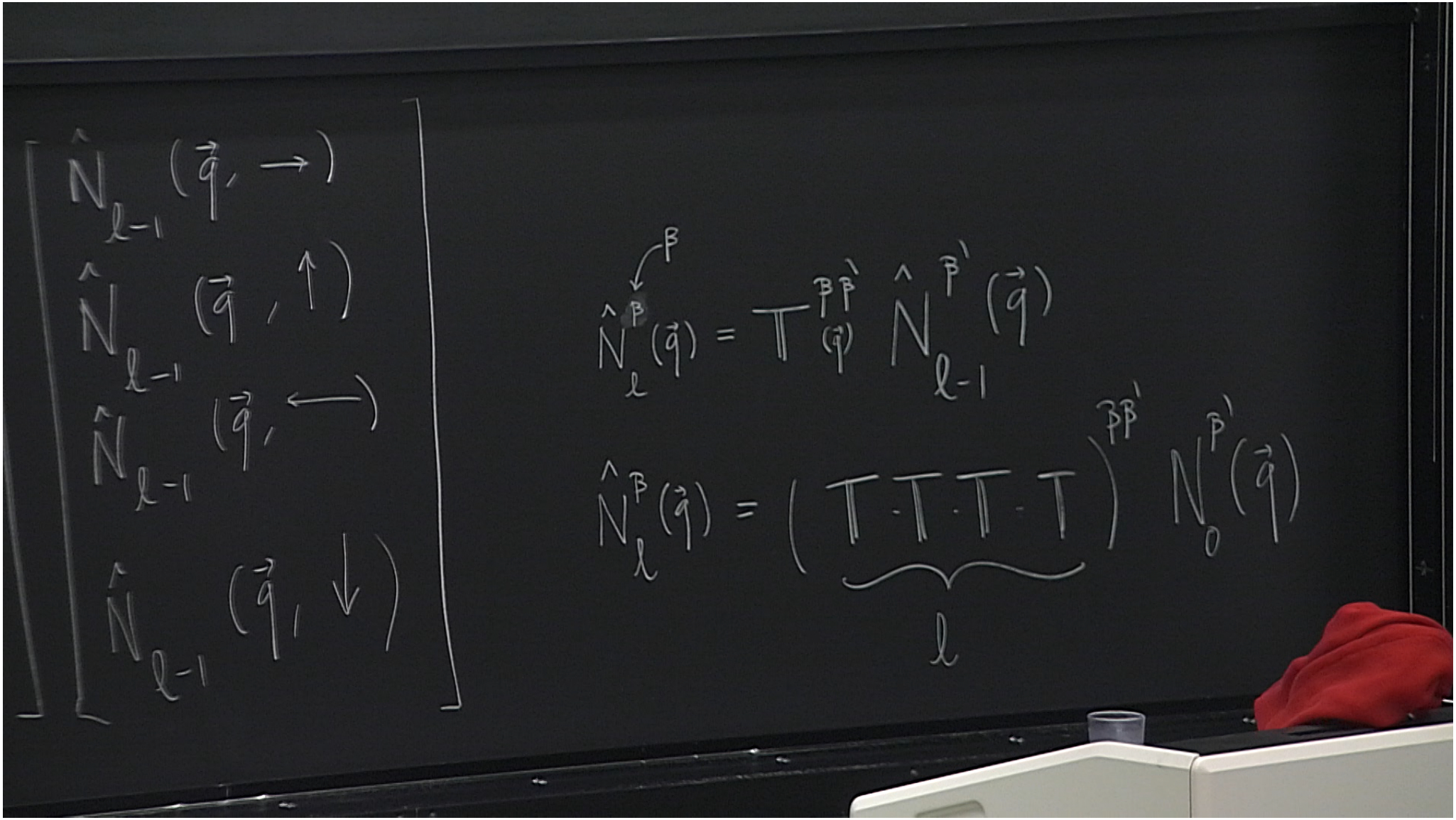
eq $(j-1)$, $\uparrow$

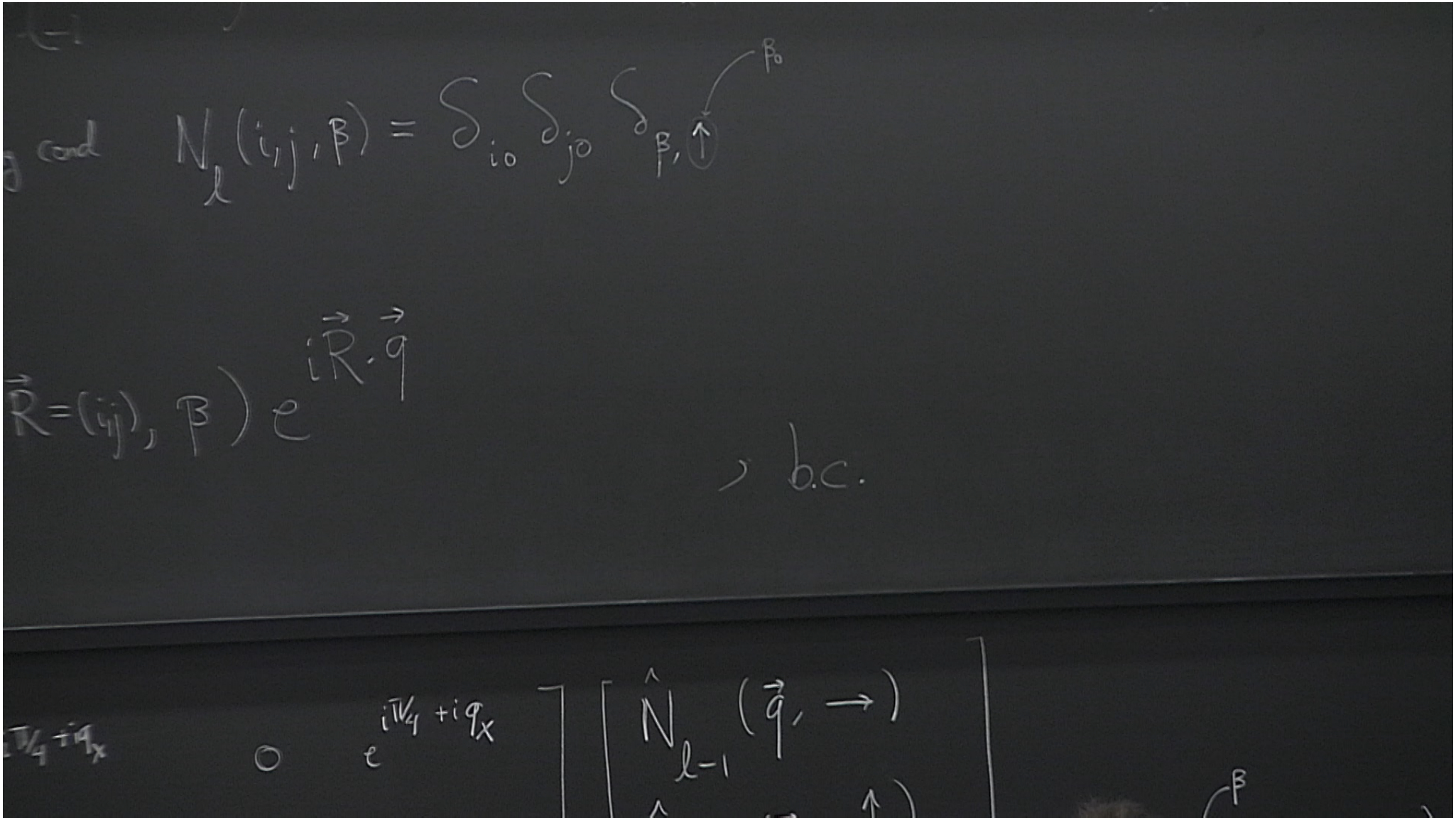
+ boundary cond $N_l(i,j,\beta) = \delta_{i0}\,\delta_{j0}\,\delta_{\beta,\uparrow}$

2) $\hat{N}_l(\vec{q},\beta) = \frac{1}{L}\sum_{\vec{R}} N_l(\vec{R}=(i,j),\beta)\, e^{i\vec{R}\cdot\vec{q}}$


$\rightarrow$) $= e^{-i\pi/4} N_{\downarrow-1}(i,j,\uparrow) + 0\, N_{\downarrow-1}(i-1,j,\leftarrow) + e^{i\pi/4} N_{\downarrow}(i-1,j,\downarrow)$
$j,\beta) = \delta_{i0}\,\delta_{j0}\,\delta_{\beta,\uparrow}$
β_0
$e^{i\vec{R}\cdot\vec{q}}$

$$\begin{bmatrix} \hat{N}_\ell(\vec{q}, \rightarrow) \\ \hat{N}_\ell(\vec{q}, \uparrow) \\ \hat{N}_\ell(\vec{q}, \leftarrow) \\ \hat{N}_\ell(\vec{q}, \downarrow) \end{bmatrix} = \begin{bmatrix} e^{iq_x} & e^{-i\pi/4 + iq_x} & 0 & e^{i\pi/4 + iq_x} \\ e^{iq_y + i\pi/4} & e^{iq_y} & e^{iq_y - i\pi/4} & 0 \\ 0 & e^{-iq_x - i\pi/4} & e^{-iq_x} & e^{+i\pi/4 - iq_x} \\ e^{-iq_y - i\pi/4} & 0 & e^{-iq_y + i\pi/4} & e^{-iq_y} \end{bmatrix}$$


$\hat{N}_{l-1}(\vec{q}, \rightarrow)$
$\hat{N}_{l-1}(\vec{q}, \uparrow)$
$\hat{N}_{l-1}(\vec{q}, \leftarrow)$
$\hat{N}_{l-1}(\vec{q}, \downarrow)$
$\hat{N}_l^{\beta}(\vec{q}) = T^{\beta\beta'}_{(\vec{q})} \hat{N}_{l-1}^{\beta'}(\vec{q})$
$\hat{N}_l^{\beta}(\vec{q}) = (T \cdot T \cdot T \cdot T)^{\beta\beta'} N_0^{\beta'}(\vec{q})$
l


cond $N_\ell(i,j,\beta) = \delta_{i0}\,\delta_{j0}\,\delta_{\beta,\uparrow}$
β_0
$\vec{R}=(ij),\beta)\,e^{i\vec{R}\cdot\vec{q}}$
, b.c.
$e^{i\nabla_4 + iq_x}$
$\hat{N}_{\ell-1}(\vec{q},\rightarrow)$
β

$$N_{\ell-1}(i,j,\rightarrow) = N_{\ell-1}(i-1,j,\rightarrow) + e^{-i\pi/4} N_{\ell-1}(i,j-1,\uparrow)$$

$(i+1)$, $\leftarrow$

$(j+1)$, $\downarrow$

$(j-1)$, $\uparrow$

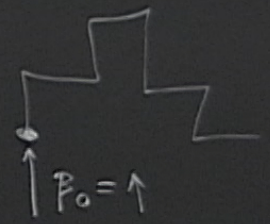
+ boundary cond $\quad N_{\ell=0}(i,j,\beta) = \delta_{i,0}\,\delta_{j,0}\,\delta_{\beta,\uparrow}$ (β_0)

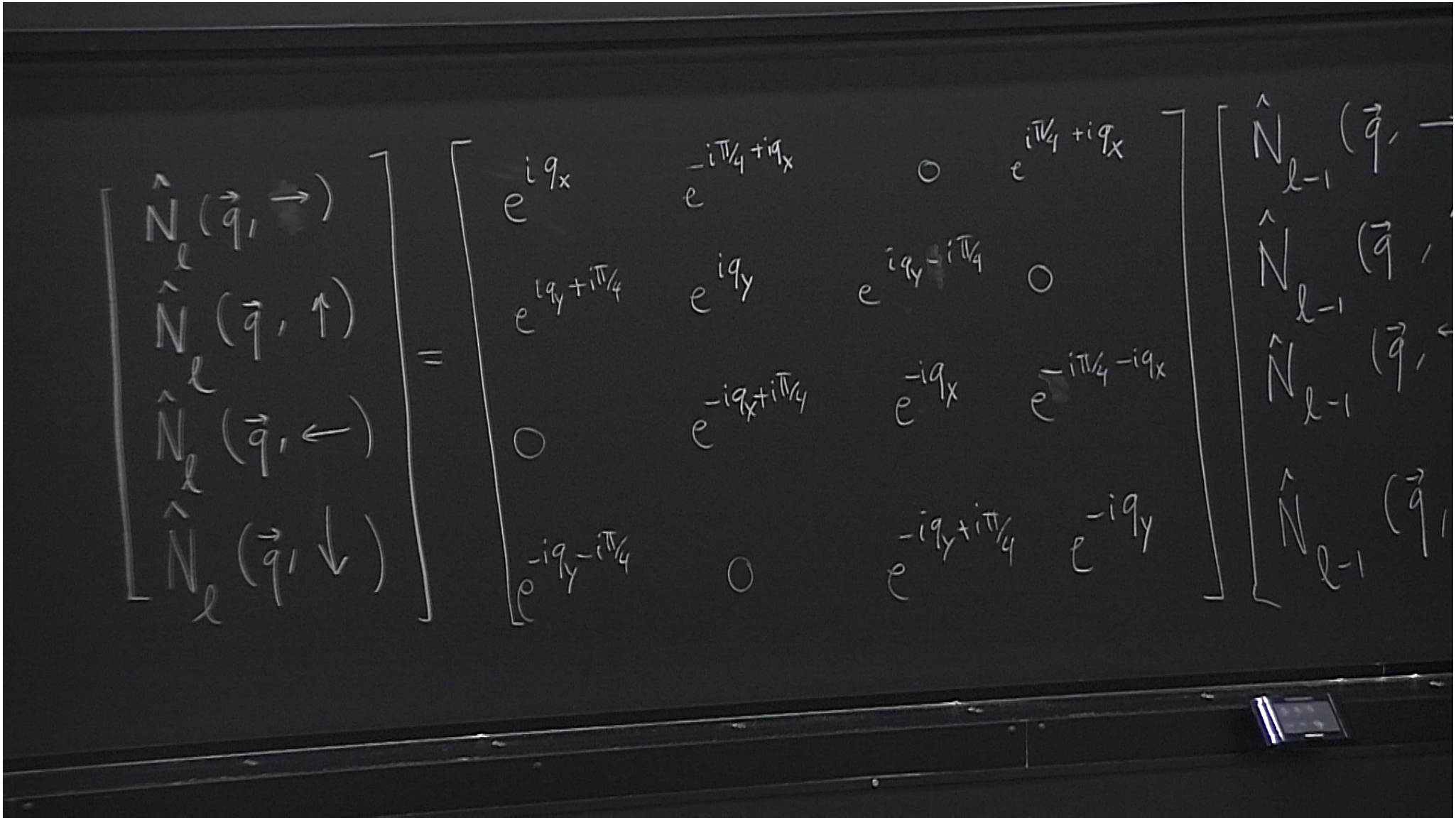
$$N_\ell(\vec{q},\beta) = \frac{1}{L}\sum_{\vec{R}} N_\ell(\vec{R}=(ij),\beta)\, e^{i\vec{R}\cdot\vec{q}}$$

ℓ ; b.c.

$$\hat{N}_{\ell}^{\beta}(\vec{q}) = \left(T^{\ell}\right)^{\beta\beta'} \hat{N}_{\ell=0}^{\beta'}(\vec{q})$$

$$\frac{1}{L}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

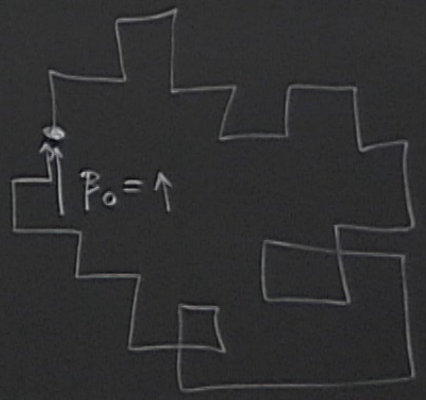



$\begin{bmatrix} \hat{N}_\ell(\vec{q}, \rightarrow) \\ \hat{N}_\ell(\vec{q}, \uparrow) \\ \hat{N}_\ell(\vec{q}, \leftarrow) \\ \hat{N}_\ell(\vec{q}, \downarrow) \end{bmatrix} = \begin{bmatrix} e^{iq_x} & e^{-i\pi/4 + iq_x} & 0 & e^{i\pi/4 + iq_x} \\ e^{iq_y + i\pi/4} & e^{iq_y} & e^{iq_y - i\pi/4} & 0 \\ 0 & e^{-iq_x + i\pi/4} & e^{-iq_x} & e^{-i\pi/4 - iq_x} \\ e^{-iq_y - i\pi/4} & 0 & e^{-iq_y + i\pi/4} & e^{-iq_y} \end{bmatrix} \begin{bmatrix} \hat{N}_{\ell-1}(\vec{q}, \\ \hat{N}_{\ell-1}(\vec{q}, \\ \hat{N}_{\ell-1}(\vec{q}, \\ \hat{N}_{\ell-1}(\vec{q}, \end{bmatrix}$

$$\hat{N}_{\ell}^{\beta}(\vec{q}) = \left(T^{\ell}\right)^{\beta\beta'} \hat{N}_{\ell=0}^{\beta'}(\vec{q})$$

$$\sum_{\beta_0} N_{\ell}(0,0,\beta_0) = N_{\ell}$$

$$\frac{1}{L}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

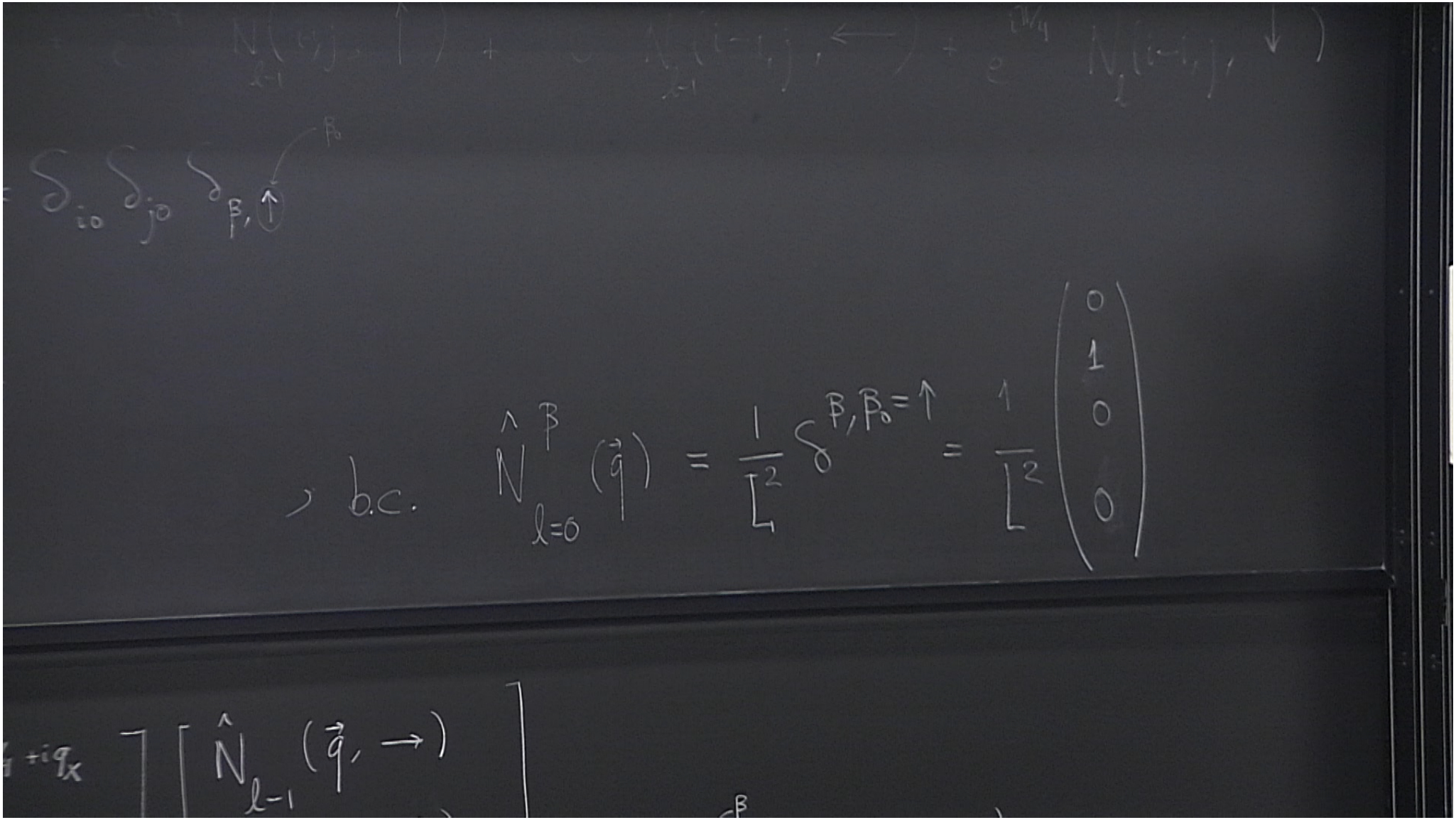


$$N_\ell(i,j,\beta) = \sum_{\vec{q}} e^{-i\vec{q}\cdot\vec{R}} \hat{N}_\ell(\vec{q},\beta)$$

$\vec{R}$

$$\hat{N}_\ell(\vec{q},\beta) = \frac{1}{L^2}\sum_{\vec{R}'} e^{i\vec{R}'\cdot\vec{q}} N_\ell(i,j,\beta)$$

$$\frac{1}{L^2}\sum_{\vec{q}} e^{i\vec{q}(\vec{R}-\vec{R}')} = \delta_{R,\vec{R}'}$$


$\delta_{i_0} \delta_{j_0} \delta_{\beta,\uparrow}$
β_0
$\hat{N}^{\beta}_{l=0}(\vec{q}) = \frac{1}{L^2} \delta^{\beta,\beta_0=\uparrow} = \frac{1}{L^2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
b.c.
$+iq_x$
$\hat{N}_{l-1}(\vec{q}, \rightarrow)$

$$N_\ell(\underbrace{i,j}_{\vec{R}},\beta) = \sum_{\vec{q}} e^{-i\vec{q}\cdot\vec{R}}\, \hat{N}_\ell(\vec{q},\beta)$$

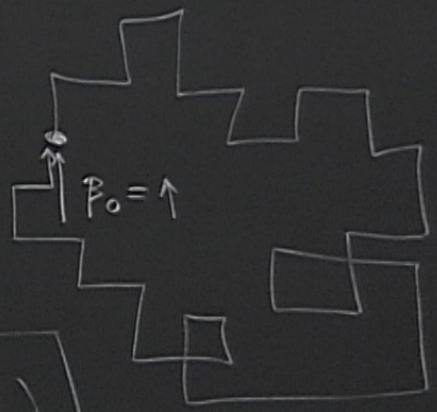
$$\hat{N}_\ell(\vec{q},\beta) = \frac{1}{L^2}\sum_{\vec{R}'} e^{i\vec{R}'\cdot\vec{q}}\, N_\ell(i,j,\beta)$$

$$\frac{1}{L^2}\sum_{\vec{q}} e^{i\vec{q}(\vec{R}-\vec{R}')} = \delta_{R,\vec{R}'}$$

$$\hat{N}_l^{\beta}(\vec{q}) = \left(T^{\ell}\right)^{\beta\beta'} \hat{N}_{\ell=0}^{\beta'}(\vec{q})$$

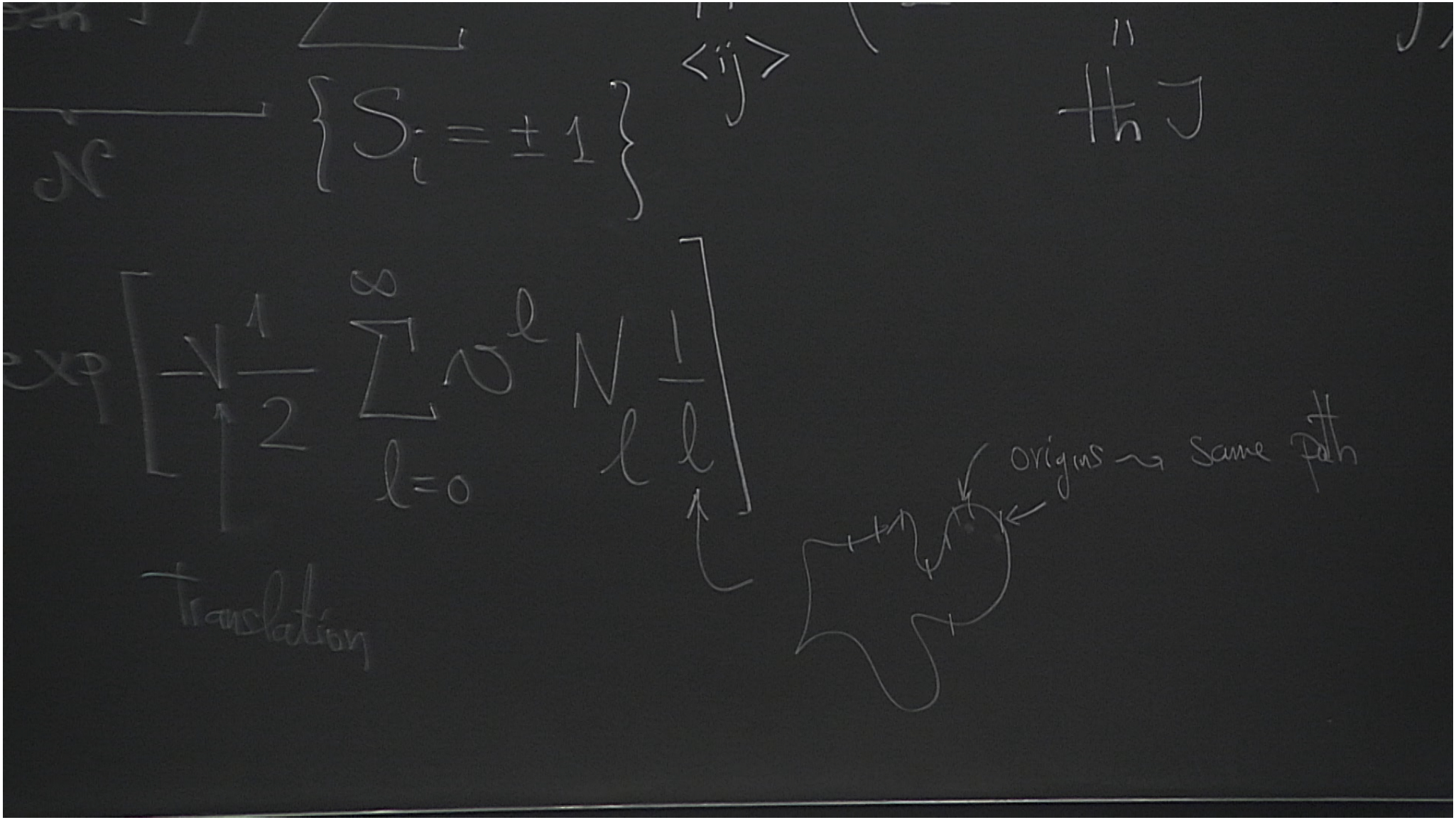
$$\sum_{\beta_0} N_l(0,0,\beta_0) = N_l$$

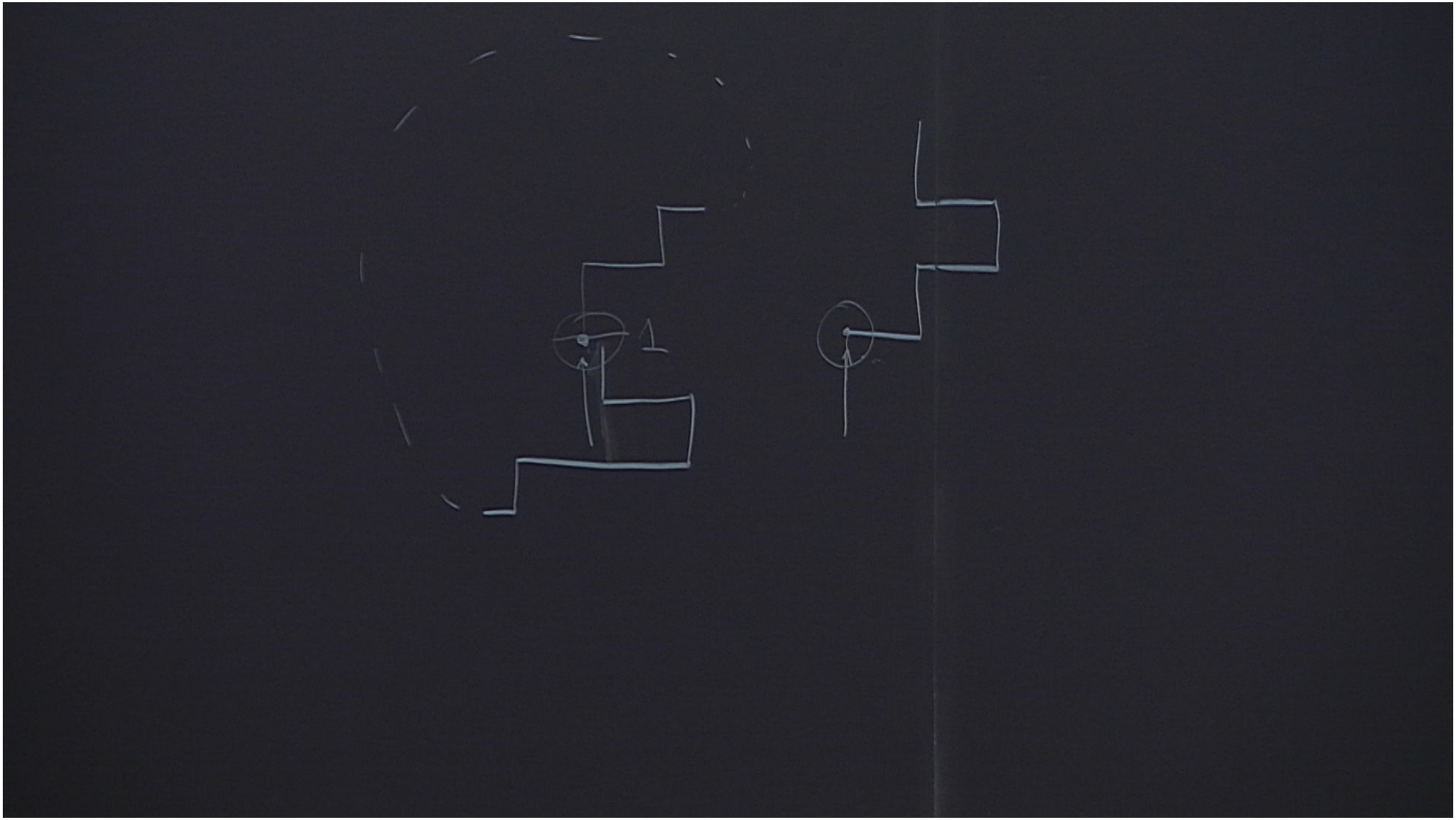
$$\frac{1}{L^2}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$



$$\boxed{N_l = \sum_{\beta_0}\sum_{\vec{q}} \hat{N}_l(\vec{q},\beta_0) = \frac{1}{L^2}\sum_{\vec{q}} \mathrm{Tr}\left(T(\vec{q})^{\ell}\right)}$$

$$\begin{bmatrix} \hat{N}_\ell(\vec{q}, \rightarrow) \\ \hat{N}_\ell(\vec{q}, \uparrow) \\ \hat{N}_\ell(\vec{q}, \leftarrow) \\ \hat{N}_\ell(\vec{q}, \downarrow) \end{bmatrix} = \overbrace{\begin{bmatrix} e^{iq_x} & e^{-i\pi/4 + iq_x} & 0 & e^{i\pi/4 + iq_x} \\ e^{iq_y + i\pi/4} & e^{iq_y} & e^{iq_y - i\pi/4} & 0 \\ 0 & e^{-iq_x + i\pi/4} & e^{-iq_x} & e^{-i\pi/4 - iq_x} \\ e^{-iq_y - i\pi/4} & 0 & e^{-iq_y + i\pi/4} & e^{-iq_y} \end{bmatrix}}^{T(\vec{q})}$$


$\{S_i = \pm 1\}$
$\langle ij \rangle$
$\sum_{l=0}^{\infty} \omega^l N \frac{1}{l\,l}$
origins ~ same path
Translation


1

$$\det J)^V \exp\left(-\frac{1}{2}\sum_{\vec{q}}\sum_{l}\underbrace{\mathrm{Tr}\left(T(\vec{q})^l\right)}_{\sum_{j=1..4}\lambda_j(\vec{q})^l}\frac{1}{l}\right)$$

$$Z_{\text{Ising 2D}} = (2\cosh J)^V \exp\left(-\frac{1}{2}\sum_{\vec{q}}\sum_{l} \text{Tr}(\ldots\right.$$

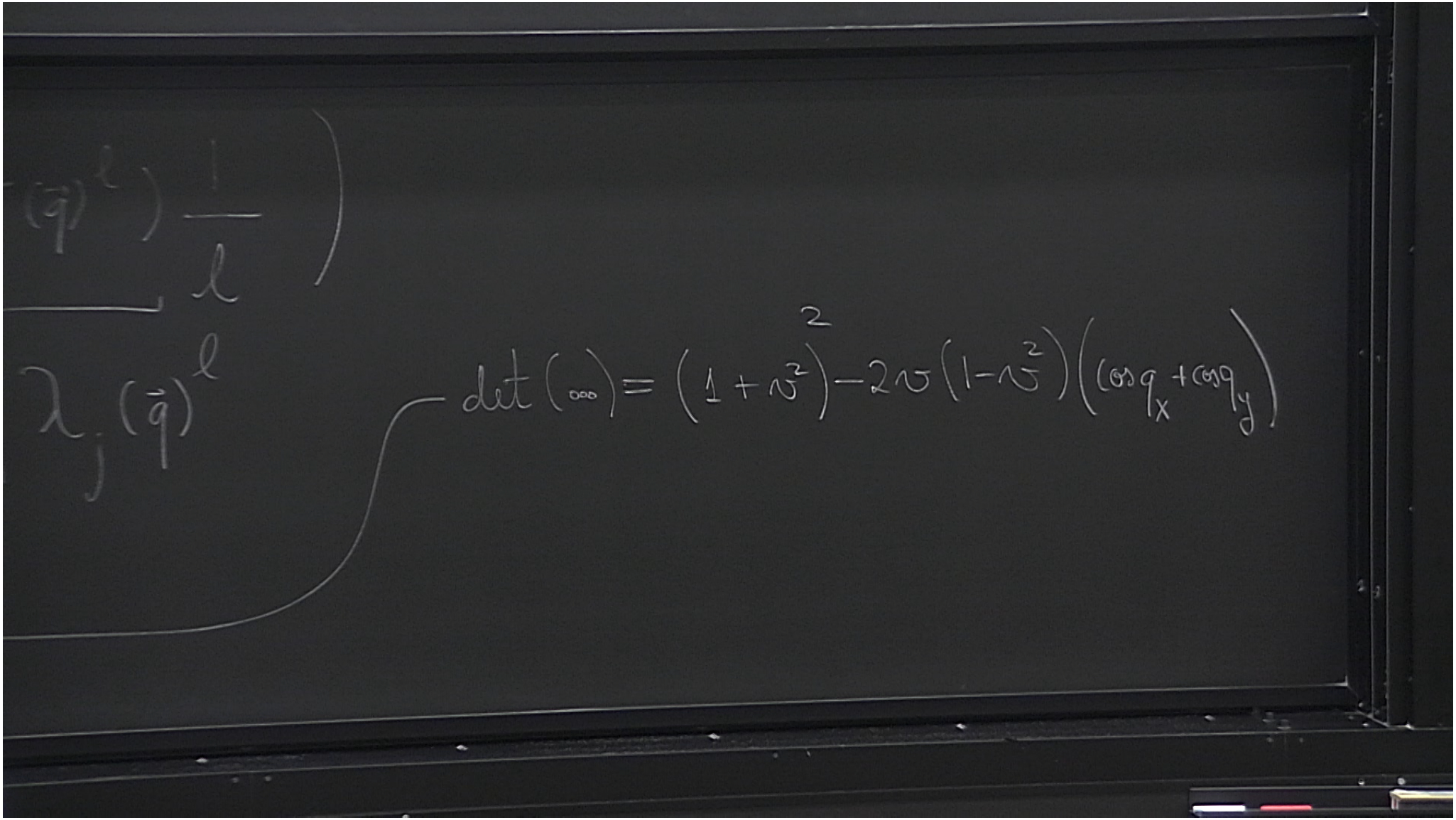
$$= \mathcal{N} \exp\left(-\frac{1}{2}\sum_{\vec{q}} \log\left(\underbrace{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)(1-\lambda_4)}_{\det(\mathbb{1}-\mathbb{T}(\vec{q}))}\right)\right)^{\sum_{j=1}}$$

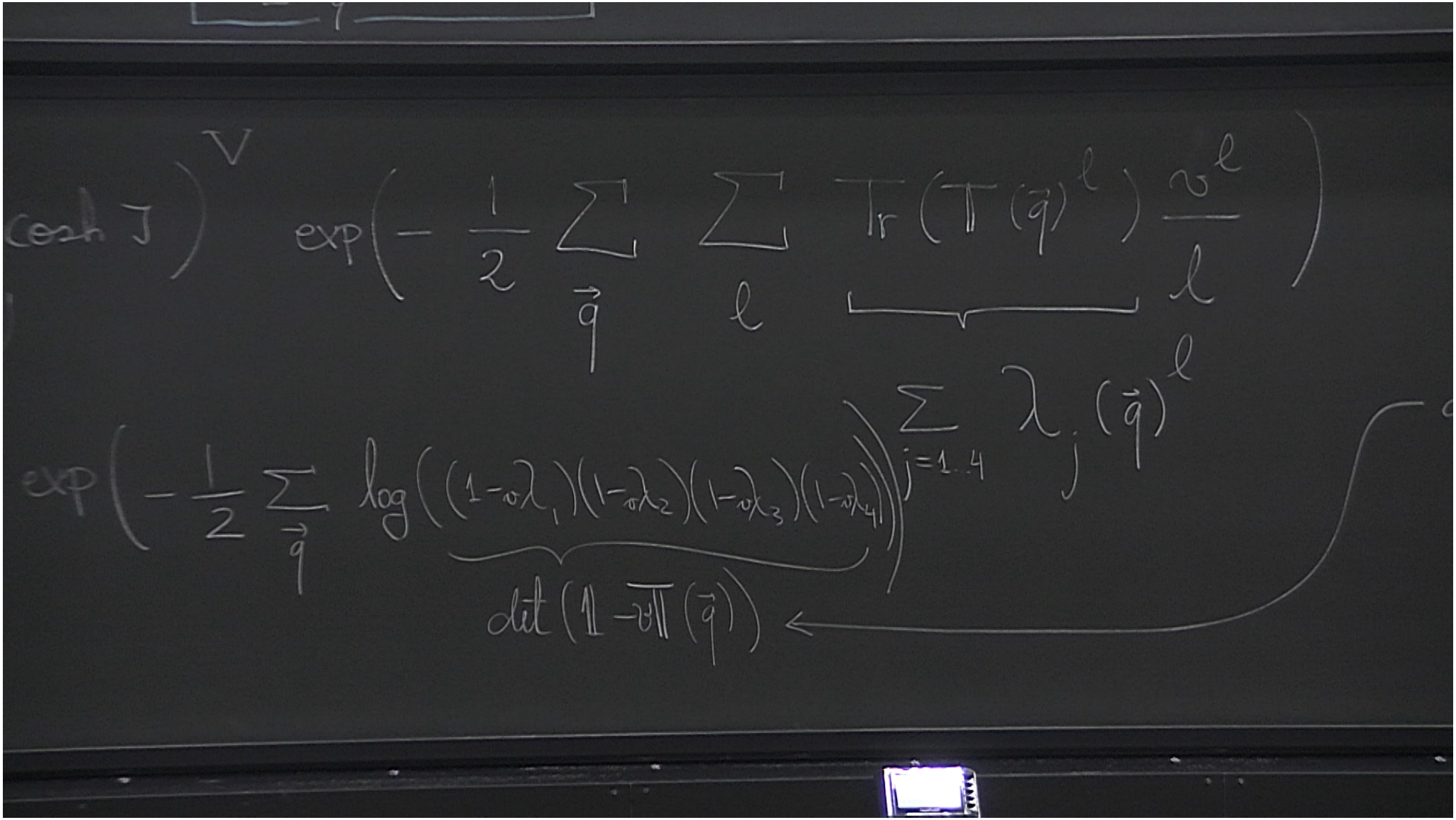
$$Z_{\text{Ising 2D}} = (2\cosh J)^V \exp\left(-\frac{1}{2}\sum_{\vec{q}}\sum_{l} \text{Tr}(\ldots\right.$$

$$= \mathcal{N} \exp\left(-\frac{1}{2}\sum_{\vec{q}} \log\left(\underbrace{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)(1-\lambda_4)}_{\det(\mathbb{1}-\Pi(\vec{q}))}\right)\right)^{\sum_{j=1}}$$

$$(2\cosh J)^V \exp\left(-\frac{1}{2}\sum_{\vec{q}}\sum_{l}\underbrace{\mathrm{Tr}\left(T(\vec{q})^l\right)}\frac{1}{l}\right)$$

$$\exp\left(-\frac{1}{2}\sum_{\vec{q}}\log\underbrace{\left((1-\lambda_1)(1-\lambda_2)(1-\lambda_3)(1-\lambda_4)\right)}_{\det\left(\mathbb{1}-\mathbb{T}(\vec{q})\right)}\right)^{\sum_{j=1..4}\lambda_j(\vec{q})^l}$$

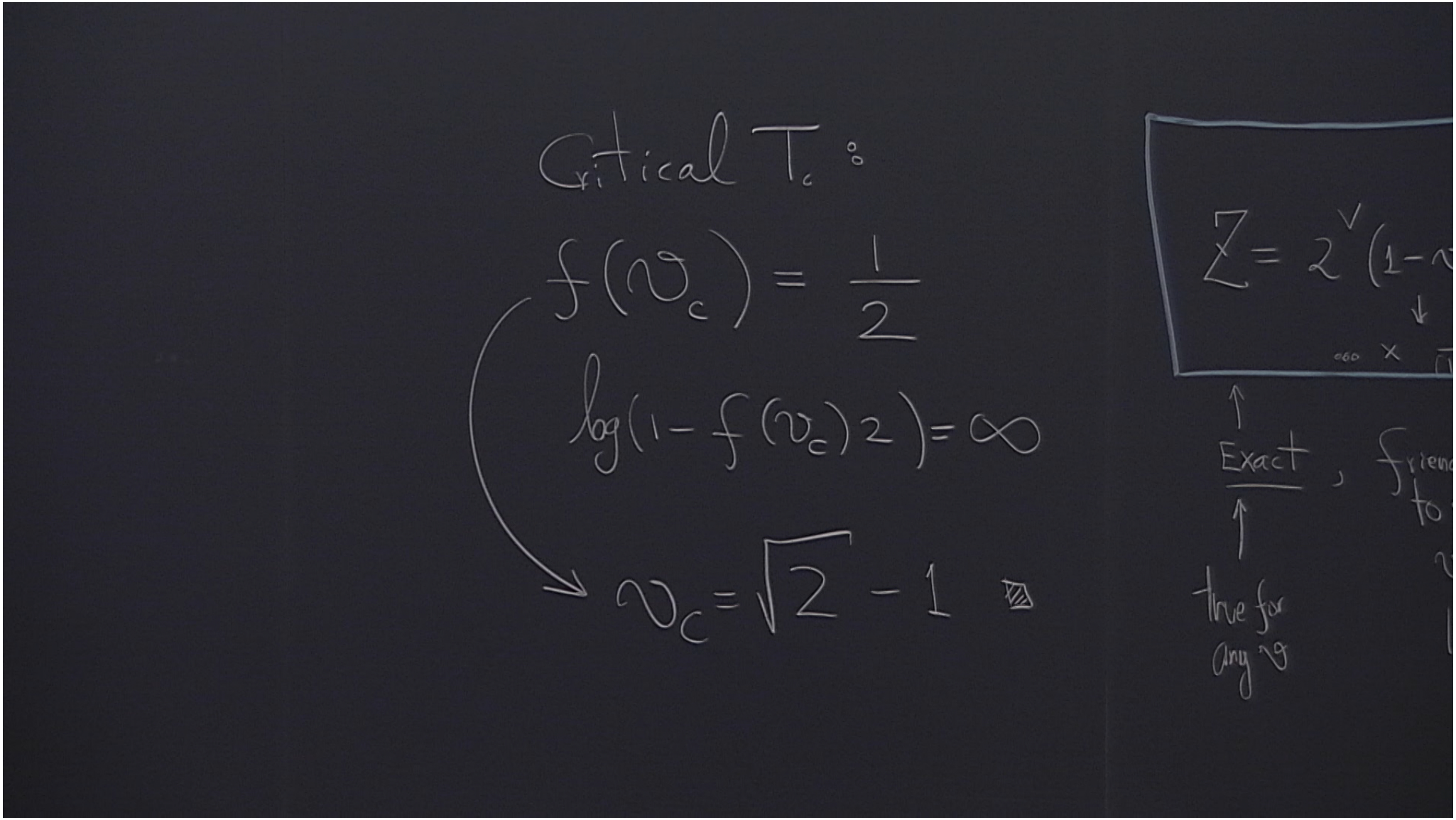

$\det(\cdots) = (1+v^2)^2 - 2v(1-v^2)(\cos q_x + \cos q_y)$


$(\cosh J)^V \exp\left(-\frac{1}{2}\sum_{\vec{q}}\sum_{l}\underbrace{\mathrm{Tr}(T(\vec{q})^l)}\frac{v^l}{l}\right)$
$\exp\left(-\frac{1}{2}\sum_{\vec{q}}\log\underbrace{\left((1-v\lambda_1)(1-v\lambda_2)(1-v\lambda_3)(1-v\lambda_4)\right)}_{\det(\mathbb{1}-v\Pi(\vec{q}))}\right)^{\sum_{j=1..4}\lambda_j(\vec{q})^l}$

$$Z = 2^{V}\left(1-v^{2}\right)^{-V/2} \exp\left(-\frac{1}{2}\sum_{\vec{q}\in BZ} \log\left(\left(1+v^{2}\right)^{2} - 2v\left(1-v^{2}\right)\left(\cos q_x + \cos q_y\right)\right)\right)$$

$$Z = 2^V (1-v^2)^{-V/2} \exp\left(-\frac{1}{2} \sum_{\vec{q} \in BZ} \log\left((1+v^2)^2 - 2v(1-v^2)(\cos q_x + \cos q_y) \right) \right)$$

$$\simeq V \int \frac{d^2 \vec{q}}{(2\pi)^2} \quad \text{for large } V$$


Critical T_c:
$f(v_c) = \frac{1}{2}$
$\log(1 - f(v_c)2) = \infty$
$v_c = \sqrt{2} - 1$
$Z = 2^V (1 - v$
Exact
true for any v

Zoom in close to $T \simeq T_c$, $v = \tanh(\beta J^{spins} \equiv J) \simeq$

$$Z|_{Sing} = \sqrt{} \times \exp\left(-\frac{V}{2}\int \frac{d^2\vec{q}}{(2\pi)^2} \log\left(\vec{q}^{\,2} + \#\right)\right.$$

↑ regular

$$+ \underbrace{\#}_{\simeq 0}(v - v_c) +$$

$$f(v) = \frac{1}{2} = + \frac{2v(1-v^2)}{(1+v^2)^2} \quad , (1+v^2)^2 = 4v(1-v^2)$$

$$Z = 2^V (1-v^2)^{-V/2} \exp\left(-\frac{1}{2} \sum_{\vec{q} \in BZ} \log\left((1+v^2)^2 - 2v(1-v^2)(\cos q_x + \cos q_y) \right) \right)$$

$(1-v^2)^{-V/2} \times \frac{1}{(1+v^2)^V}$; $\sum_{\vec{q} \in BZ} 1$; $- \frac{2v(1-v^2)}{(1+v^2)^2} \equiv f(v)$

Exact , friendly to small v expansion $\log(1 + v(\dots) + v^2(\dots))$

true for any v

$$\simeq V \int \frac{d^2\vec{q}}{(2\pi)^2} \quad \text{for large } V$$

$$Z = 2^V (1-v^2)^{-V/2} \exp\left(-\frac{1}{2} \sum_{\vec{q}\in BZ} \log\left((1+v^2)^2 - 2v(1-v^2)(\cos q_x + \cos q_y) \right) \right)$$

$$\to 1 \times \frac{1}{(1+v^2)^V} \qquad \to 1 \qquad - \frac{2v(1-v^3)}{(1+v^2)^2} \equiv f(v)$$

Exact, friendly to small v expansion

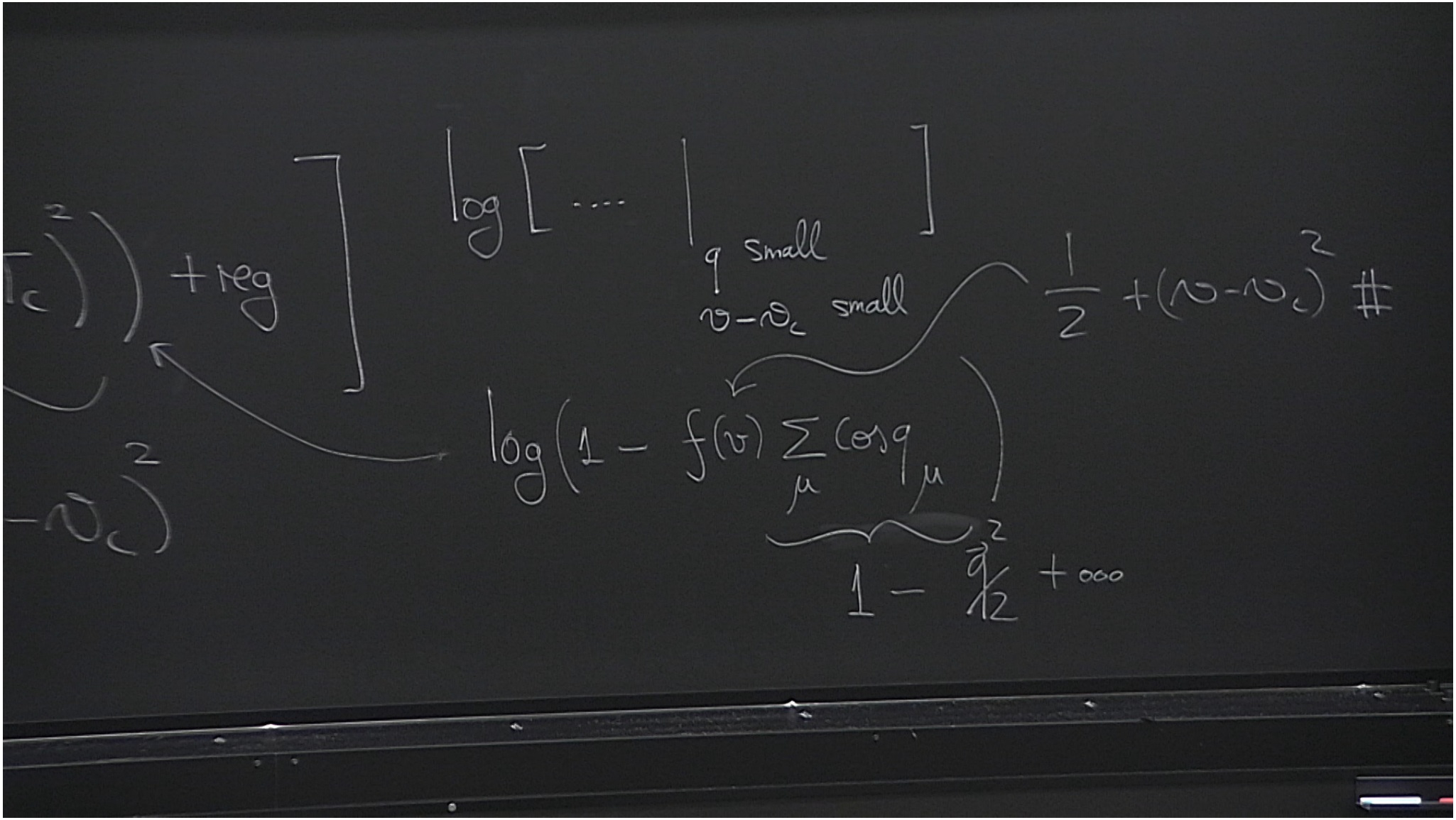
$$\log(1 + v(\dots) + v^2(\dots))$$

true for any v

$$\simeq V \int \frac{d^2\vec{q}}{(2\pi)^2} \quad \text{for large } V$$

$$f(v) \simeq \frac{1}{2} + \#\,(v - v_c) + \#\,(v-v_c)^2 + \dots$$

$v_c = 0$


$\log\left[\ldots\right]$
q small
$v - v_c$ small
$\frac{1}{2} + (v - v_c)^2 + \#$
$\log\left(1 - f(v)\sum_\mu (\sigma)_\mu q_\mu\right)$
$1 - \frac{q^2}{2} + \ldots$
$+ \text{reg}$

$$v = \tanh\left(\beta J^{\text{spins}} \equiv J\right) \simeq v_c$$

$$-\frac{V}{2}\int \frac{d^2\vec{q}}{(2\pi)^2} \log\left(\vec{q}^{\,2} + \#\left(T - T_c\right)^2\right) + \text{reg} \Big]$$

$$+ \underbrace{\approx \#}_{\parallel\ 0}\left(v - v_c\right) + \tilde{\#}\left(v - v_c\right)^2$$

Zoom in close to $T \simeq T_c$, $v = \tanh(\beta J^{sp} \equiv J) \simeq$

$$Z\Big|_{Sing} = \sqrt{\quad} \times \exp\left(-\frac{V}{2}\int \frac{d^2\vec{q}}{(2\pi)^2} \log\left(\vec{q}^{\,2} + \right.\right.$$

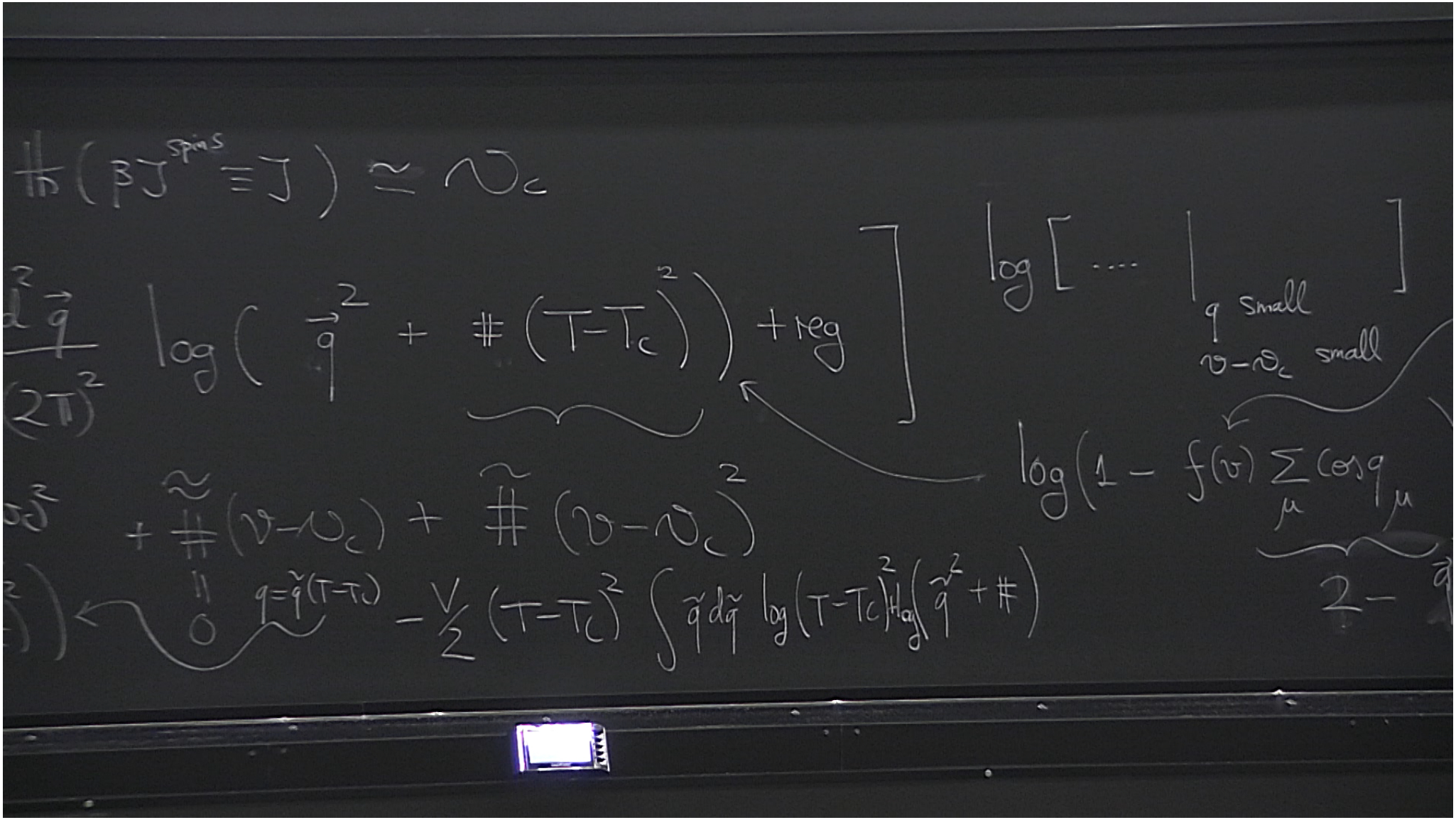
$\sqrt{\quad}$ regular

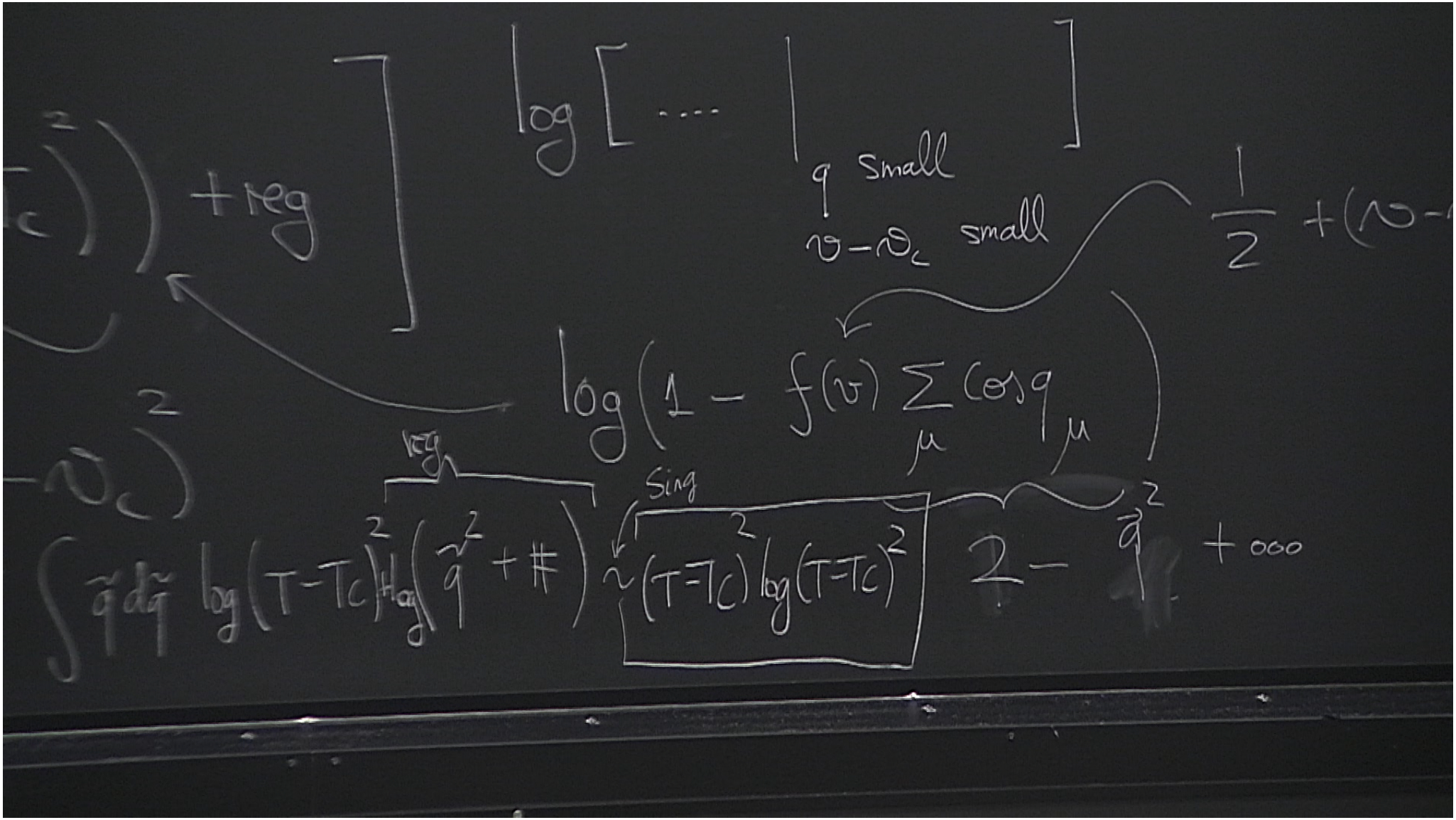
$$\log Z = \text{reg} - \frac{V}{2}\int \frac{2\pi q\, dq}{(2\pi)^2} \log\left(q^2 + \#(T-T_c)\right)$$

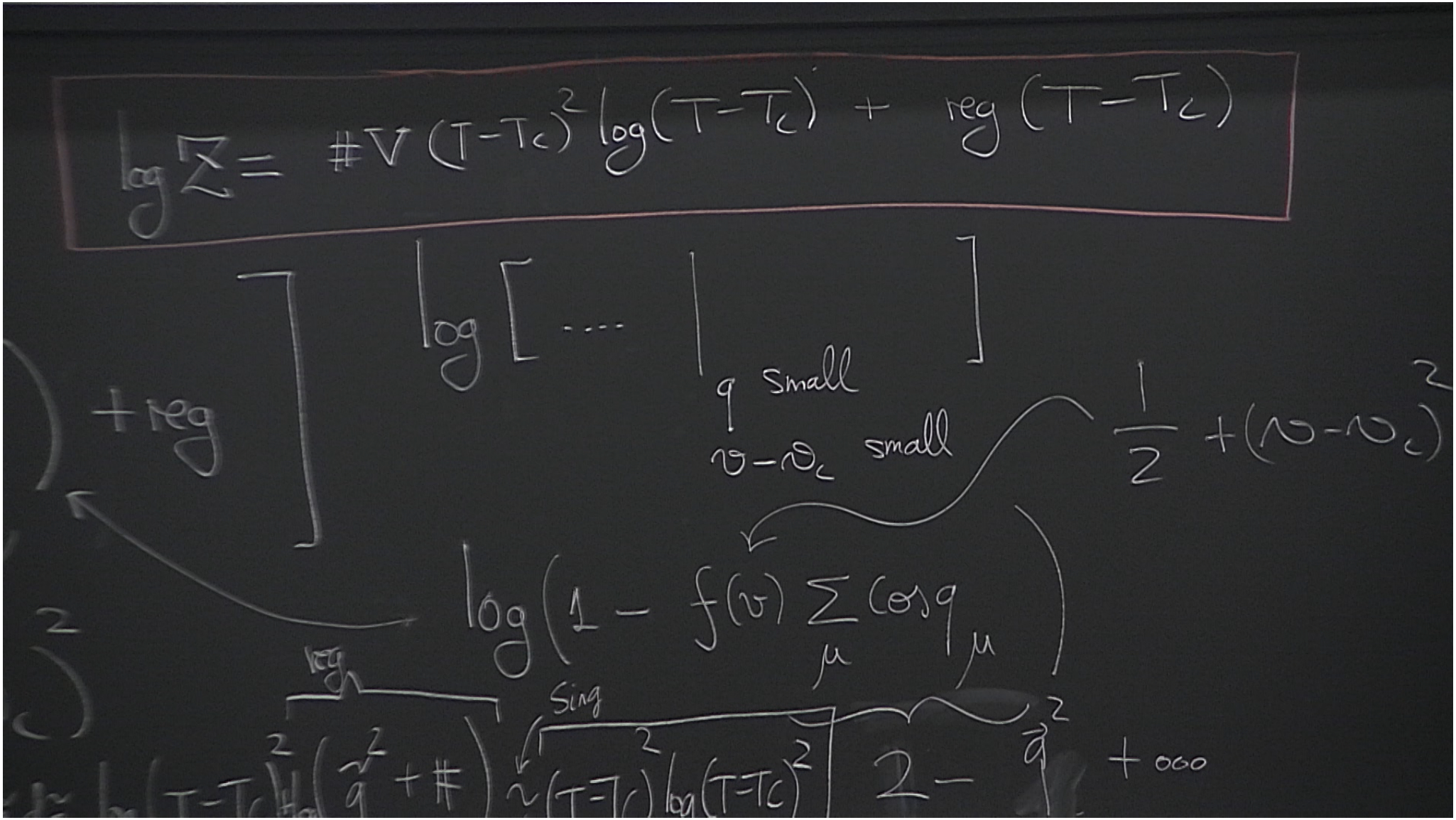
$\#(v-v_c)^2 \sim \#(T-T_c)^2$

$$+ \#(v-v_c) + \approx$$

$\#(v-v_c) = 0$


$\frac{1}{n}(\beta J^{spins} \equiv J) \simeq \nu_c$
$\int \frac{d^3\vec{q}}{(2\pi)^2} \log\left(\vec{q}^2 + \#(T-T_c)\right) + \text{reg}$
$\log[\ldots]$
q small
$\nu - \nu_c$ small
$\log(1 - f(\nu)\sum_\mu \cos q_\mu$
$+ \#(\nu-\nu_c) + \#(\nu-\nu_c)^2$
$q=\tilde{q}(T-T_c)$
$-\frac{V}{2}(T-T_c)^3 \int \tilde{q}\,d\tilde{q} \log(T-T_c)^2 H_{\alpha}\left(\frac{\tilde{q}^2}{\ldots} + \#\right)$
$2-$


+ reg
log [....]
q small
v − v_c small
1/2 + (v −
log (1 − f(v) Σ_μ (σ)q_μ
reg
Sing
∫ q dq log (T − T_c) H_α (q² + #) ~ (T − T_c)² log (T − T_c)²
2 − q² + ...


$\log Z = \#V (T-T_c)^2 \log(T-T_c) + \text{reg}\,(T-T_c)$
$\log\left[\cdots \right]$
q small
$v - v_c$ small
$+\text{reg}$
$\log\left(1 - f(v) \sum_\mu \cos q_\mu\right)$
$\frac{1}{2} + (v-v_c)^2$
reg
Sing
$+ \cdots$

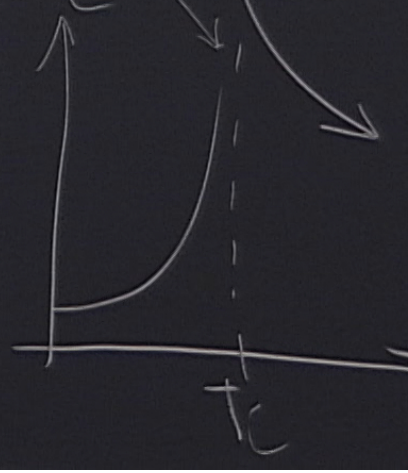
$$\langle E \rangle \sim \frac{\partial}{\partial \beta} \log Z$$

$$\sim \# \frac{\partial}{\partial T} \log Z$$

$$\frac{E}{V} \simeq \#\,(T - T_c) \log (T - T_c)$$

$$C_v \equiv \frac{\partial}{\partial T}\left(\frac{E}{V}\right) \simeq \# \log (T - T_c) + \text{reg}$$

phase transition



Critical T_c :

$$f(v_c) = \frac{1}{2}$$

$$\log(1 - f(v_c)2$$

$$v_c = \sqrt{2}$$

$\langle E \rangle \sim \frac{\partial}{\partial \beta} \log Z$

$\sim \# \frac{\partial}{\partial T} \log Z$

$\simeq \#(T - T_c) \log(T - T_c)$

$\left(\frac{E}{N}\right) \simeq \# \log(T - T_c) + \text{reg}$

$v \simeq 0$ ↑↑↑↑ ferro

$v \simeq 1$ ↓↑↑↓ paramagnetic

Critical T_c :

$f(v_c) = \frac{1}{2}$

$\log(1 - f(v_c)2) = \infty$

$v_c = \sqrt{2} - 1$

phase transition

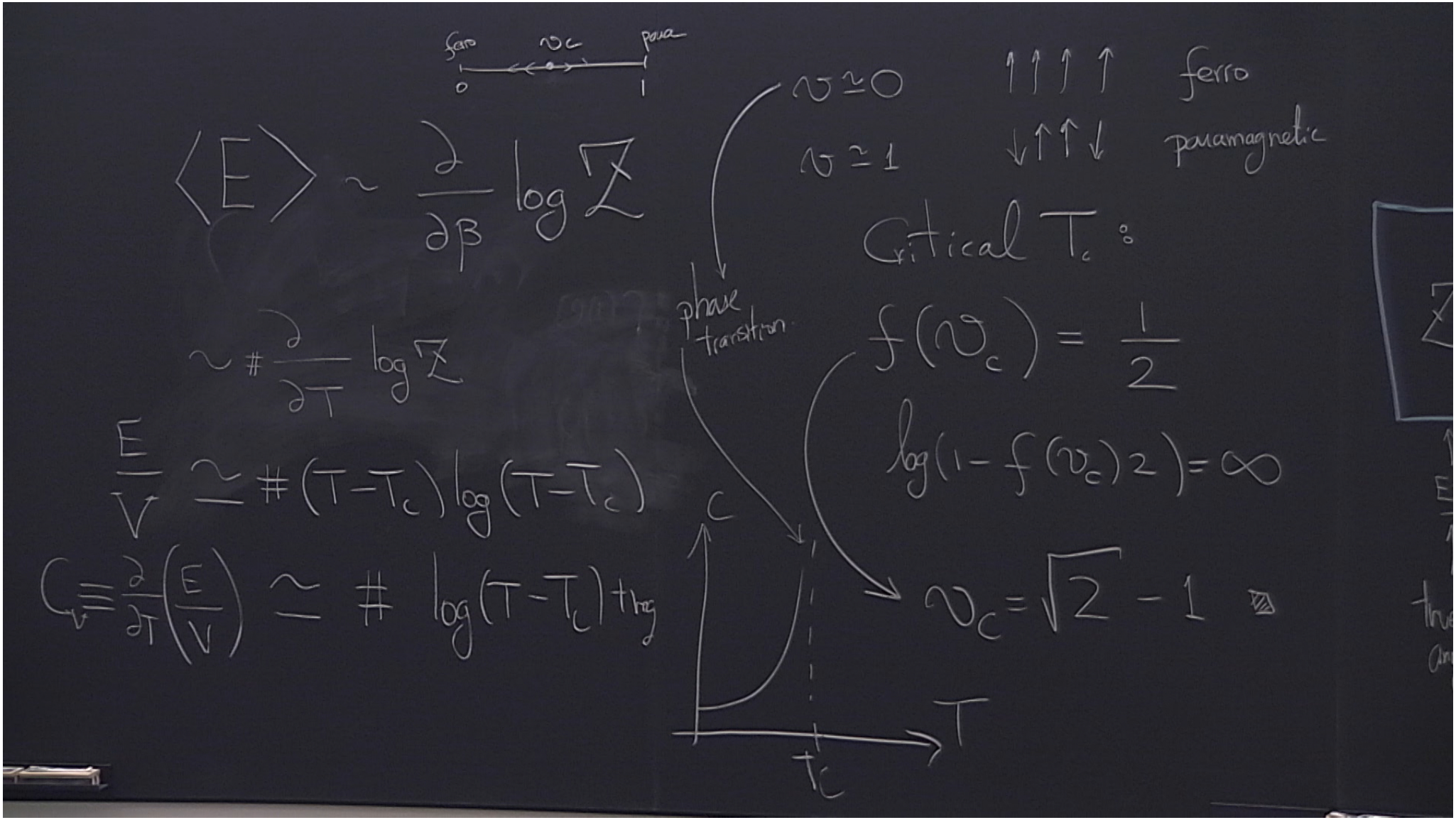
C

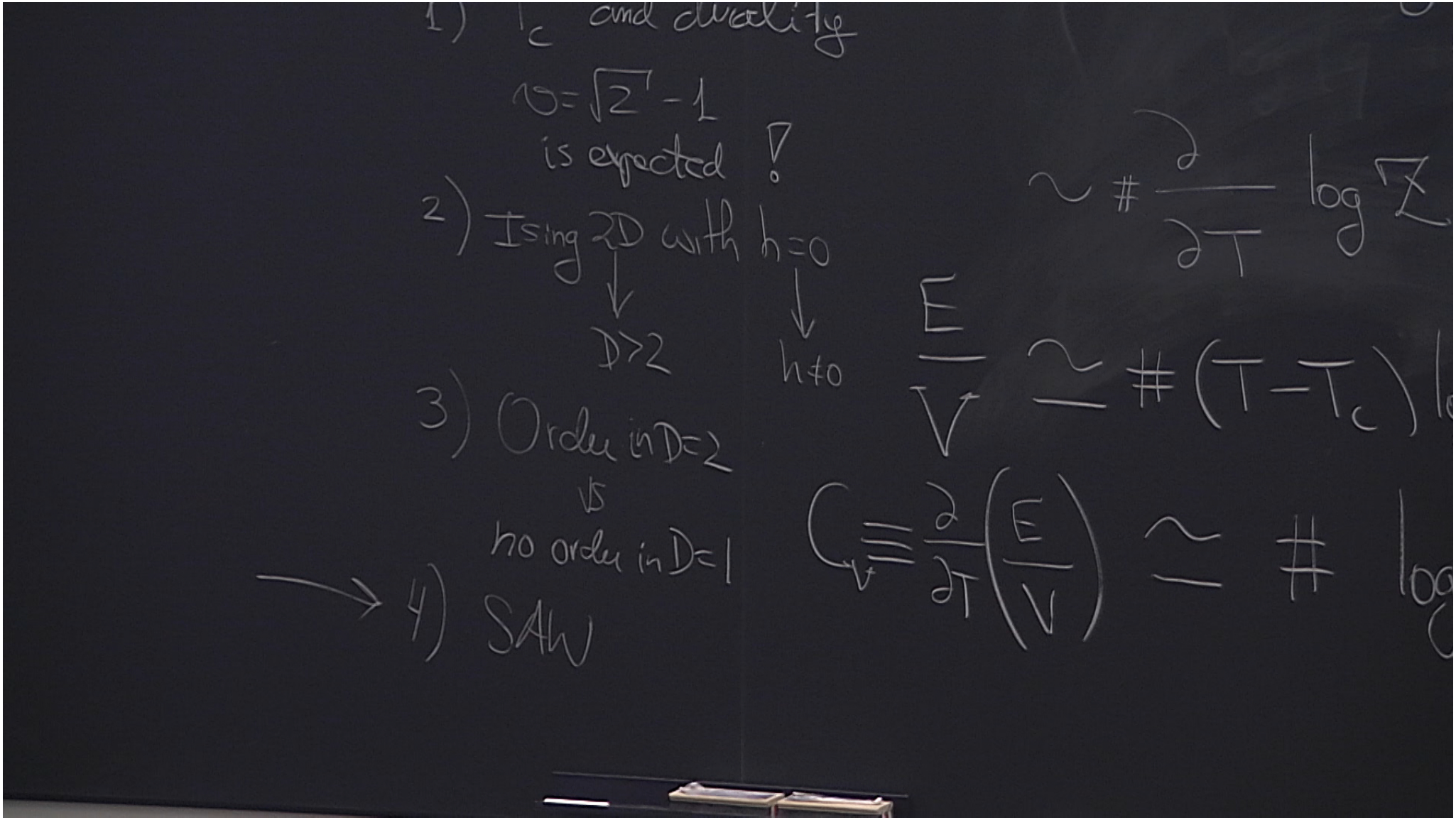
$Z =$

Exa

true

any


ferro
v_c
para
0
1
$\langle E \rangle \sim \frac{\partial}{\partial \beta} \log Z$
$v \simeq 0$
$v \simeq 1$
ferro
paramagnetic
Critical T_c :
phase transition
$\sim \# \frac{\partial}{\partial T} \log Z$
$f(v_c) = \frac{1}{2}$
$\frac{E}{V} \simeq \# (T - T_c) \log (T - T_c)$
$\log(1 - f(v_c)2) = \infty$
$C_v \equiv \frac{\partial}{\partial T}\left(\frac{E}{V}\right) \simeq \# \log(T - T_c) + \text{reg}$
C
T
T_c
$v_c = \sqrt{2} - 1$


1) T_c and duality
$\nu = \sqrt{2} - 1$
is expected !
2) Ising 2D with $h=0$
$D>2$
$h \neq 0$
3) Order in $D=2$
vs
no order in $D=1$
4) SAW
$\sim \# \frac{\partial}{\partial T} \log Z$
$\frac{E}{V} \simeq \# (T - T_c)$
$C_V \equiv \frac{\partial}{\partial T} \left(\frac{E}{V} \right) \simeq \#$