Title: PSI 2019/2020 - Statistical Mechanics (Vieira) - Lecture 6

Speakers: Pedro Vieira

Collection: PSI 2019/2020 - Statistical Mechanics (Vieira)

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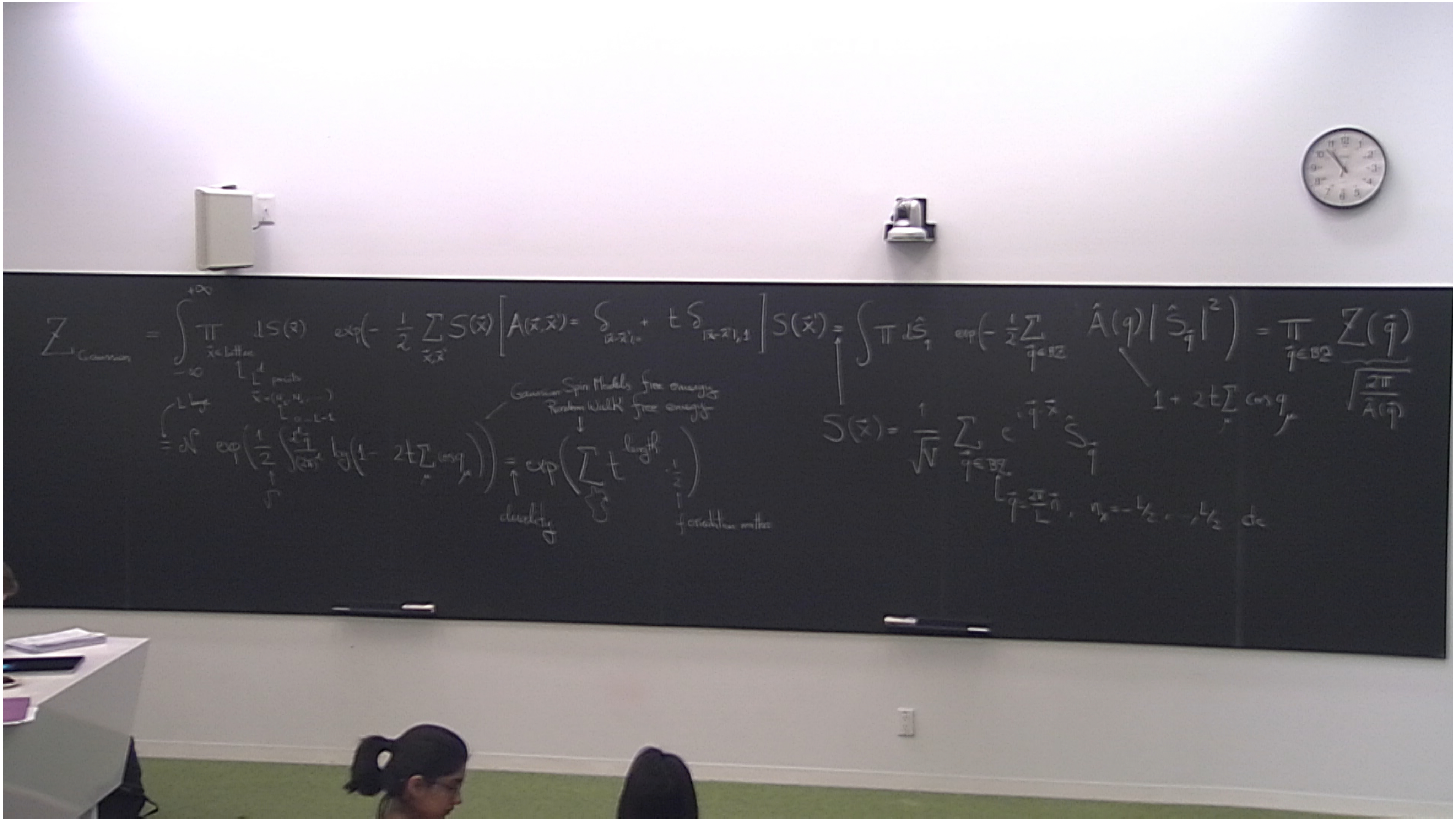
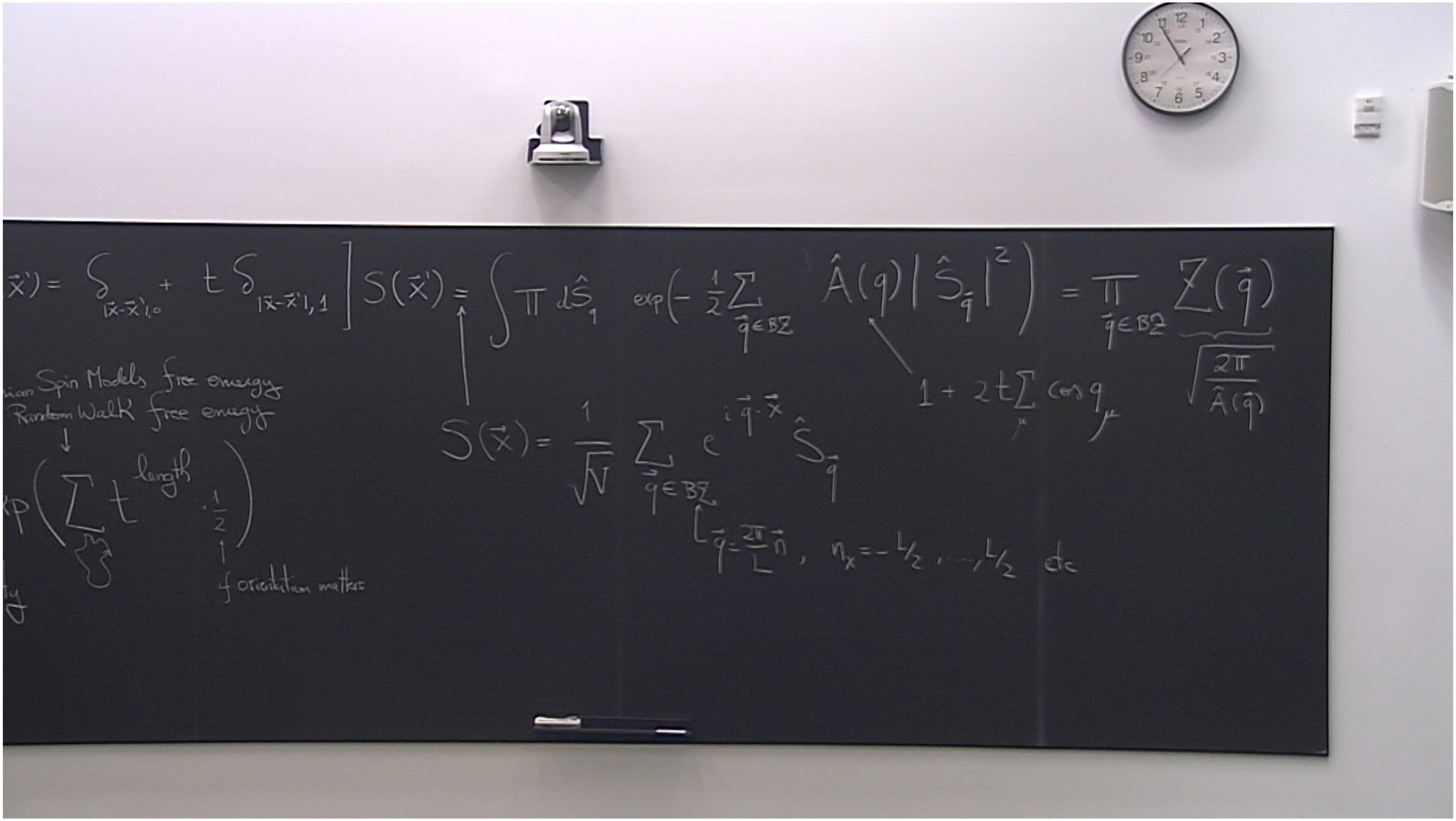
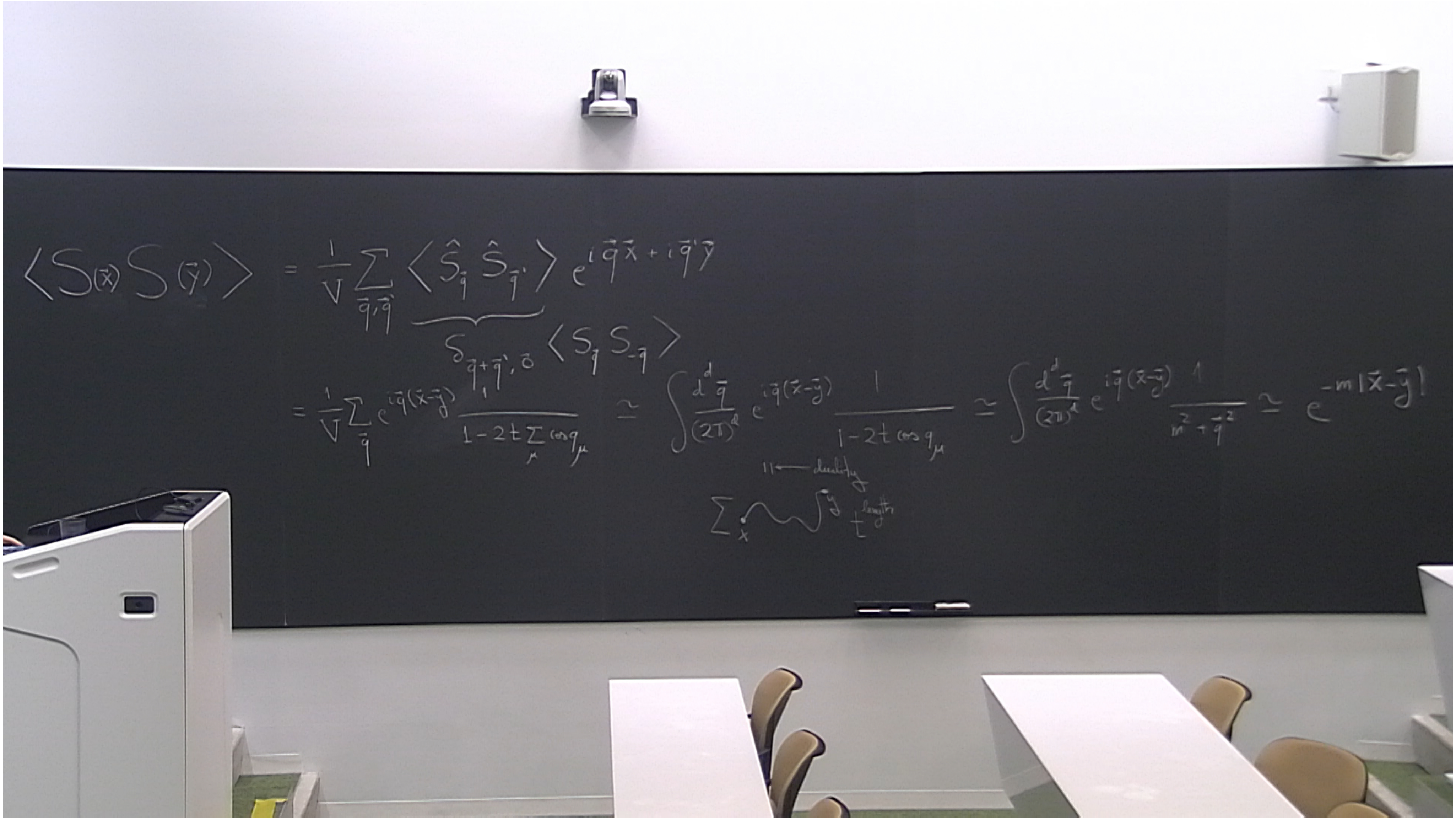
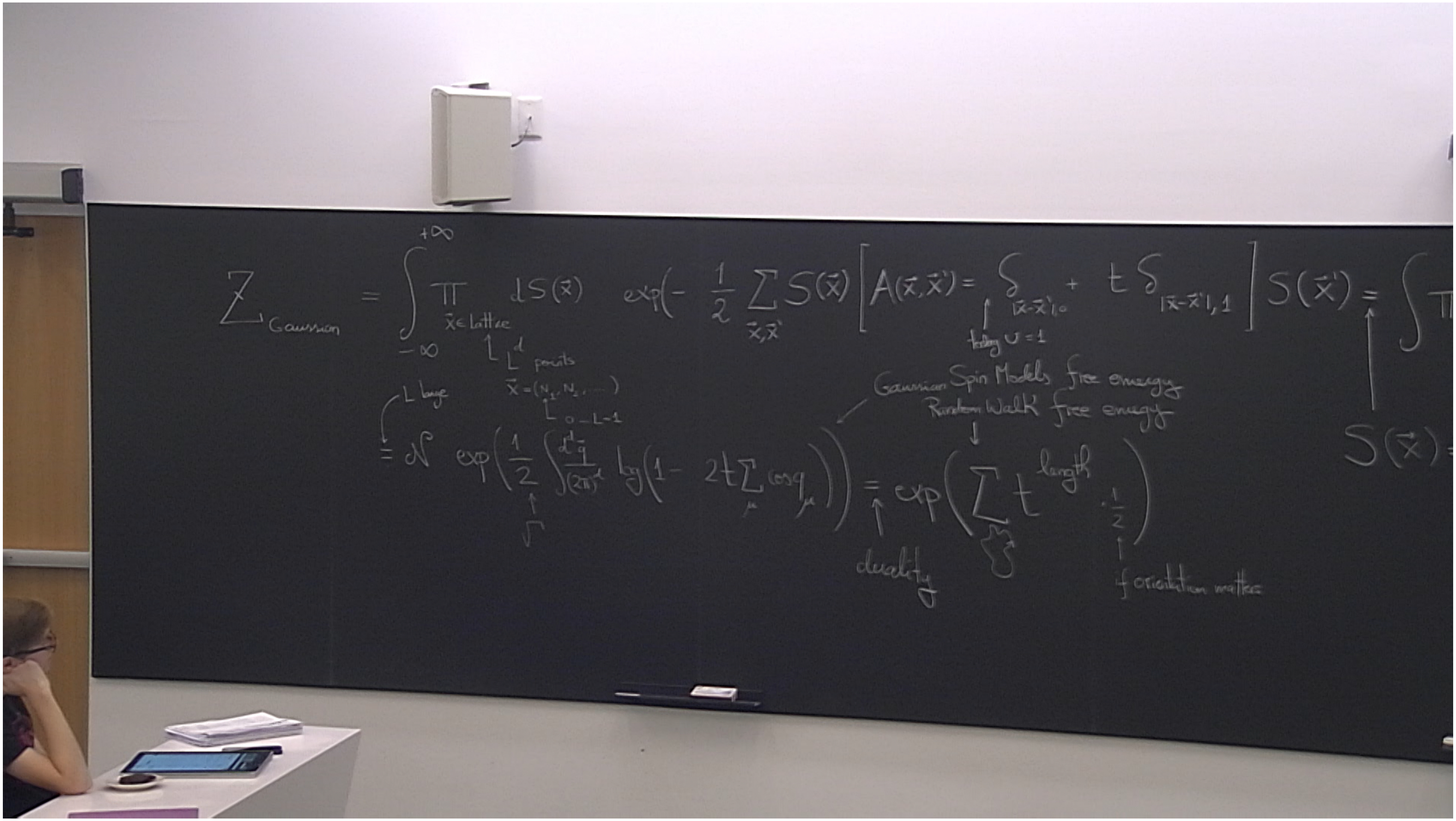

Gaussian Spin Models free energy
Random Walk free energy



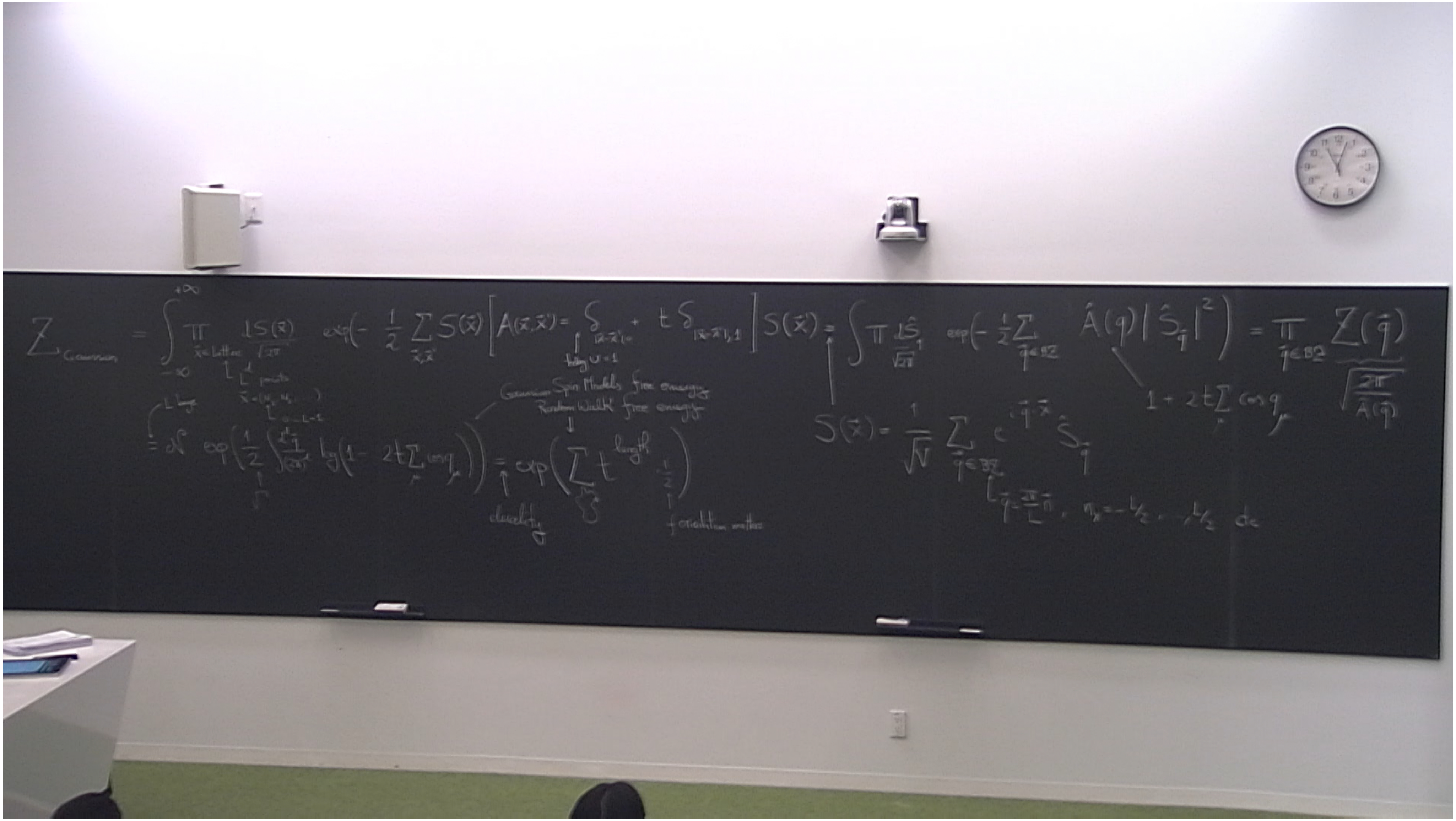
$\vec{x}') = \delta_{|\vec{x}-\vec{x}'|,0} + t\,\delta_{|\vec{x}-\vec{x}'|,1}$
$S(\vec{x}') = \int \prod d\hat{S}_q \exp\left(-\frac{1}{2}\sum_{\vec{q}\in BZ} \hat{A}(q)|\hat{S}_{\vec{q}}|^2\right) = \prod_{\vec{q}\in BZ} \underbrace{Z(\vec{q})}_{\sqrt{\frac{2\pi}{\hat{A}(\vec{q})}}}$
$1 + 2t\sum_\mu \cos q_\mu$
sian Spin Models free energy
Random Walk free energy
$\exp\left(\sum t^{\text{length}} \cdot \frac{1}{2}\right)$
if orientation matters
ty
$S(\vec{x}) = \frac{1}{\sqrt{N}} \sum_{\vec{q}\in BZ} e^{i\vec{q}\cdot\vec{x}} \hat{S}_{\vec{q}}$
$\vec{q} = \frac{2\pi}{L}\vec{n}$, $n_x = -L/2, \ldots, L/2$ etc



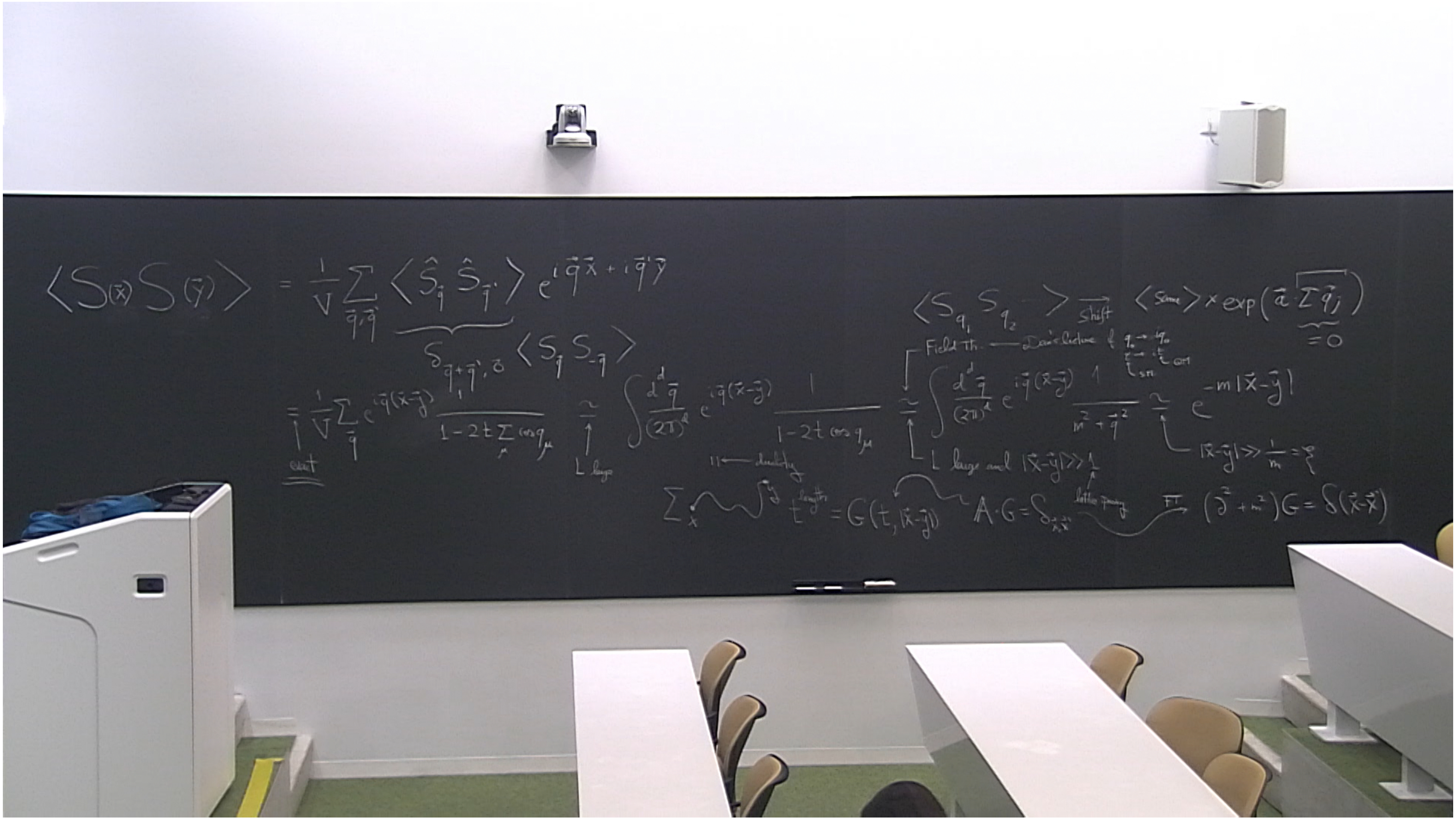
$\langle S(\vec{x}) S(\vec{y}) \rangle = \frac{1}{V} \sum_{\vec{q},\vec{q}'} \underbrace{\langle \hat{S}_{\vec{q}} \hat{S}_{\vec{q}'} \rangle}_{\delta_{\vec{q}+\vec{q}',\vec{0}} \langle S_{\vec{q}} S_{-\vec{q}} \rangle} e^{i\vec{q}\vec{x} + i\vec{q}'\vec{y}}$
$= \frac{1}{V} \sum_{\vec{q}} e^{i\vec{q}(\vec{x}-\vec{y})} \frac{1}{1 - 2t \sum_\mu \cos q_\mu} \simeq \int \frac{d^d\vec{q}}{(2\pi)^d} e^{i\vec{q}(\vec{x}-\vec{y})} \frac{1}{1-2t\cos q_\mu} \simeq \int \frac{d^d\vec{q}}{(2\pi)^d} e^{i\vec{q}(\vec{x}-\vec{y})} \frac{1}{m^2 + \vec{q}^2} \simeq e^{-m|\vec{x}-\vec{y}|}$
duality
$\sum_{x \to y} t^{\text{length}}$



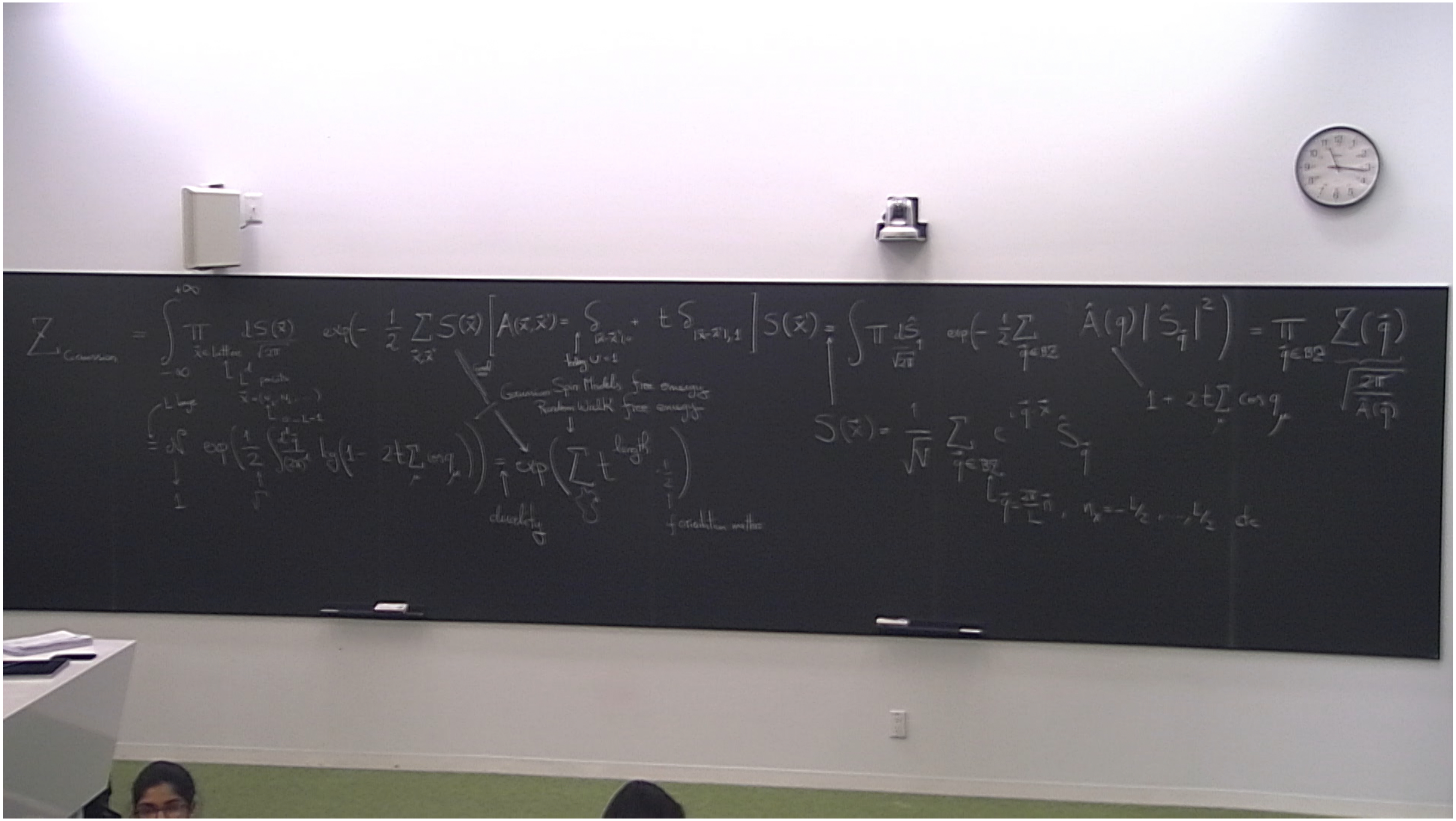
Z Gaussian = ∫ Π dS(x) exp(- 1/2 Σ S(x) [A(x,x') = δ + t δ] S(x') =
x ∈ lattice
L^d points
x = (N_1, N_2, ...)
0 ... L-1
L large
taking υ = 1
Gaussian Spin Models free energy
Random Walk free energy
= N exp(1/2 ∫ d^dq/(2π)^d log(1 - 2t Σ cos q_μ)) = exp(Σ t^length · 1/2)
duality
orientation matters
S(x) =



$Z_{\text{Gaussian}} = \int_{-\infty}^{+\infty} \prod_{\vec{x} \in \text{lattice}} \frac{dS(\vec{x})}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\sum_{\vec{x},\vec{x}'} S(\vec{x}) \left[A(\vec{x},\vec{x}') = \delta_{|\vec{x}-\vec{x}'|=0} + t\,\delta_{|\vec{x}-\vec{x}'|,1}\right] S(\vec{x}')\right) = \int \prod \frac{d\hat{S}_{\vec{q}}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\sum_{\vec{q}\in BZ} \hat{A}(q)\,|\hat{S}_{\vec{q}}|^2\right) = \prod_{\vec{q}\in BZ} \underbrace{Z(\vec{q})}_{\sqrt{\frac{2\pi}{\hat{A}(\vec{q})}}}$
taking $u=1$
L^d points
$\vec{x} = (n_1, n_2, \dots)$
$0 \dots L-1$
L large
$S(\vec{x}) = \frac{1}{\sqrt{N}} \sum_{\vec{q}\in BZ} e^{i\vec{q}\cdot\vec{x}} \hat{S}_{\vec{q}}$
$\vec{q} = \frac{2\pi}{L}\vec{n}$, $n_x = -L/2, \dots, L/2$ etc
$1 + 2t\sum_{\mu} \cos q_{\mu}$
$= \mathcal{N} \exp\left(\frac{1}{2}\int \frac{d^dq}{(2\pi)^d} \log\left(1 - 2t\sum_{\mu}\cos q_{\mu}\right)\right) = \exp\left(\sum t^{\text{length}} \cdot \frac{1}{2}\right)$
Gaussian Spin Models free energy
Random Walk free energy
duality
orientation matters



$\langle S(\vec{x}) S(\vec{y}) \rangle = \frac{1}{V} \sum_{\vec{q},\vec{q}'} \underbrace{\langle \hat{S}_{\vec{q}} \hat{S}_{\vec{q}'} \rangle}_{\delta_{\vec{q}+\vec{q}',\vec{0}} \langle S_{\vec{q}} S_{-\vec{q}} \rangle} e^{i\vec{q}\vec{x} + i\vec{q}'\vec{y}}$
$= \frac{1}{V} \sum_{\vec{q}} e^{i\vec{q}(\vec{x}-\vec{y})} \frac{1}{1-2t\sum_\mu \cos q_\mu} \simeq \int \frac{d^d\vec{q}}{(2\pi)^d} e^{i\vec{q}(\vec{x}-\vec{y})} \frac{1}{1-2t\cos q_\mu} \simeq \int \frac{d^d\vec{q}}{(2\pi)^d} e^{i\vec{q}(\vec{x}-\vec{y})} \frac{1}{m^2+\vec{q}^2} \simeq e^{-m|\vec{x}-\vec{y}|}$
exact
L large
L large and $|\vec{x}-\vec{y}| \gg 1$
lattice spacing
$|\vec{x}-\vec{y}| \gg \frac{1}{m} = \xi$
Field Th. — Dan's lecture
$\langle S_{q_1} S_{q_2} \cdots \rangle \xrightarrow[\text{shift}]{} \langle \text{same} \rangle \times \exp(\vec{a} \cdot \underbrace{\sum \vec{q}_j}_{=0})$
|| ← duality
$\sum_x t^{\text{length}} = G(t, |\vec{x}-\vec{y}|)$
$A \cdot G = \delta_{\vec{x},\vec{x}'}$
FT
$(\partial^2 + m^2) G = \delta(\vec{x}-\vec{x}')$



Gaussian Spin Models free energy
Random Walk free energy
duality
orientation matters

Today (This week : Parisi, Feynman)

* Gaussian = RW
* High T expansions in gen models.
* Comments on XY model.

$$Z = \int \prod_{\vec{x}} d\mu(S(\vec{x})) \quad \exp\left(t \sum_{\langle \vec{x}, \vec{x}' \rangle} S_{\vec{x}} S_{\vec{x}'} \right)$$

$$\int d\mu(S) = \int dS \, e^{-S^2/2}$$

$$\| $$

$$\frac{1}{2} \sum_{S=\pm 1} \quad \text{if } d\mu(S) = dS \left(\delta(S+1) + \delta(S-1) \right)$$

$$Z = \int \prod_{\vec{x}} d\mu(S(\vec{x})) \quad \exp\left(t \sum_{\langle \vec{x}, \vec{x}' \rangle} S_{\vec{x}} S_{\vec{x}'} \right)$$

$$\int d\mu(S) = \int \frac{dS}{\sqrt{2\pi}} e^{-S^2/2} \qquad \int d\mu(S)\, 1 = 1$$

$$= \frac{1}{2} \sum_{S=\pm 1}$$

if $d\mu(S) = dS \left(\frac{\delta(S+1)}{2} + \frac{\delta(S-1)}{2} \right)$

$$\boxed{Z = \int \prod_{\vec{x}} d\mu(S(\vec{x})) \quad \exp\Big(t \sum_{\langle \vec{x}, \vec{x}' \rangle} S_{\vec{x}} S_{\vec{x}'}\Big)}$$

$=$

$\uparrow$ t high

$$\int d\mu(S) = \int \frac{dS}{\sqrt{2\pi}}\, e^{-S^2/2} \longrightarrow \int d\mu(S)\, 1 = 1$$

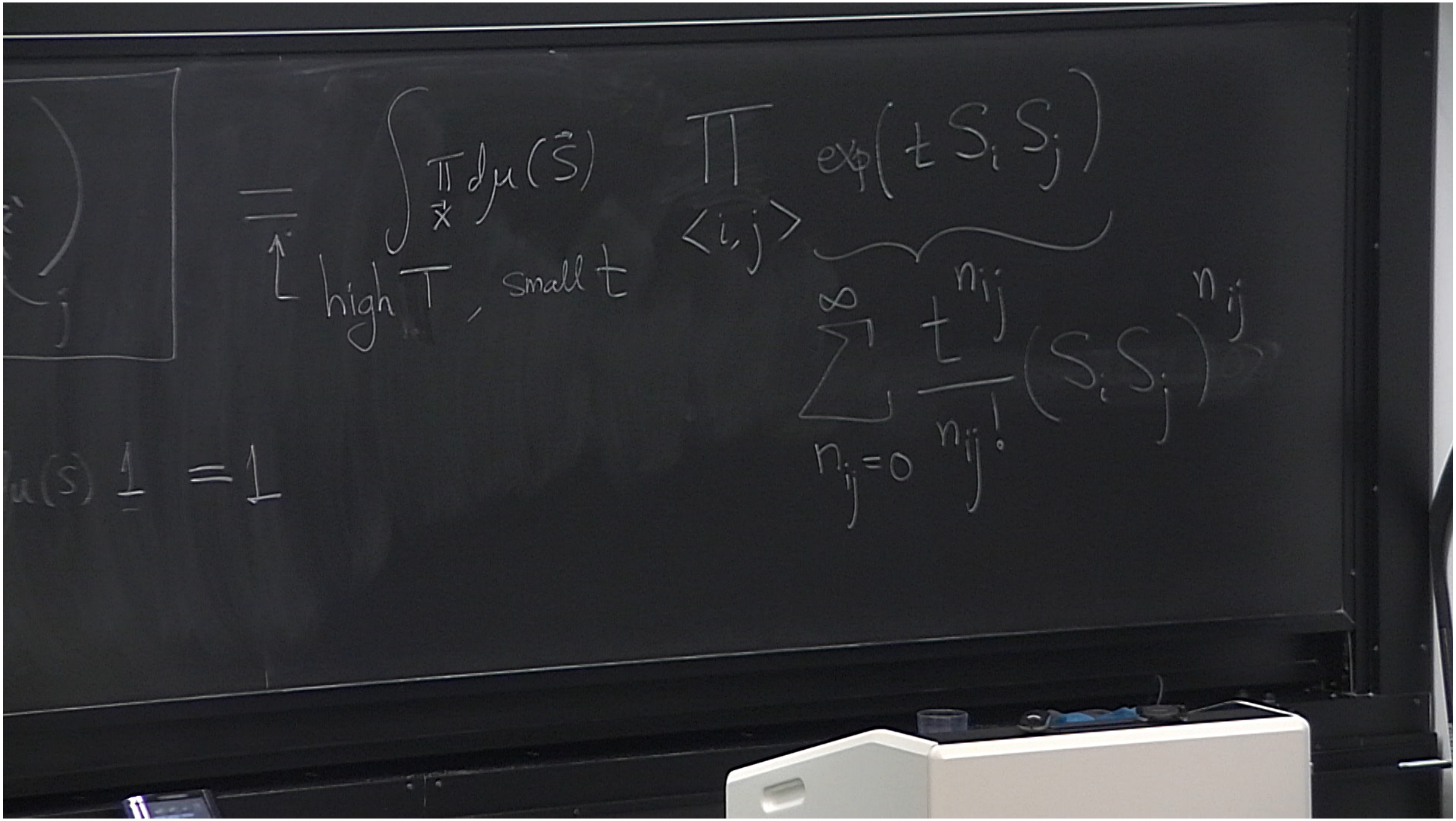
$$\frac{1}{2}\sum_{S=\pm 1} \quad \text{if } d\mu(S) = dS\left(\frac{\delta(S+1)}{2} + \frac{\delta(S-1)}{2}\right)$$

$$\boxed{Z = \int \prod_{\vec{x}} d\mu(S(\vec{x})) \; \exp\Big(t \sum_{\underbrace{\langle \vec{x}, \vec{x}' \rangle}_{\langle i,j \rangle}} S_{\vec{x}} S_{\vec{x}'} \Big)}$$

$i \qquad j$

$$\int d\mu(S) = \int \frac{dS}{\sqrt{2\pi}} \, e^{-S^2/2} \qquad \int d\mu(S)\, 1 =$$

$$\frac{1}{2} \sum_{S=\pm 1} \qquad \text{if } d\mu(S) = dS \left(\frac{\delta(S+1)}{2} + \frac{\delta(S-1)}{2} \right)$$



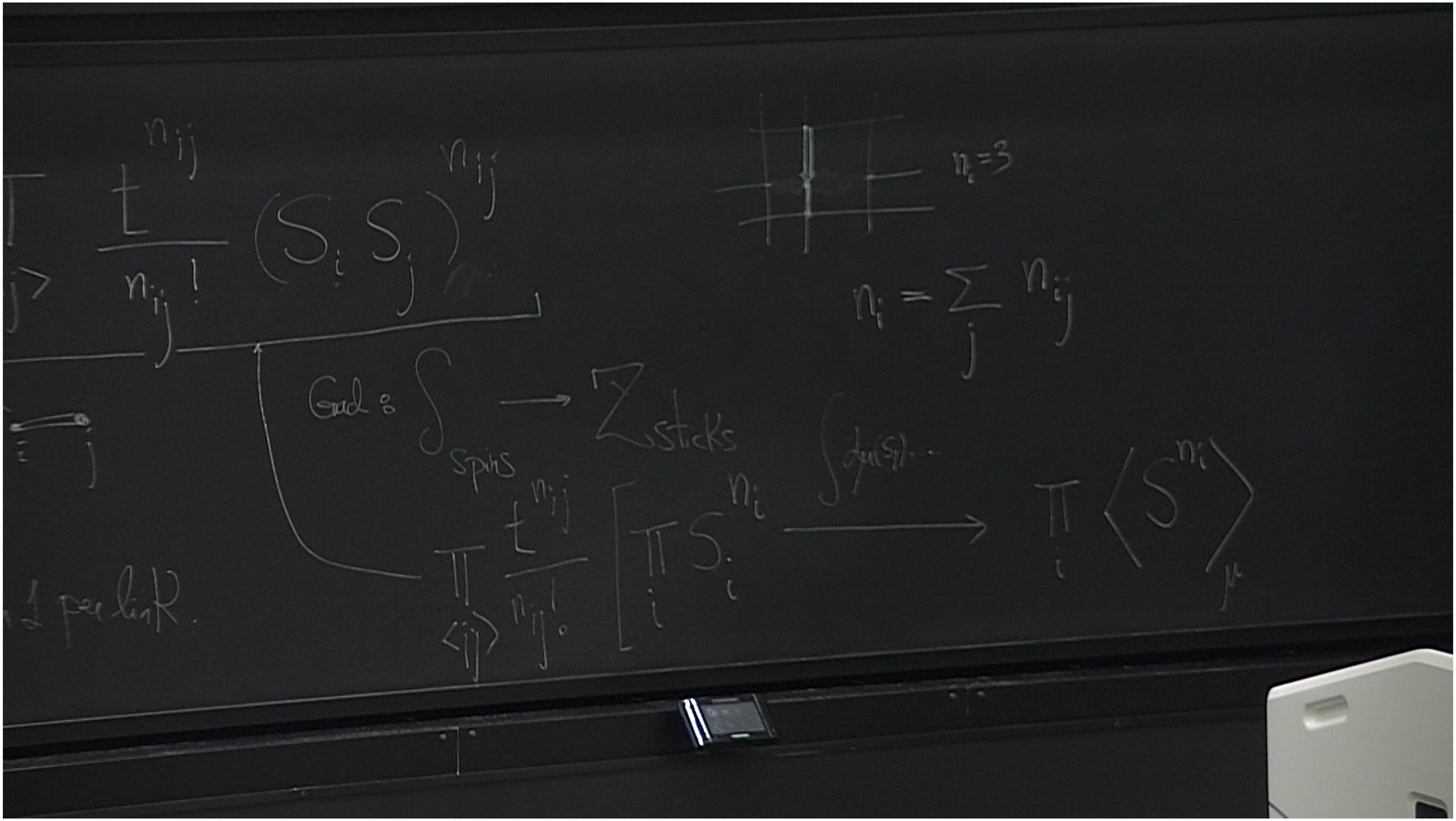
$= \int \prod_{\vec{x}} d\mu(\vec{S}) \prod_{\langle i,j \rangle} \underbrace{\exp(t S_i S_j)}_{\sum_{n_{ij}=0}^{\infty} \frac{t^{n_{ij}}}{n_{ij}!} (S_i S_j)^{n_{ij}}}$
high T, small t
$\mu(s)\, 1 = 1$

$$Z = \sum_{\{n_{ij} = 0, 1, 2, \ldots \infty\}} \boxed{\int d\mu(S)} \; \underbrace{\prod_{\langle ij \rangle} \frac{t^{n_{ij}}}{n_{ij}!} (S_i S_j)^{n_{ij}}}$$

$$\sum_{\text{sticks}} \text{ (allowing more than 1 per link.}$$

Goal: $\displaystyle \int_{\text{spins}} \longrightarrow Z_{\text{sticks}}$

$$\prod_{\langle ij \rangle} \frac{t^{n_{ij}}}{n_{ij}!} \; \prod_i S_i^{n_i}$$



$\frac{t^{n_{ij}}}{n_{ij}!}(S_i S_j)^{n_{ij}}$
$n_i = 3$
$n_i = \sum_j n_{ij}$
Goal : $\int_{\text{spins}} \longrightarrow \sum_{\text{sticks}}$
$\prod_{\langle ij \rangle} \frac{t^{n_{ij}}}{n_{ij}!} \left[\prod_i S_i^{n_i} \right] \longrightarrow \prod_i \langle S^{n_i} \rangle_\mu$
per link.

$$n_i = \sum_j n_{ij}$$

$$\frac{\int d\mu(s)\, S^{n_i}}{\int d\mu(s)\, 1} \equiv M_{n_i}$$

$$Z_{\text{sticks}}$$

$$\left[\prod_i S_i^{n_i} \xrightarrow{\int d\mu(s_i)\ldots} \prod_i \left\langle S^{n_i} \right\rangle_\mu \right]$$

$\sum_{\text{stick}}$ (allowing more than 1 per link.

$\prod_{\langle ij\rangle} \frac{t^{n_{ij}}}{n_{ij}!}\left[\prod_i S_i^{n_i} \xrightarrow{\int d\mu(S_i)\ldots} \prod_i \langle S^{n_i}\rangle_\mu\right]$

$Z = \sum_{\substack{\text{config of}\\ \Omega \text{ sticks}}} \Omega(\text{config})$

$\sum_{\langle i,j\rangle} n_{ij} = \Omega$

stick between i & j

sticks leaving site i, $n_i = \sum_j n_{ij}$, $\frac{1}{2}\sum_i n_i = \Omega$

$\Omega = \prod_{\langle ij\rangle} \frac{t^{n_{ij}}}{n_{ij}!} \prod_i M_{n_i}$

Ising

$\langle S^n\rangle_\mu = \delta_{n,\text{even}}$

in Gaussian model $M_n = (n-1)!!$ if n is even

$= 0$ if n is odd

$= \frac{\int ds\, s^n e^{-s^2/2}}{\int ds\, e^{-s^2/2}}$

$$Z = \sum_{\substack{\text{config of} \\ \Omega \text{ sticks}}} \Omega(\text{config})$$

\# stick between i & j

$$\sum_{\langle i,j \rangle} n_{ij} = \Omega$$

\# sticks leaving site i, $n_i = \sum_j n_{ij}$

$$\Omega = \prod_{\langle ij \rangle} \frac{t^{n_{ij}}}{n_{ij}!} \prod_i M_{n_i}$$

Ising

$$\langle S^n \rangle_\mu \doteq \delta_{n,\text{even}}$$

sticks

$$\int d\mu(S_i)\ldots$$

$$\frac{t^{n_{ij}}}{n_{ij}!}\left[\prod_i S_i^{n_i}\right] \longrightarrow \prod_i \left\langle S^{n_i}\right\rangle_\mu$$

$n_{ij} = \Omega$

site i, $n_i = \sum_j n_{ij}$, $\frac{1}{2}\sum_i n_i = \Omega$

$$\frac{\int ds\, s^n e^{-s^2/2}}{\int ds\, e^{-s^2/2}} = 0 \quad \text{if } n \text{ is odd}$$

Ising

$\delta_{n,\text{even}}$

$$\int dx\, e^{-x^2/2} = \sqrt{2\pi}$$

$$\int dx\, e^{-a x^2/2} = \sqrt{2\pi/a}$$

$$-2\frac{\partial}{\partial a}(\ldots)\Big/(\ldots)\Big|_{a=1} = \langle x^2\rangle = -2\frac{\partial}{\partial a}\frac{1}{\sqrt{a}} = \frac{1}{a^{3/2}}\Big|_{a=1} = 1$$

* High T expansions in gen models

* comments on XY model

$\#\partial^2 + \#$

$+ f(\vec{x})(2dt+1)$

$\Delta \cdot f \longrightarrow a^2 \partial^2 f$

$$\int dx\, e^{-x^2/2} = \sqrt{2\pi}$$

$$\int dx\, e^{-a x^2/2} = \sqrt{2\pi/a}$$

$$-2\frac{\partial}{\partial a}(\dots)\Big/(\dots)\Big|_{a=1} = \langle x^2\rangle = -2\frac{\partial}{\partial a}\frac{1}{\sqrt{a}} = \frac{1}{a^{3/2}}\Big|_{a=1} = 1$$

$$\langle x^4\rangle = -2\frac{\partial}{\partial a}\left(-2\frac{\partial}{\partial a}(\dots)\right)\Big/(\dots)\Big|_{a=1} = 2\times\frac{3}{2}\times\frac{1}{a^{5/2}}\Big|_{a=1} = 3$$

$\langle x^2\rangle = 1$

$\langle x^4\rangle = 3$

$\langle x^6\rangle = 5\cdot 3$

$\langle x^8\rangle = 7\cdot 5\cdot 3$

$$\langle x^n\rangle = (n-1)!!$$

sticks (allowing more than 1 per link.

$$\prod_{\langle ij \rangle} \frac{\ }{n_{ij}!} \left[\prod_i \ \ \right]$$

$$Z = \sum_{\substack{\text{Config of} \\ \cancel{\#}\text{ sticks}}} \Omega(\text{config})$$

$$\sum_{\langle i,j \rangle} n_{ij} = \Omega$$

stick between i&j

$\boxed{(n_i - 1)!!}$

sticks leaving site i, $n_i =$

$$\Omega = \prod_{\langle ij \rangle} \frac{t^{n_{ij}}}{n_{ij}!} \prod_i M_{n_i}$$

Ising

$$\langle S^n \rangle_\mu = \delta_{n,\text{even}}$$

$\langle x^6 \rangle = 5 \cdot 3$

$\langle x^8 \rangle = 7 \cdot 5 \cdot 3$

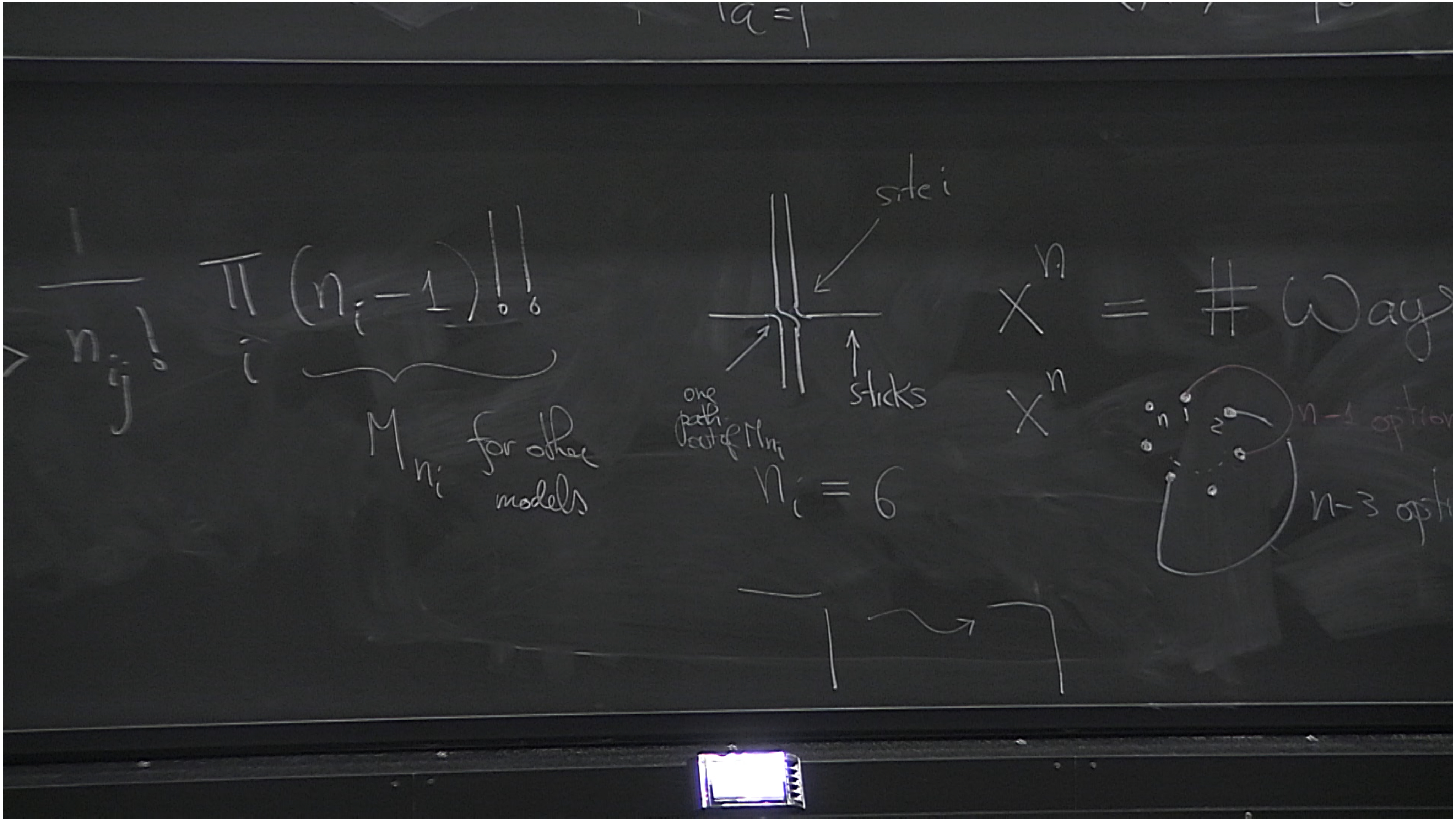
x^n = # Ways of contracting n endpoints.

x^n

$n-1$ options

$n-3$ options

$= (n-1)!!$


site i
sticks
one path out of Mn_i
for other models
n_i = 6
x^n = # Way
x^n
n-1 options

$$\langle x^8 \rangle = 7\cdot 5\cdot 3$$

site i

sticks

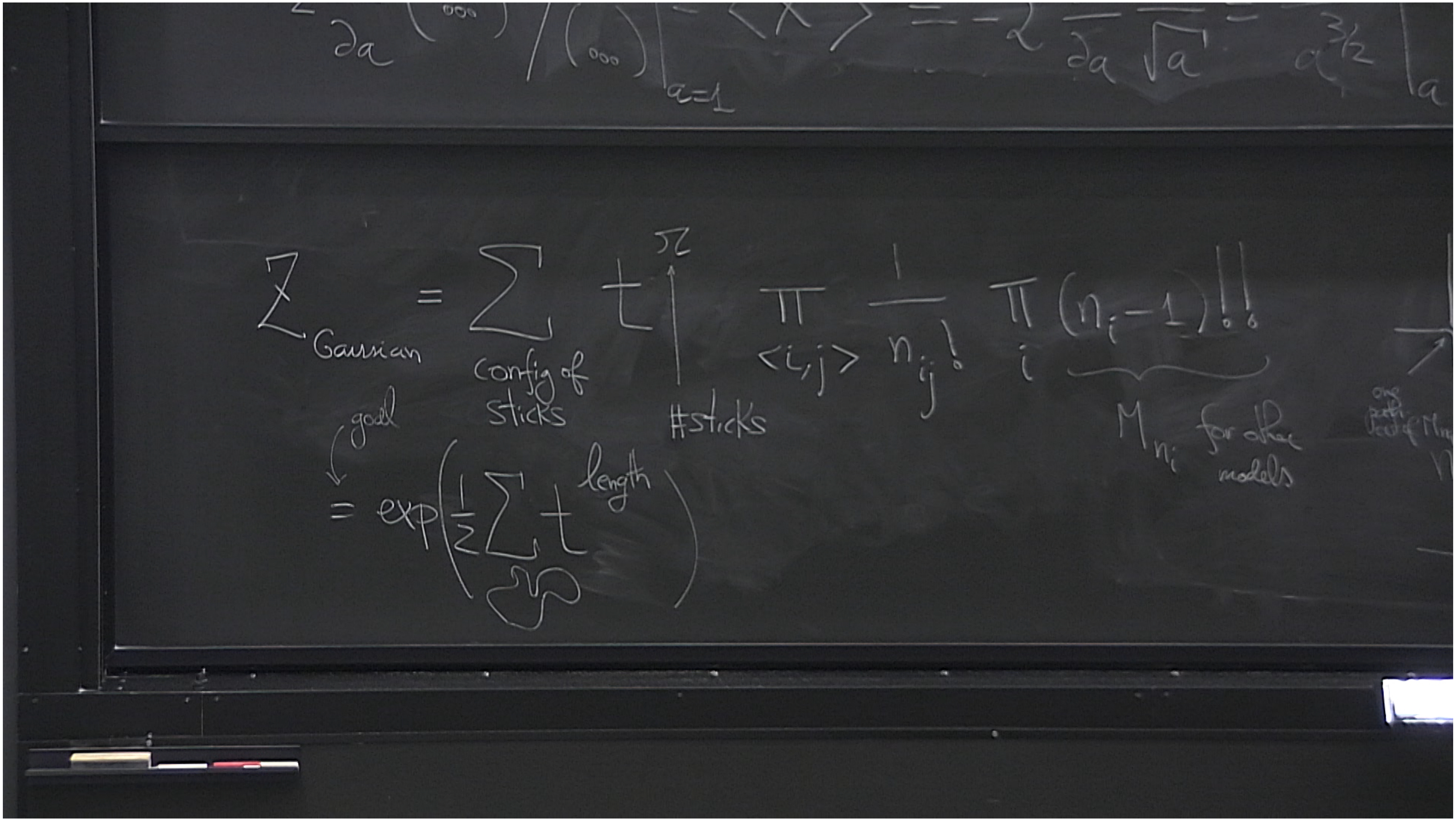
$$x^n = \#\text{ ways of contracting } n \text{ endpoints}$$

why

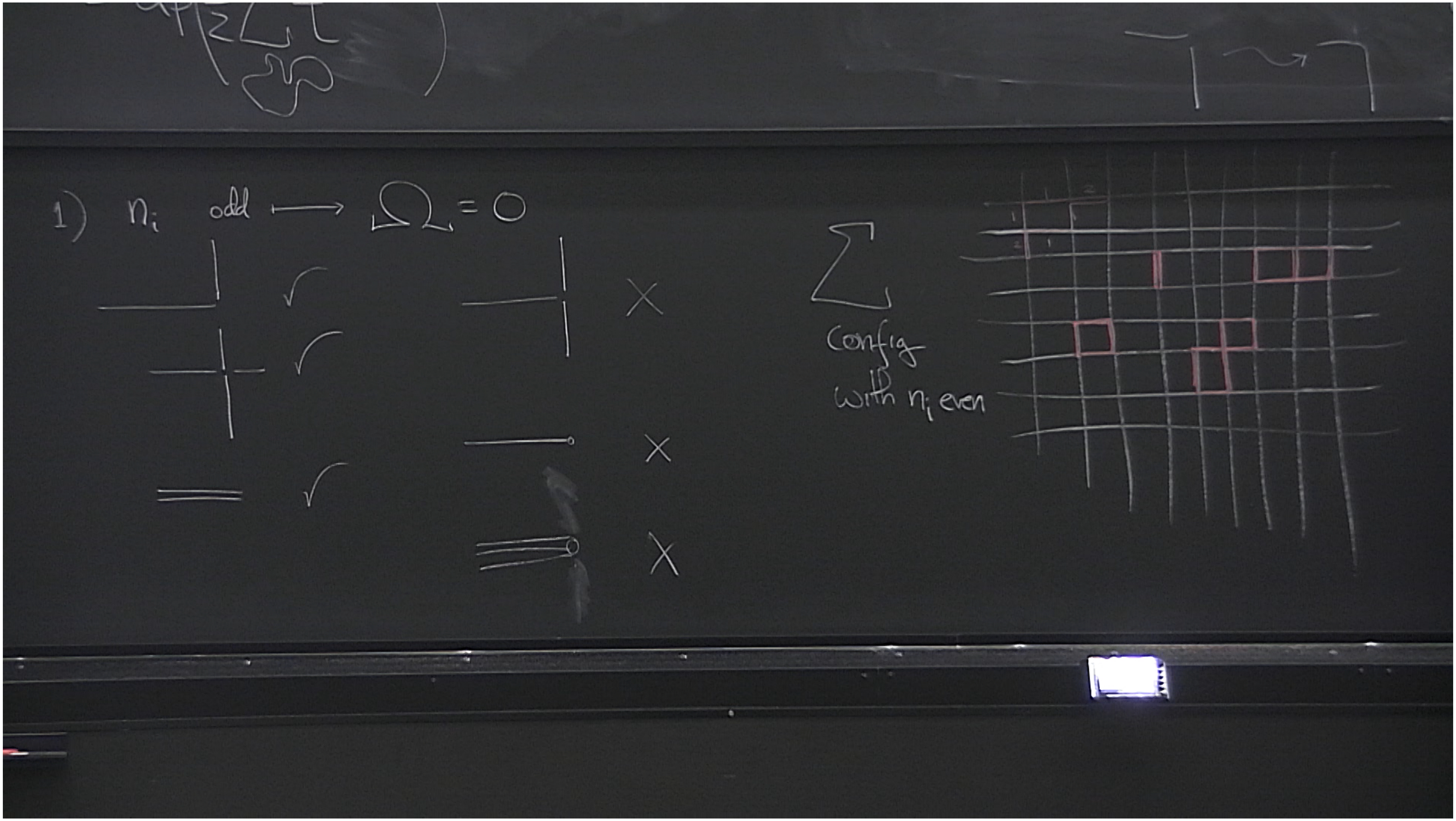
$$\sum_{\text{stick}} \longmapsto \sum_{\text{paths}}$$

$$x^n \quad n-1 \text{ options}, \quad n-3 \text{ options} \quad = (n-1)!!$$

$= 6$



$\partial_a \quad \Big|_{a=1} = \langle x \rangle = -2\,\partial_a \frac{1}{\sqrt{a}} = a^{3/2}$
$Z_{\text{Gaussian}} = \sum_{\text{config of sticks}} t^{\#\text{sticks}} \prod_{\langle i,j \rangle} \frac{1}{n_{ij}!} \prod_i \underbrace{(n_i - 1)!!}_{M_{n_i} \text{ for other models}}$
goal
$= \exp\left(\frac{1}{2} \sum t^{\text{length}}\right)$


1) n_i odd ⟶ $\Omega = 0$
$\sum$
config with n_i even

1) empty $\Omega = 0$, $\Omega = 1$ $\qquad \frac{1}{1!} = 1$

2) config : $n_{ij} = 1$, $n_i = 2$, $\underbrace{(1)!!}_{M_2} = 1$

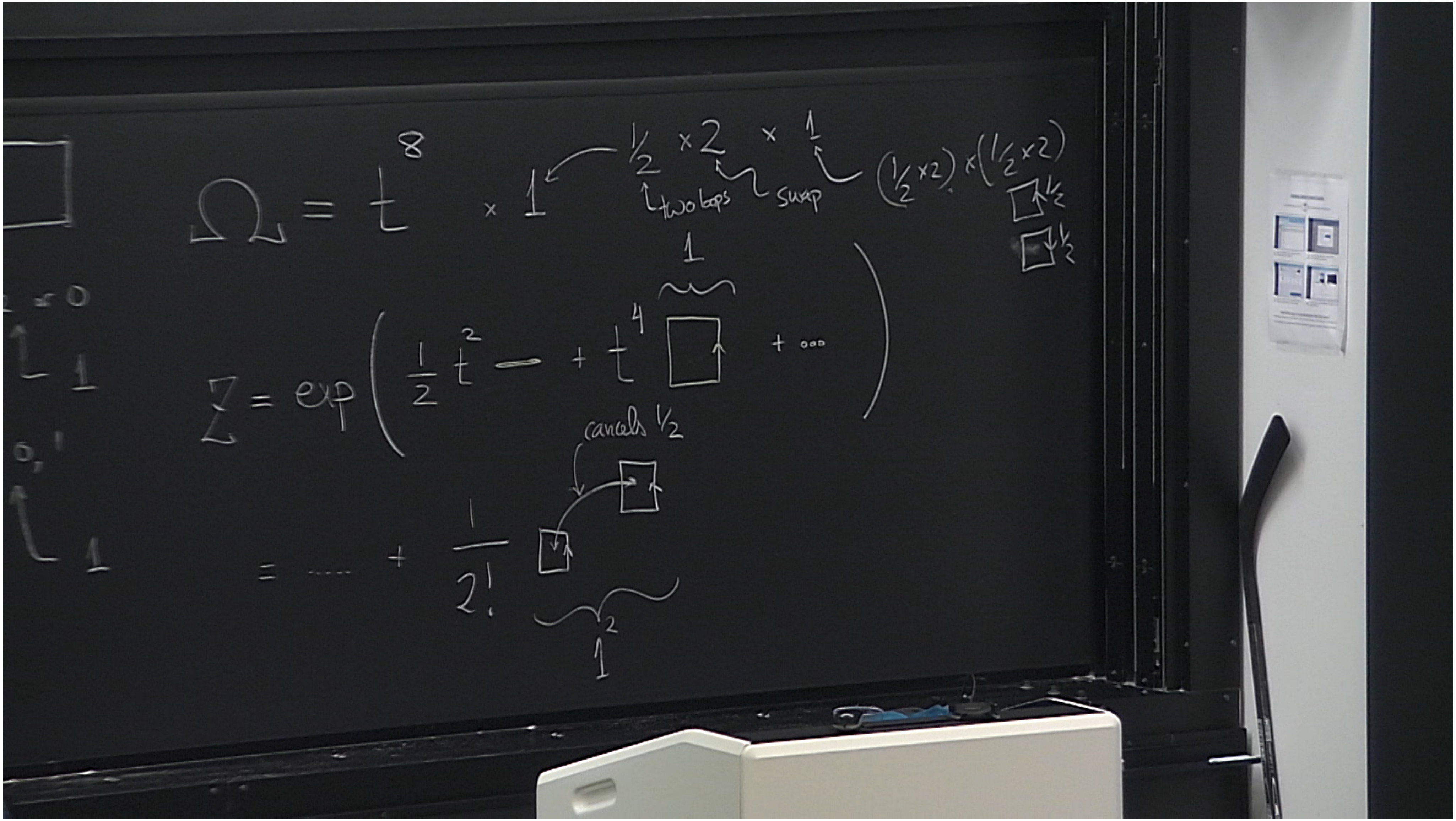
$$\Omega = t^4 \times 1$$

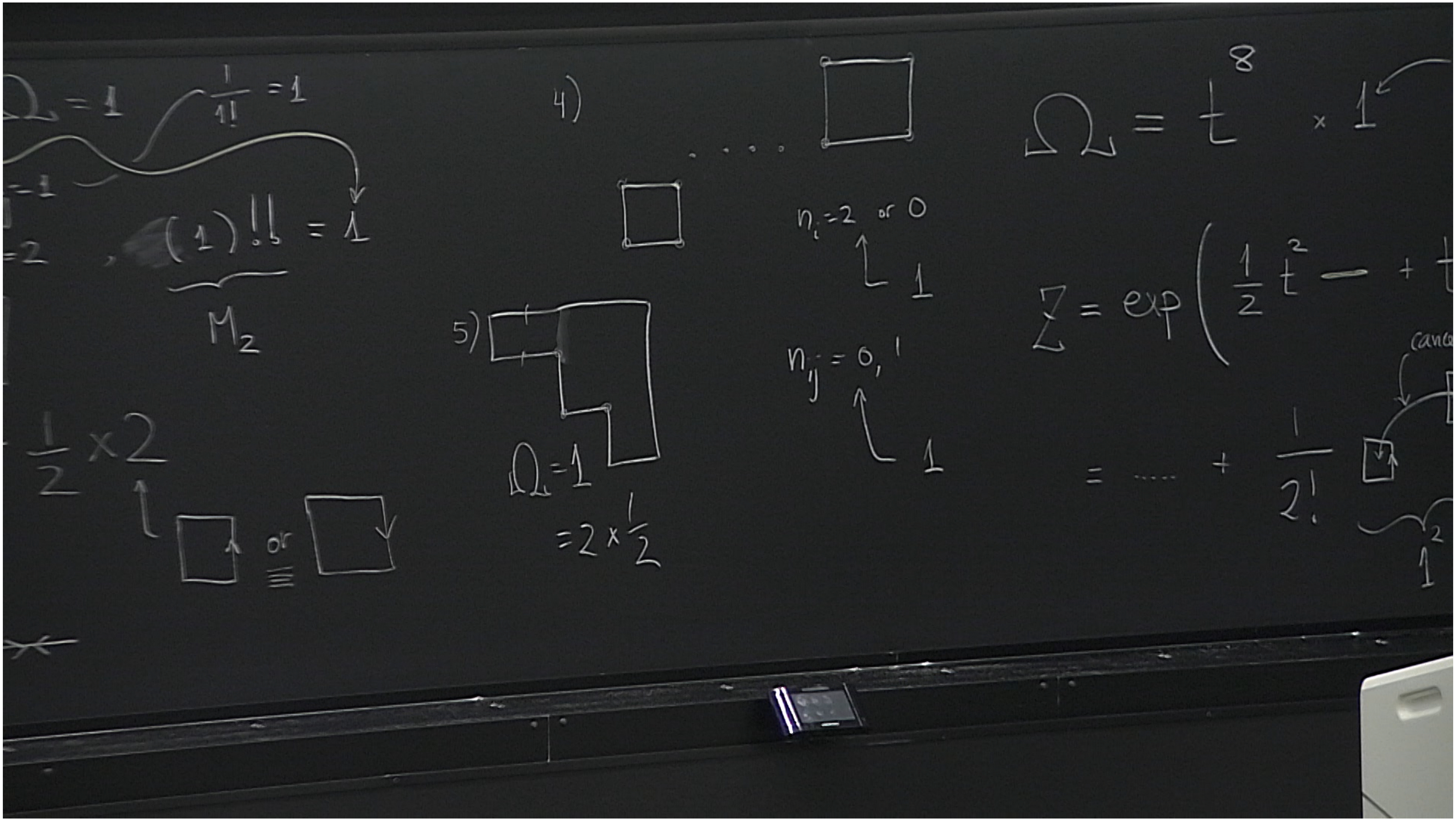
$$\frac{1}{2} \times 2$$

or

works.

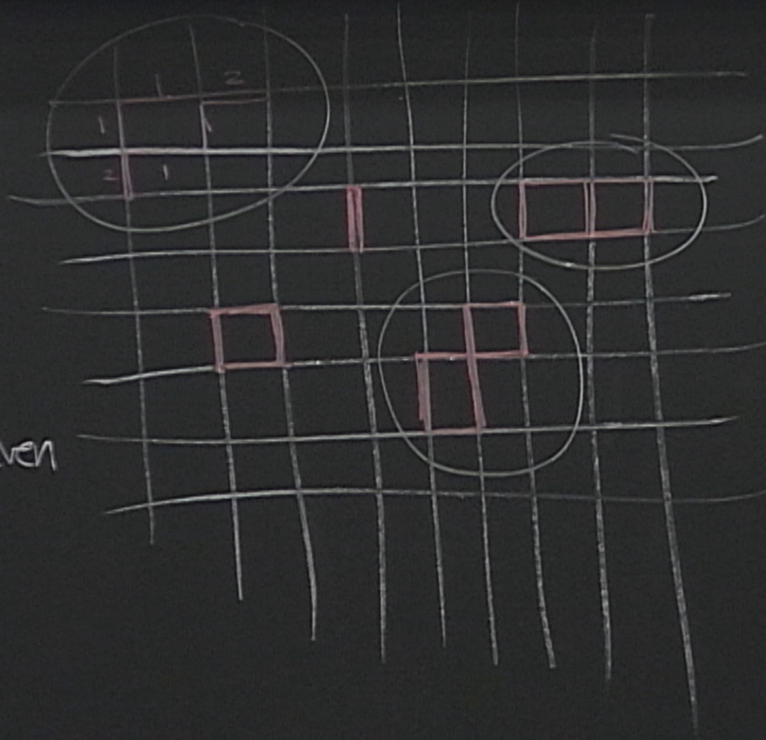
3) $\Omega = t^2 \times \frac{1}{2}$


Ω = t^8 × 1
1/2 × 2 × 1
two loops
swap
(1/2 × 2) × (1/2 × 2)
1/2
1/2
1
Z = exp(1/2 t^2 + t^4 + ...)
cancels 1/2
= .... + 1/2!
1^2
1
1



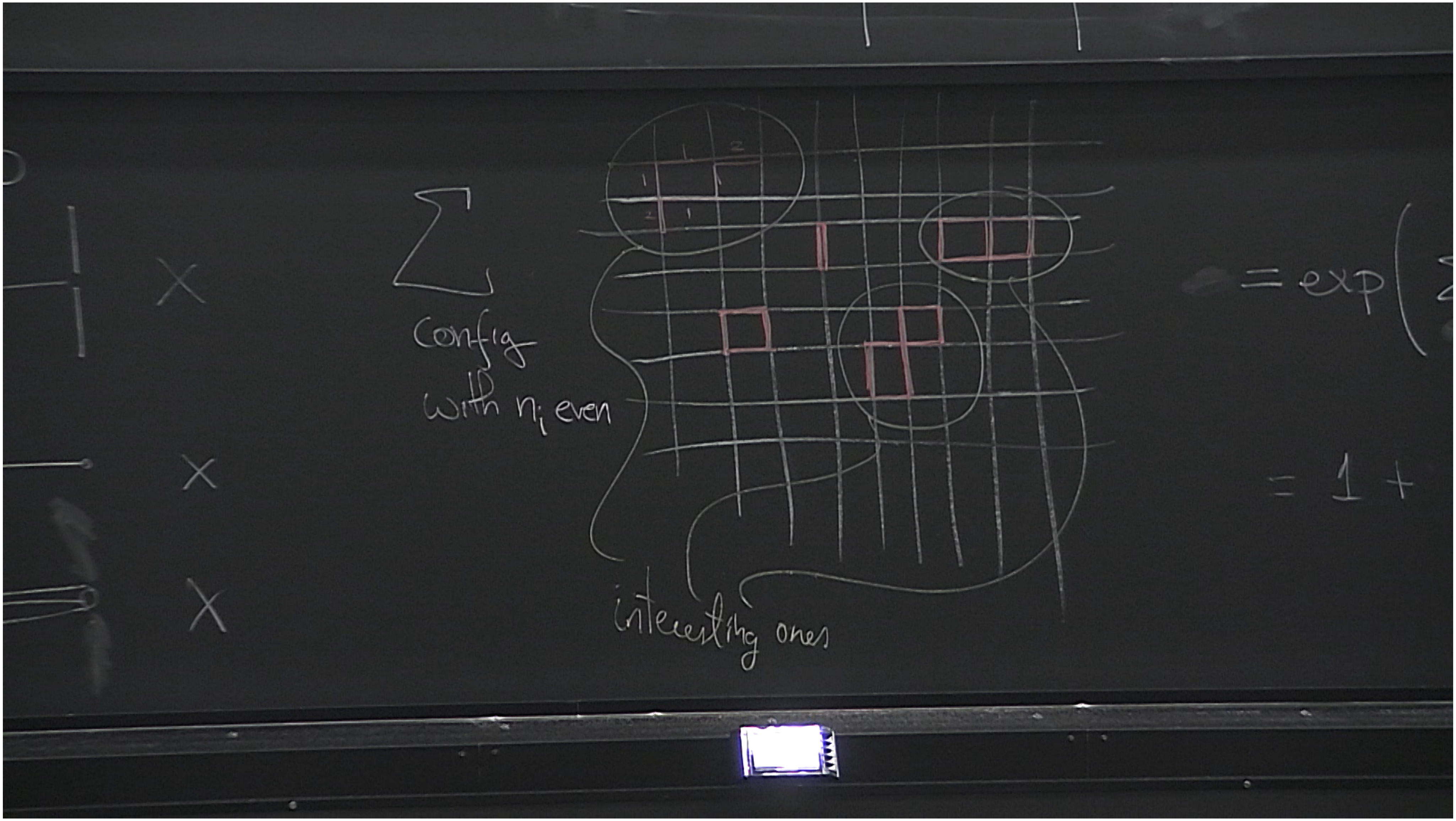
Ω = 1
1/1! = 1
(1)!! = 1
M2
1/2 x2
or
4)
5)
Ω = 1
= 2 x 1/2
n_i = 2 or 0
1
n_ij = 0, 1
1
Ω = t^8 x 1
Z = exp(1/2 t^2 + t
= .... + 1/2!

$$\sum_{\substack{\text{config} \\ \text{with } n_i \text{ even}}}$$

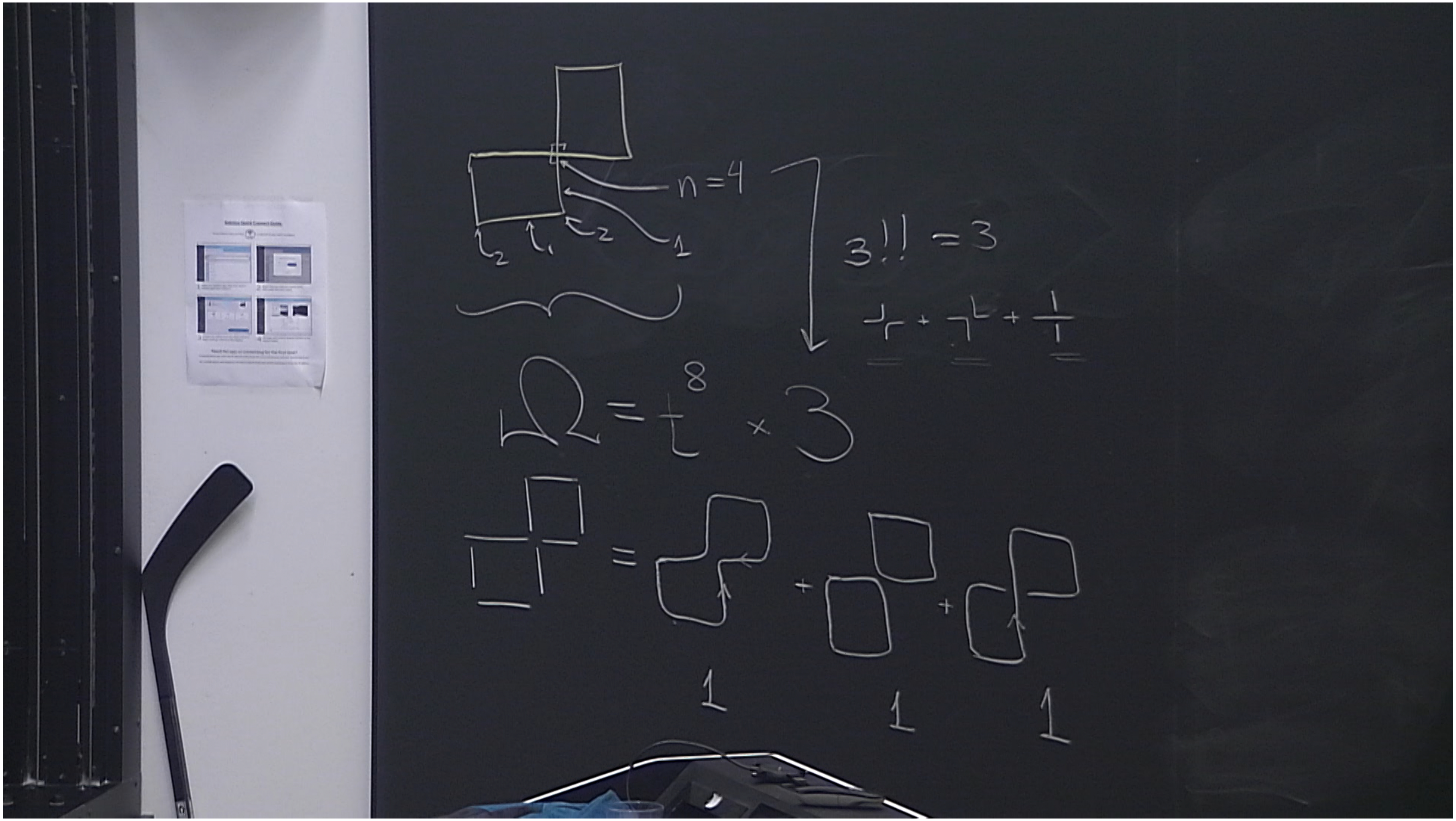


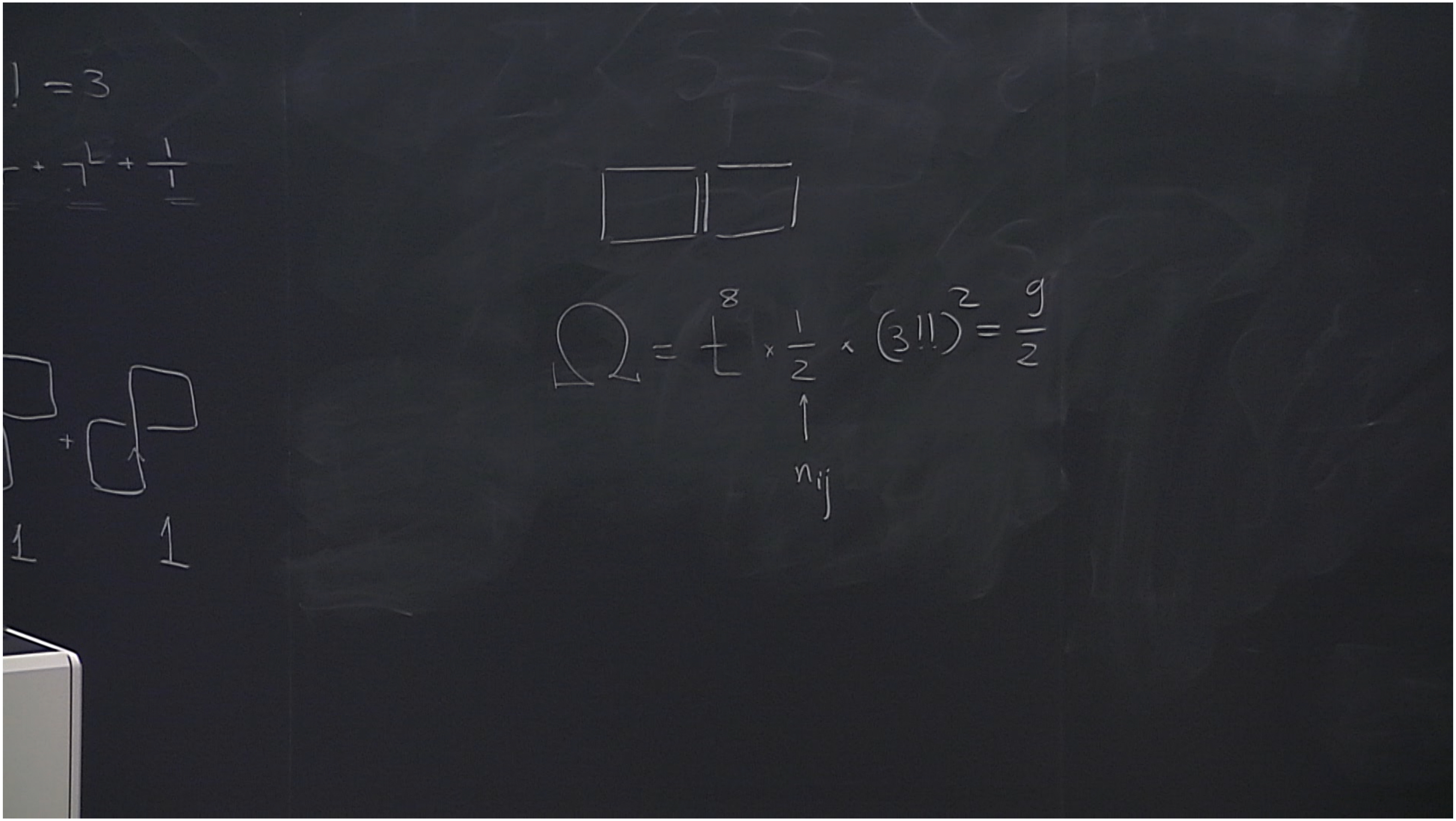
$$= \exp\left(\sum t^{\text{length}} \frac{1}{2} \right.$$

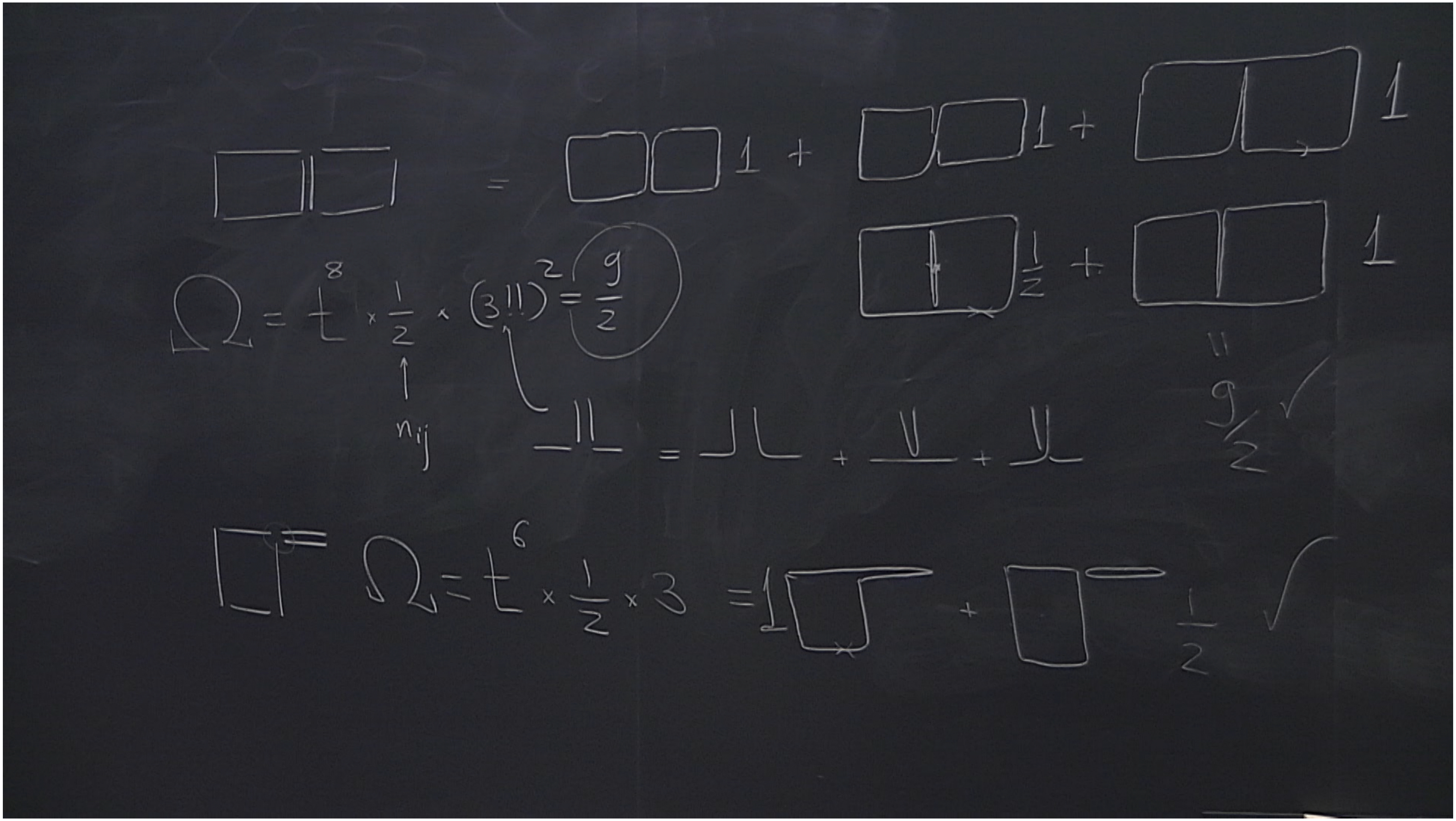
$$= 1 + \ldots$$

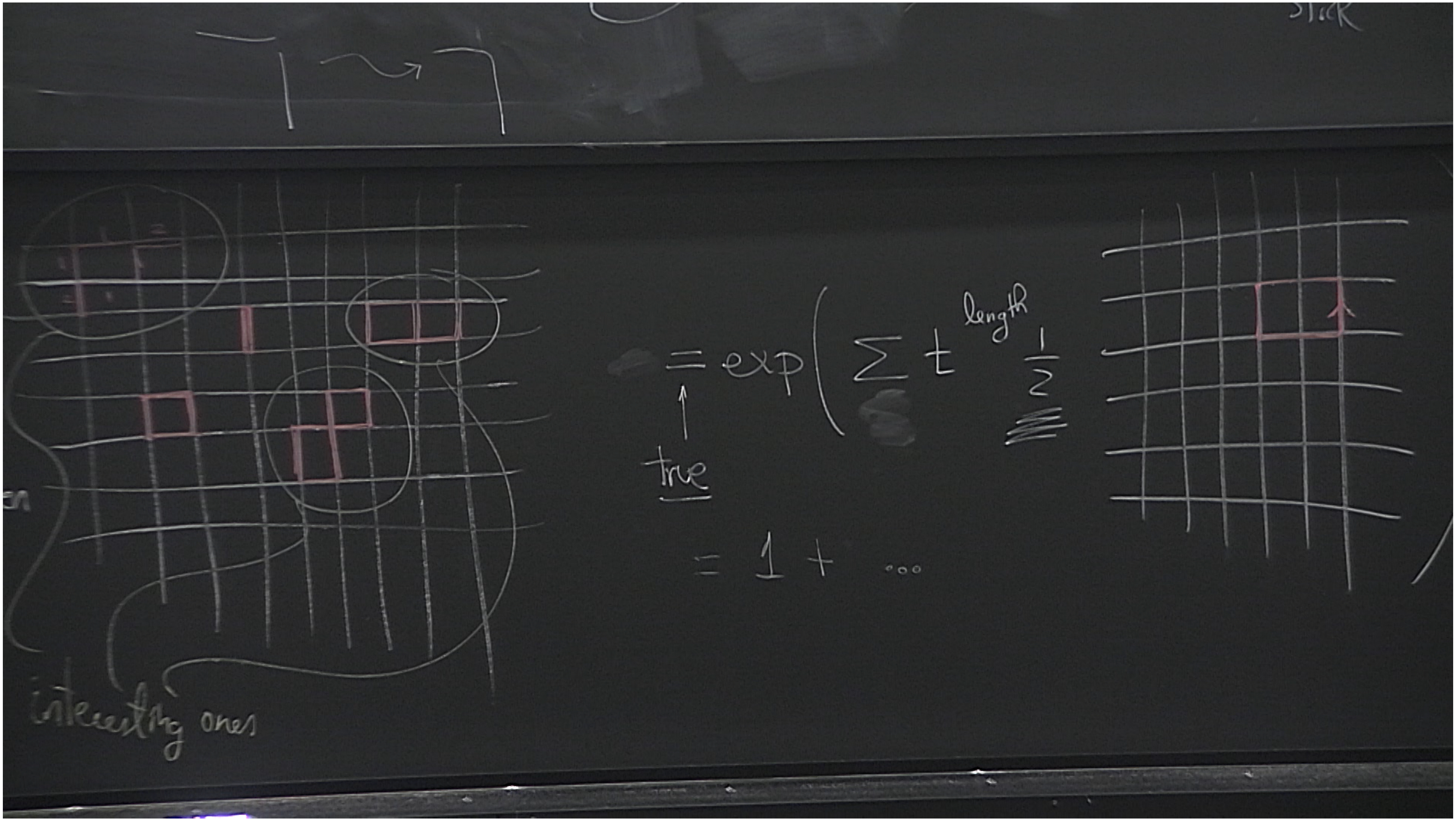


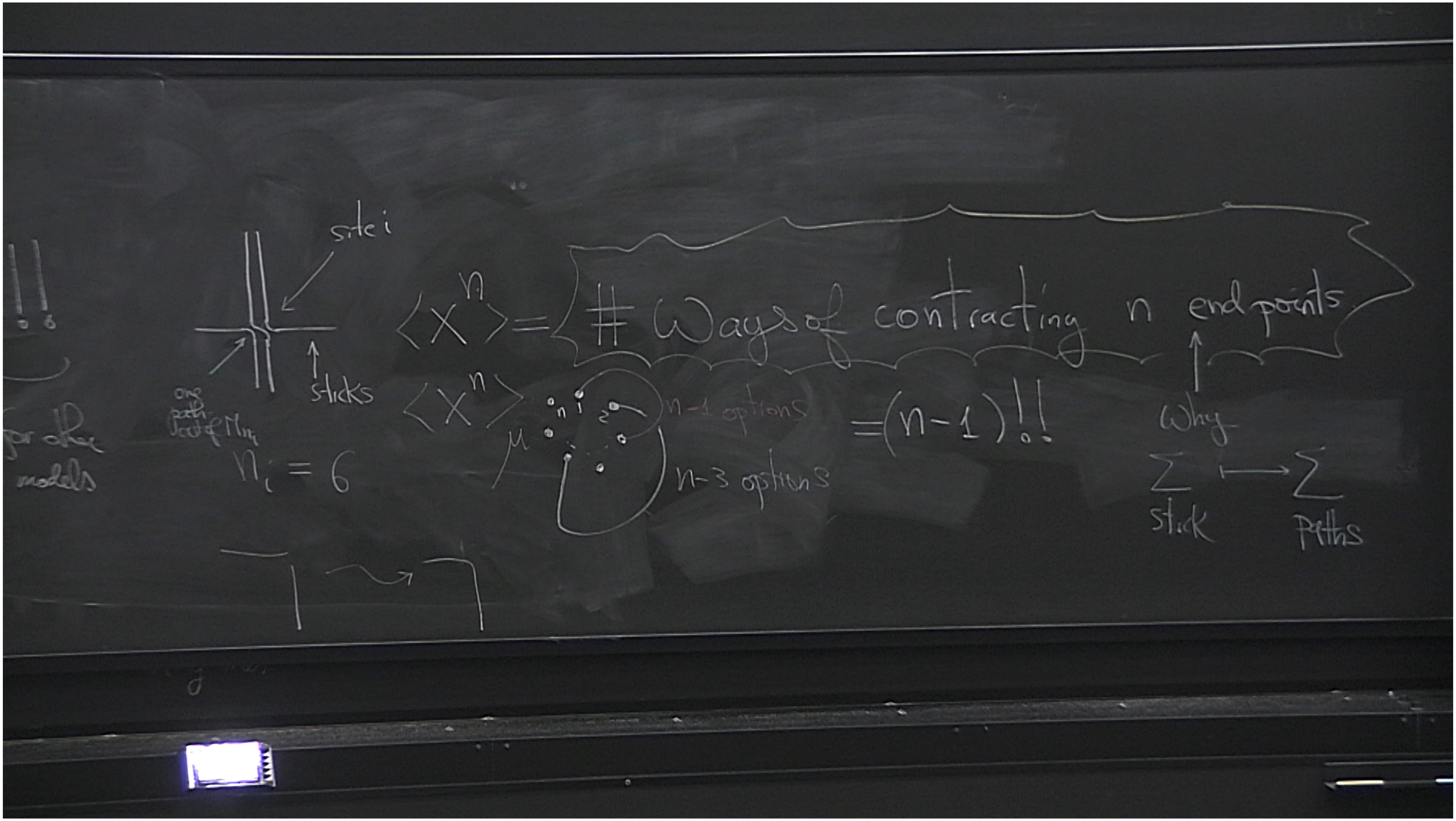
$\sum$
config with n_i even
interesting ones
$= \exp($
$= 1 +$

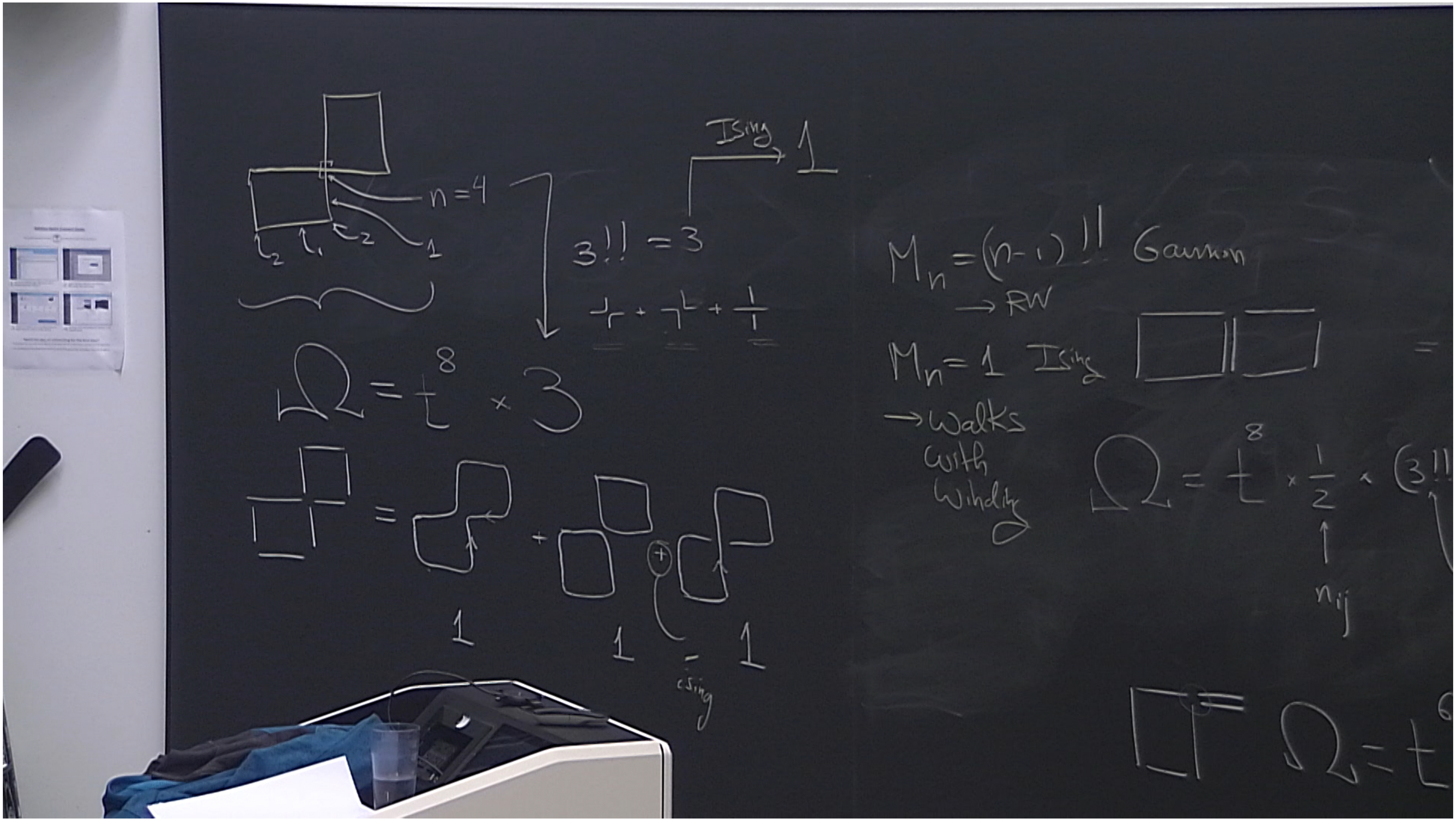

n=4
1
3!! = 3
Ω = t^8 × 3
1
1
1


! = 3
Ω = t^8 × 1/2 × (3!!)^2 = 9/2
n_ij
1
1


1 + 1 + 1
1/2 + 1
Ω = t^8 × 1/2 × (3!!)^2 = 9/2
n_ij
= 9/2
Ω = t^6 × 1/2 × 3 = 1 + 1/2


= exp(Σ t^length 1/2
true
= 1 + ...
interesting ones


site i
sticks
for other models
$n_i = 6$
$\langle X^n \rangle$ = # Ways of contracting n end points
$\langle X^n \rangle$
n-1 options
n-3 options
$= (n-1)!!$
Why
$\sum_{\text{stick}} \longmapsto \sum_{\text{paths}}$


Ising
1
n=4
1
t2 t1 t2
3!! = 3
Ω = t8 × 3
1
1
+
−
1
Ising
Mn = (n-1)!! Gaussian
→ RW
Mn = 1 Ising
→ walks with winding
Ω = t8 × 1/2 × (3!!
nij
Ω = t6