

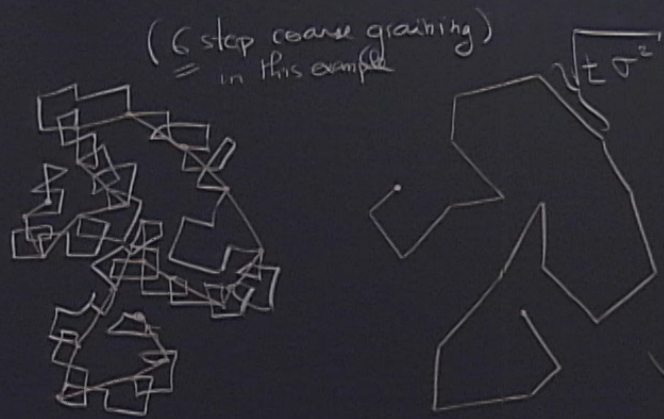
Title: PSI 2019/2020 - Statistical Mechanics (Vieira) - Lecture 4

Speakers: Pedro Vieira

Collection: PSI 2019/2020 - Statistical Mechanics (Vieira)

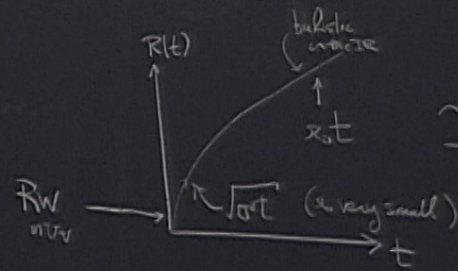
Date: October 10, 2019 - 10:45 AM

URL: <http://pirsa.org/19100027>



rescale to be
able to compare

$$P_{\text{rob}_t}(\vec{R}) = \int \prod_{i=1}^d d\vec{r}_i$$



$$\sim \exp(-)$$

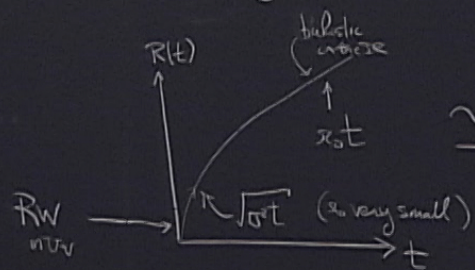
int out / coarse grain

$$R(t) \equiv \left\langle \left(\sum_{i=1}^t \vec{r}_i \right)^2 \right\rangle^{1/2} = \sqrt{t^2 \langle \vec{r}_i^2 \rangle + t (\langle \vec{r}_i^2 \rangle - \langle \vec{r}_i \rangle^2)} = \sqrt{t\sigma^2 + t\sigma_0^2} \rightarrow \text{Gaussian}$$

no sum over i

initial prob dist with moments $c_1, c_2, c_3, c_4, \dots$

$$P_{\text{rob}_t}(\vec{R}) = \int \prod_{i=1}^t d\vec{x}_i \text{prob}(\vec{x}_i) \delta\left(\vec{R} - \sum_{i=1}^t \vec{x}_i\right)$$

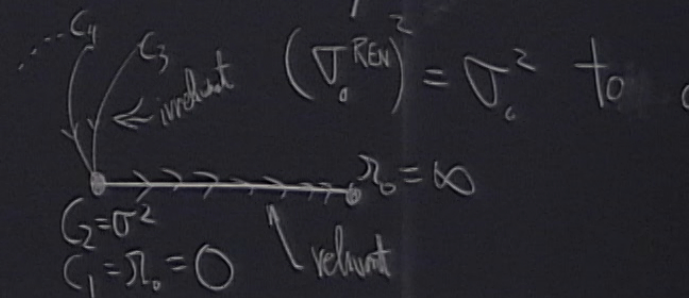


$$\approx \exp\left(-\frac{(\vec{R} - t \vec{x}_0)^2}{t \sigma^2}\right) \quad \xrightarrow{\langle \vec{x} \rangle} \quad \exp\left(-\frac{(\vec{x} - \vec{x}_0^{\text{REN}})^2}{2 \sigma_0^2}\right)$$

$\vec{R} = \sqrt{n} \vec{x}$

$$= \sqrt{t \sigma^2 + t^2 \sigma_0^2}$$

→ Gaussian fixed point in the IR



initial prob dist with moments $C_1, C_2, C_3, C_4, \dots$ in UV

$$P_{\text{rob}_t}(\vec{R}) = \int \frac{t}{\pi} d^d \vec{x}_i \text{prob}(\vec{x}_i) \delta\left(\vec{R} - \sum_{i=1}^t \vec{x}_i\right)$$

$\vec{x}_0^{\text{REN}} = \sqrt{t} \vec{x}_0$
 Ren prob dist with C_1, C_2 and $C_{n \geq 2} = 0$ in the IR

$$\approx \exp\left(-\frac{(\vec{R} - t \vec{x}_0)^2}{t \sigma_0^2}\right) = \exp\left(-\frac{(\vec{x} - \vec{x}_0^{\text{REN}})^2}{2 \sigma_0^2}\right)$$

$\vec{R} = \sqrt{t} \vec{x}$
 $\sigma_0^2 = \langle \vec{x}^2 \rangle - \langle \vec{x} \rangle^2$
 $(\sigma_0^{\text{REN}})^2 = \sigma_0^2$ to compare

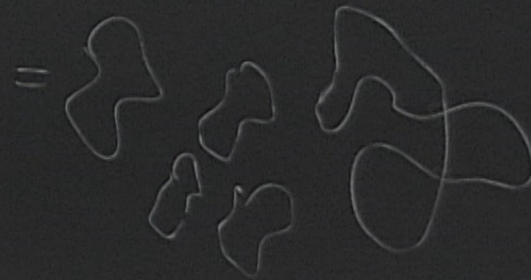
$\sqrt{t} \sigma^2 + t \sigma_0^2$
 $\rightarrow$ Gaussian fixed point in the IR

Diagrams:
 - A graph of $R(t)$ vs t showing a curve starting at the origin. A tangent line at $t=0$ is labeled $\sqrt{t}$ (as very small). A point on the curve is labeled $x_0 t$ with a bracket indicating a ballistic curve.
 - A flow diagram showing a sequence of points C_1, C_2, C_3, C_4 on a horizontal axis. Arrows indicate a flow from C_4 towards C_1 , labeled "invariant". Below the axis, $C_2 = \sigma^2$ and $C_1 = x_0 = 0$ are marked, with an arrow labeled "relevant" pointing towards C_1 .

Today * Gen fn of paths.

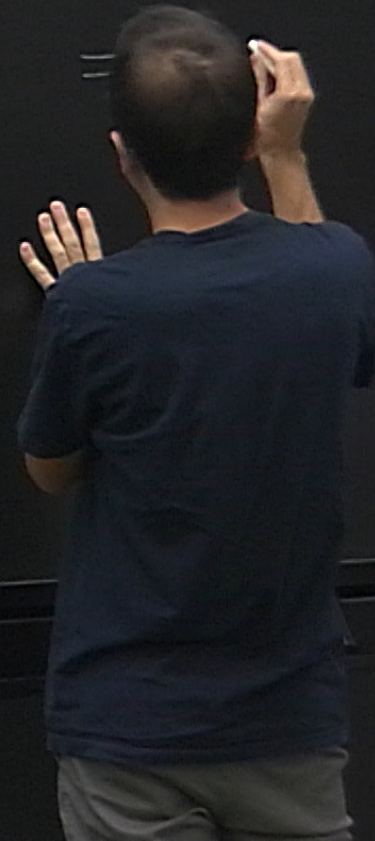
* $P_t(\vec{z})$ exactly and then take $|\vec{z}| \gg 1$

* Compute a partition fn $Z = \sum \text{closed RWs}$



RW gen fn

$$G(\mu, \vec{R}) \equiv \sum_t \mu^t P_t(\vec{R}) =$$



$$= \left(\sum_{\text{length}} e^{-\beta \rho \text{ length}} \right)$$

molecules of length l connecting

total

$\vec{R}$

RW gen fn

$$G(\mu, \vec{R}) \equiv \sum_t \mu^t P_t(\vec{R}) = \left(\sum_{\text{length}} e^{\mu \cdot \text{length}} \right)$$

in RW pit When $\mu = 1$

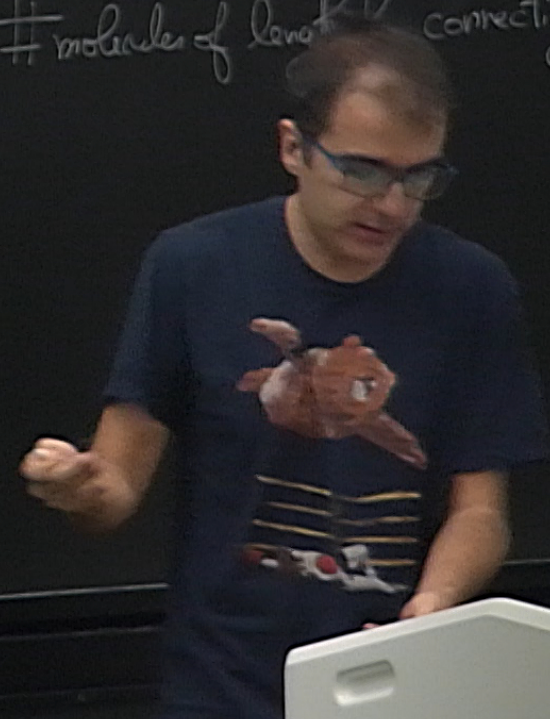
G = average time spent at $\vec{R}$ in an ∞ walk.

$$\sum_t \mu^t P_t(\vec{R}) = \left(\sum_{\text{length}} e^{-\beta \overbrace{E}^{\text{length}}} \right)^{\overbrace{\mu}^{\text{entropy}}}$$

molecules of length l connecting

in 3D $\simeq 1.52$
 $= \infty$ in $d=1, 2$.

an ∞ walk.



$= \infty$ in $d=1,2$

$$G(\mu, \vec{R}) = \sum_{t=0}^{\infty} \mu^t \left[\frac{1}{(2\pi t \sigma^2)^{d/2}} \exp\left(-\frac{R^2}{2t\sigma^2}\right) \right]$$

$\vec{x}_0 = 0$
today.

so in $d=12$.

$$-\frac{R^2}{2t\sigma^2} = \int \frac{d^d \vec{q}}{(2\pi)^d} e^{i\vec{q}\vec{R} - \vec{q}^2 \frac{t\sigma^2}{2}}$$

spot at $\vec{R}$ in an ∞ walk.

$$\ln 3D \simeq 1.52$$

$$= \infty \text{ in } d=1,2.$$

$$\begin{aligned} & \stackrel{\substack{\uparrow \\ \vec{x}_0=0 \\ \text{today.}}}{=} \sum_{t=0}^{\infty} \mu^t \left[\frac{1}{(2\pi t\sigma^2)^{d/2}} \exp\left(-\frac{R^2}{2t\sigma^2}\right) = \int \frac{d^d \vec{q}}{(2\pi)^d} e^{i\vec{q}\vec{R} - \vec{q}^2 \frac{t\sigma^2}{2}} \right] \\ &= \int \frac{d^d \vec{q}}{(2\pi)^d} \frac{e^{i\vec{q}\vec{R}}}{1 - \mu e^{-\vec{q}^2 \sigma^2/2}} \end{aligned}$$

$\vec{R}$ in a walk. $\ln \{D\} \simeq 1.52$
 $= \infty$ in $d=1,2$.

$$\mu^{\pm} \left[\frac{1}{(2\pi t \sigma^2)^{d/2}} \exp\left(-\frac{R^2}{2t\sigma^2}\right) = \int \frac{d^d \vec{q}}{(2\pi)^d} e^{i\vec{q}\vec{R} - \vec{q}^2 \frac{t\sigma^2}{2}} \right]$$

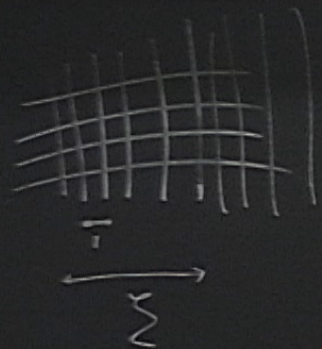
$$= \int \frac{d^d \vec{q}}{(2\pi)^d} \frac{e^{i\vec{q}\vec{R}}}{1 - \mu e^{-\vec{q}^2 \frac{t\sigma^2}{2}}} \underset{\substack{\sim \\ \text{large } \vec{R}}}{=} \# \int \frac{d^d \vec{q}}{(2\pi)^d} \frac{e^{i\vec{q}\vec{R}}}{\vec{q}^2 + m^2}$$

$$= \infty \text{ in } d=1, 2.$$

$$P\left(-\frac{R^2}{2\pm\sigma^2}\right) = \int \frac{d^d \vec{q}}{(2\pi)^d} e^{i\vec{q}\vec{R} - \vec{q}^2 \frac{\pm\sigma^2}{2}}$$

$\frac{e^{i\vec{q}\vec{R}}}{-i e^{-\vec{q}^2 \frac{\pm\sigma^2}{2}}} \approx \# \int \frac{d^d \vec{q}}{(2\pi)^d} \frac{e^{i\vec{q}\vec{R}}}{\vec{q}^2 + \underbrace{\frac{m^2}{2}}_{\substack{\text{inverse cor length} \\ m^2(K, \sigma)}}} \sim e^{-m|\vec{R}|}$

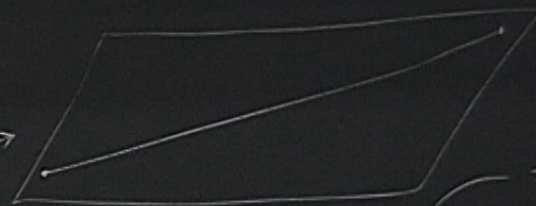
universality
 at large $\vec{R}$
 $|\vec{R}| \gg \frac{1}{m}$
 large $\vec{R}$ ($\vec{q}$ small)
 $|\vec{R}| \gg \text{lattice spacing}$



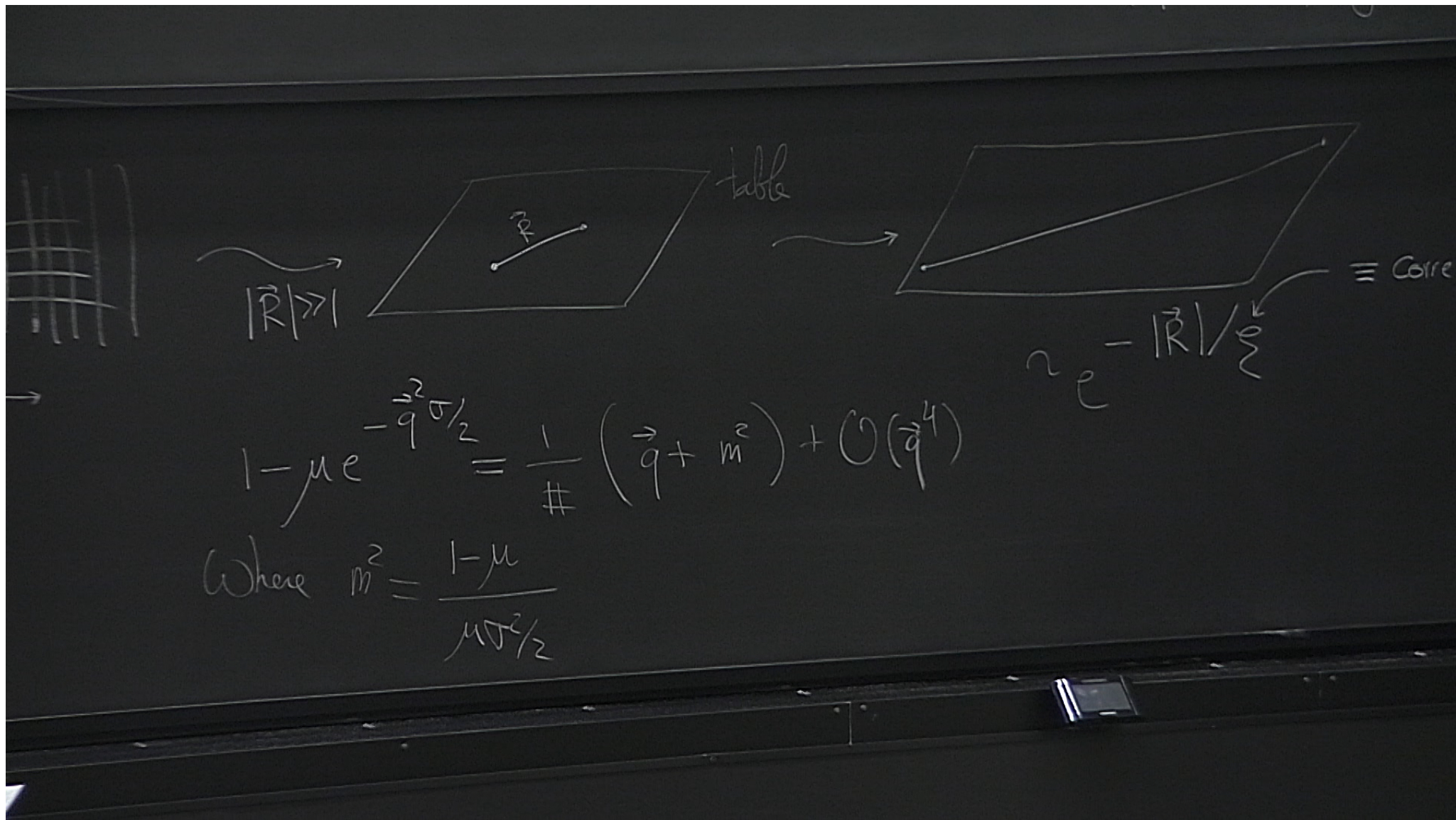
$|\vec{R}| \gg 1$



table



$\equiv$ correlation length



$$P(\vec{r}) = \begin{cases} p_i & \text{if } \vec{r} = \vec{r}_i \\ 0 & \text{otherwise} \end{cases} \quad \left(\sum_i p_i = 1 \right)$$

e.g. $p_i = \frac{1}{2d}$ 

if $r_i = \pm \hat{e}_\mu$

$$P_{t+1}(\vec{r}) = \sum_i p_i P_t(\vec{r} - \vec{r}_i)$$

$$P_0(\vec{r}) = \delta_{\vec{r}, \vec{0}}$$

$$\sum_{t=0}^{\infty} \mu^t (eq)$$

$$RHS = \sum_i p_i G(\mu, \vec{R} - \vec{z}_i)$$

$$LHS = \frac{1}{\mu} \left[G(\mu, \vec{R}) - \delta_{\vec{R}, \vec{0}} \right]$$

$$\sum_{\vec{R}} A_{\vec{R}} G(\mu, \vec{R})$$

$$\sum_{\vec{R}} A_{\vec{R}, \vec{R}'} \cdot \underbrace{G(\mu, \vec{R}')}_{\text{vector}} = \delta_{\vec{R}, 0}$$

matrix

$$A_{\vec{R}, \vec{R}'} = \delta_{\vec{R}, \vec{R}'} - \sum_i \mu P_i \delta_{\vec{R}, \vec{R}' - \vec{\pi}_i}$$

$$\sum_{\vec{R}} A_{\vec{R}, \vec{R}'} \cdot \underbrace{G(\mu, \vec{R})}_{\text{vector}} = \delta_{\vec{R}, \vec{0}} \quad , \quad G = \begin{pmatrix} A^{-1} \end{pmatrix}_{\vec{R} \vec{R}'} \delta_{\vec{R}', \vec{0}} \\ = \begin{pmatrix} A^{-1} \end{pmatrix}_{\vec{R} \vec{0}}$$

$$A_{\vec{R}, \vec{R}'} = \delta_{\vec{R}, \vec{R}'} - \sum_i \mu P_i \delta_{\vec{R}, \vec{R}' + \vec{\pi}_i}$$

$$\sum_{\vec{R}} A_{\vec{R}, \vec{R}'} \cdot \underbrace{G(\mu, \vec{R})}_{\text{vector}} = \delta_{\vec{R}, \vec{0}} \quad , \quad G_{\vec{R} \leftarrow \vec{R}''} = \begin{pmatrix} A^{-1} \end{pmatrix}_{\vec{R} \vec{R}'} \delta_{\vec{R}', \vec{0}} \quad \begin{matrix} \uparrow \\ \vec{R}'' \end{matrix} \\
 \qquad \qquad \qquad \underbrace{\qquad}_{\text{matrix}} \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \begin{matrix} \vec{R}' \leftarrow \vec{R}'' \\ \uparrow \\ \vec{0} \end{matrix} \\
 \qquad \qquad \qquad G(\mu, \vec{R}' \leftarrow \vec{R}'') \\
 A_{\vec{R}, \vec{R}'} = \delta_{\vec{R}, \vec{R}'} - \sum_i \mu P_i \delta_{\vec{R}, \vec{R}' + \vec{x}_i} \quad \leftarrow \text{translation invariant}$$

$$\mu [G(\mu, \vec{R}) \circ \vec{R}, \vec{0}]$$

$$A_{\vec{R}, \vec{R}'} = \delta_{\vec{R}, \vec{R}'}$$

$$G = A^{-1}$$

check

claim : $G(\mu, \vec{R}) = \int \frac{d^d \vec{q}}{(2\pi)^d} \frac{e^{i\vec{q} \cdot \vec{R}}}{1 - \mu \sum_i P_i e^{i\vec{q} \cdot \vec{r}_i}}$

$|\vec{q}| \leq \pi \leftarrow \text{Brillouin Zone}$

$$A_{\vec{R}, \vec{R}'} = \delta_{\vec{R}, \vec{R}'} - \sum_i \mu P_i \delta_{\vec{R}, \vec{R} + \vec{r}_i} \leftarrow \text{translation invariant}$$

$$P(\vec{r}) = \begin{cases} P_i & \text{if } \vec{r} = \vec{r}_i \\ 0 & \text{otherwise} \end{cases} \quad \left(\sum_i P_i = 1 \right)$$

$\vec{r} \in \text{cubic lattice}$ vectors w/ integer entries (eg $\vec{r}_i = (1, 0, 0, \dots)$)



e.g. $P_i = \frac{1}{2d}$

if $r_i = \pm \hat{e}_\mu$

$$P_{t+1}(\vec{r}) = \sum_i P_i P_t(\vec{r} - \vec{r}_i)$$

$$P_0(\vec{r}) = \delta_{\vec{r}, \vec{0}}$$

$$A_{\vec{R}, \vec{R}'} = \delta_{\vec{R}, \vec{R}'} - \sum_i \mu P_i \delta_{\vec{R}, \vec{R}' + \vec{r}_i} \leftarrow \text{translation invariant}$$

check $\sum_{\vec{R}'} A_{\vec{R}\vec{R}'} G_{\vec{R}'}$

$$\int \frac{d^d q}{(2\pi)^d} \left(\underbrace{\sum_{\vec{R}'} \delta_{\vec{R}, \vec{R}'} - \mu \sum_i P_i \delta_{\vec{R}, \vec{R}' + \vec{r}_i}}_{e^{i\vec{q}\cdot\vec{R}} - \mu \sum_i P_i e^{+i\vec{q}\cdot\vec{R} - i\vec{q}\cdot\vec{r}_i}} \right) e^{i\vec{q}\cdot\vec{R}'} \frac{1}{1 - \mu \sum_i P_i e^{i\vec{q}\cdot\vec{r}_i}}$$

$$A_{\vec{R}, \vec{R}'} = \delta_{\vec{R}, \vec{R}'} - \sum_i \mu P_i \delta_{\vec{R}, \vec{R}'} - \dots$$

$$\vec{R}) = \int \frac{d^d \vec{q}}{(2\pi)^d} \frac{e^{i\vec{q} \cdot \vec{R}}}{1 - \mu \sum_i P_i e^{i\vec{q} \cdot \vec{R}_i}}$$

$|\vec{q}| \leq \pi \leftarrow \text{Brillouin Zone}$

∇

check $\sum_{\vec{R}'} A_{\vec{R}, \vec{R}'} G_{\vec{R}'}$

$\int \frac{d^d q}{(2\pi)^d} \left(\sum_{\vec{R}'} \delta_{\vec{R}, \vec{R}'} - \mu \sum_i P_i \delta_{\vec{R}, \vec{R} + \vec{R}_i} \right)$

$e^{i\vec{q} \cdot \vec{R}} - \mu \sum_i P_i e^{i\vec{q} \cdot \vec{R} + i\vec{q} \cdot \vec{R}_i}$

$e^{i\vec{q} \cdot \vec{R}} (1 - \mu \sum_i P_i e^{i\vec{q} \cdot \vec{R}_i})$

$$A_{\vec{R}, \vec{R}'} = \delta_{\vec{R}, \vec{R}'} - \sum_i \mu P_i \delta_{\vec{R}, \vec{R} + \vec{r}_i} \leftarrow \text{translation invariant}$$

check $\sum_{\vec{R}'} A_{\vec{R}\vec{R}'} G_{\vec{R}'}$

$$\int \frac{d^d q}{(2\pi)^d} \left(\sum_{\vec{R}'} \delta_{\vec{R}, \vec{R}'} - \mu \sum_i P_i \delta_{\vec{R}, \vec{R} + \vec{r}_i} \right) e^{i\vec{q} \cdot \vec{R}'} \frac{1}{1 - \mu \sum_i P_i e^{-i\vec{q} \cdot \vec{r}_i}}$$

$$e^{i\vec{q} \cdot \vec{R}} - \mu \sum_i P_i e^{+i\vec{q} \cdot \vec{R} - i\vec{q} \cdot \vec{r}_i}$$

$$e^{i\vec{q} \cdot \vec{R}} (1 - \mu \sum_i P_i e^{-i\vec{q} \cdot \vec{r}_i})$$

$$A_{\vec{R}, \vec{R}'} = \delta_{\vec{R}, \vec{R}'} - \sum_i \mu P_i \delta_{\vec{R}, \vec{R} + \vec{r}_i} \leftarrow \text{translation invariant}$$

check $\sum_{\vec{R}'} A_{\vec{R}\vec{R}'} G_{\vec{R}'} = 1 \delta_{\vec{R}, \vec{0}} \leftarrow \vec{R} \in \mathbb{Z}^d$
 $\uparrow$
 $1, \text{ lattice spacing}$

$$\int \frac{d^d \vec{q}}{(2\pi)^d}$$

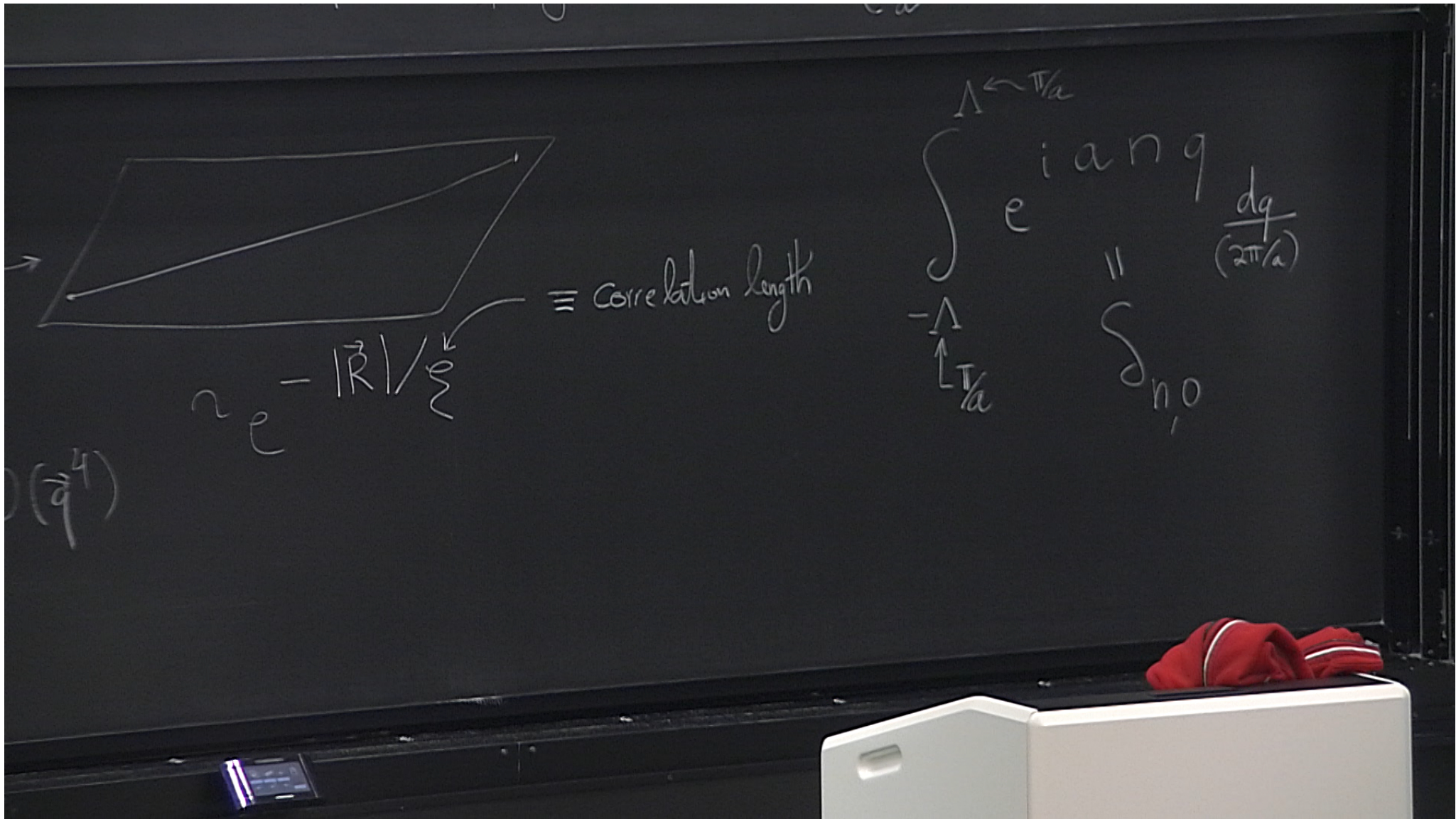
$$\left(\sum_{\vec{R}'} \delta_{\vec{R}, \vec{R}'} - \mu \sum_i P_i \delta_{\vec{R}, \vec{R} + \vec{r}_i} \right) e^{i\vec{q} \cdot \vec{R}'}$$

$$1 - \mu \sum_i P_i e^{-i\vec{q} \cdot \vec{r}_i}$$

$$e^{i\vec{q} \cdot \vec{R}} - \mu \sum_i P_i e^{+i\vec{q} \cdot \vec{R} - i\vec{q} \cdot \vec{r}_i}$$

$$e^{i\vec{q} \cdot \vec{R}} \left(1 - \mu \sum_i P_i e^{-i\vec{q} \cdot \vec{r}_i} \right)$$





$$G(\mu, \vec{R}) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{1 - \mu \sum_i P_i e^{-i\vec{q} \cdot \vec{r}_i}} \left[\frac{1}{(2\pi)^d} \left(\sum_{\vec{R}'} \delta_{\vec{R}, \vec{R}'} - \mu \sum_i P_i \delta_{\vec{R}, \vec{R} + \vec{r}_i} \right) e^{i\vec{q} \cdot \vec{R}} \right]$$

$|\vec{q}| \leq \pi \leftarrow \text{Brillouin Zone}$

$$e^{i\vec{q} \cdot \vec{R}} \left(1 - \mu \sum_i P_i e^{-i\vec{q} \cdot \vec{r}_i} \right)$$

$\vec{R} \gg 1$, $\vec{q}$ is small

$$1 - \mu \sum_i P_i e^{-i\vec{q} \cdot \vec{r}_i} = (1 - \mu) + \mu \sum_i P_i \underbrace{\pi_i^\mu \pi_i^\nu}_{\substack{\text{no sum over } i \\ \text{C}_2^{\mu\nu} = \sigma^2 \delta^{\mu\nu}}} q_\mu q_\nu$$

1) $\sum_i P_i = 1$
 2) $\sum_i P_i \vec{r}_i = \vec{r}_0 = 0$

$$1 - i\vec{q} \cdot \vec{r}_i - \frac{(\vec{q} \cdot \vec{r}_i)^2}{2} + \dots$$

$$|q_\mu| \ll \pi \leftarrow \text{Brillouin Zone}$$

$$\vec{R} \gg 1, \quad \vec{q} \text{ is small}$$

$$1 - \mu \sum_i p_i e^{-i\vec{q} \cdot \vec{R}_i} = (1 - \mu) + \frac{\mu}{2}$$

$$1) \sum_i p_i = 1$$

$$2) \sum_i p_i \vec{R}_i = \vec{R}_0 = 0$$

$$1 - i\vec{q} \cdot \vec{R}_i - \frac{(\vec{q} \cdot \vec{R}_i)^2}{2} + \dots$$

$\Gamma \leftarrow$ Brillouin Zone $\rightarrow$ 

$$e^{i\vec{q}\cdot\vec{R}} \left(1 - \mu \sum_i p_i e^{-i\vec{q}\cdot\vec{R}_i} \right)$$

small

$$i\vec{q}\cdot\vec{R}_i = (1-\mu) + \underbrace{\mu \sum_i p_i \pi_i^\mu \pi_i^\nu}_{C_2^{\mu\nu} = \sigma^2 \delta^{\mu\nu}} q_\mu q_\nu = \# \left(\vec{q}^2 + \dots \right)$$

$$-i\vec{q}\cdot\vec{R}_i - \frac{(\vec{q}\cdot\vec{R}_i)^2}{2} + \dots$$

$$i\vec{q} \cdot \vec{R} \left(1 - \mu \sum_i p_i e^{-i\vec{q} \cdot \vec{R}_i} \right)$$



$$q_\mu q_\nu = \# \left(\vec{q}^2 + \underbrace{m^2}_{\approx 0} \right) \quad \text{QED} \quad \square$$

$|\vec{R}| \gg \text{lattice spacing}$

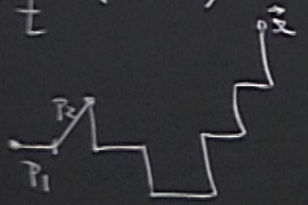
Another derivation:

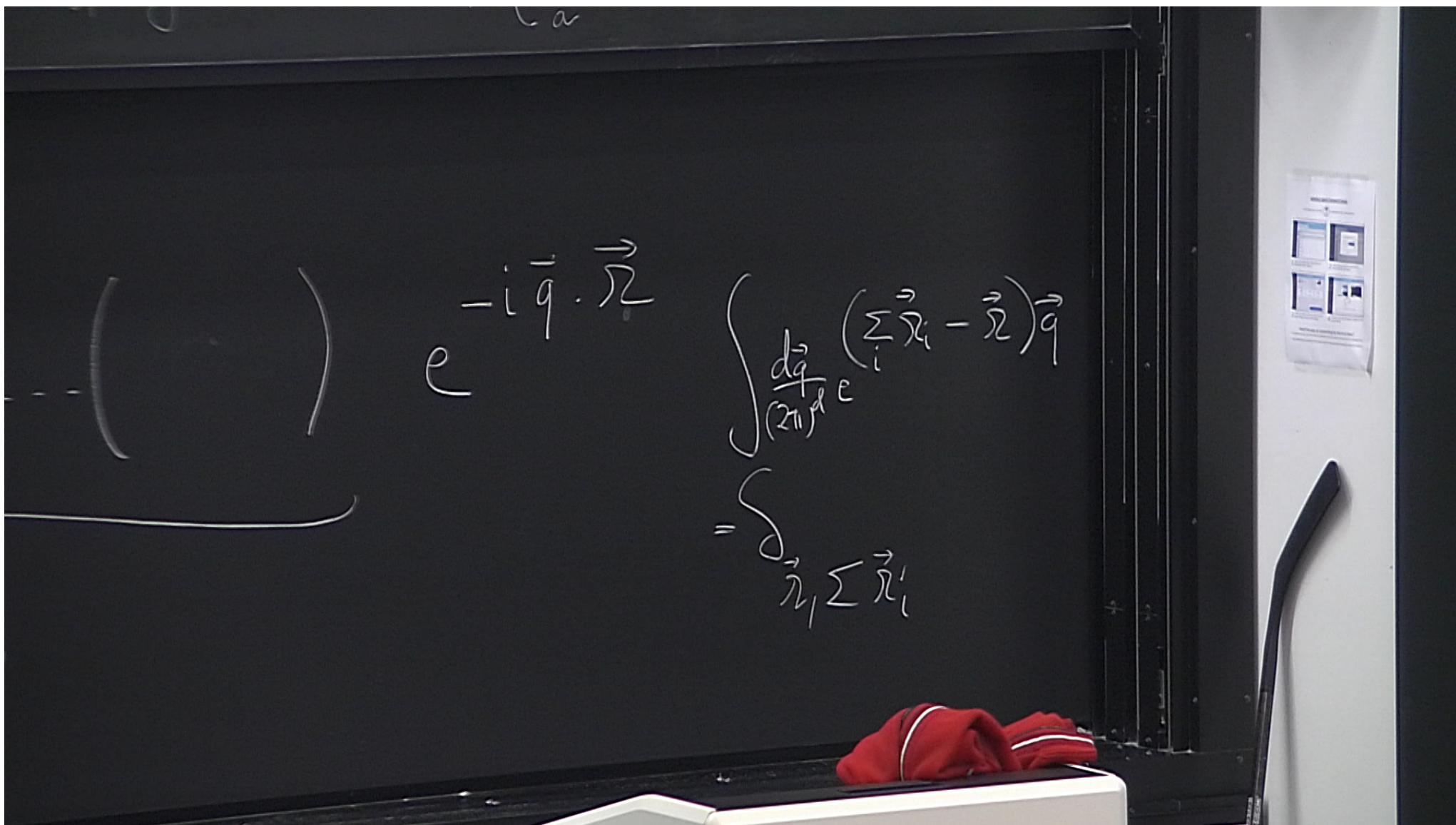
$$P_t(\vec{r}) = \int_{\text{BZ}} \frac{d^d \vec{q}}{(2\pi)^d} \underbrace{\left(\sum_i p_i e^{i\vec{q} \cdot \vec{r}_i} \right) \left(\sum p_i e^{i\vec{q} \cdot \vec{r}_2} \right) \dots \left(\right)}_{t \text{ factors}}$$

$|\vec{R}| \gg \text{lattice spacing}$

Another derivation:

$$P_t(\vec{r}) = \int_{\text{BZ}} \frac{d^d \vec{q}}{(2\pi)^d} \underbrace{\left(\sum_i p_i e^{i\vec{q} \cdot \vec{r}_i} \right)}_{\text{all options for step 1}} \underbrace{\left(\sum p_i e^{i\vec{q} \cdot \vec{r}_i} \right) \dots \left(\right)}_{\substack{\text{step 2} \\ t \text{ factors}}}$$





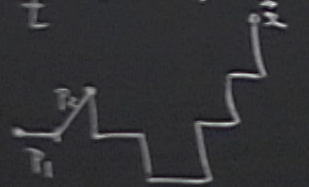
today.

$$(2\pi)^d \left(1 - \mu e^{-q^2/2} \right)$$

large $\vec{R}$ ($\vec{q}$ small)
 $|\vec{R}| \gg \text{lattice spacing}$

Another derivation:

$$P_t(\vec{R}) = \int_{\text{BZ}} \frac{d^d \vec{q}}{(2\pi)^d} \underbrace{\left(\sum_i p_i e^{i\vec{q} \cdot \vec{r}_i} \right)}_{\substack{\text{all options for step 1} \\ \parallel \\ A^t}} \underbrace{\left(\sum p_i e^{i\vec{q} \cdot \vec{r}_i} \right) \dots \left(\quad \right)}_{\substack{\text{step 2} \\ t \text{ factors}}} e^{i\vec{q} \cdot \vec{R}}$$



$$G = \int \frac{d^d q}{(2\pi)^d} e^{-i\vec{q} \cdot \vec{R}} \sum p$$

$$e^{i\vec{q} \cdot \vec{x}_1} \dots e^{-i\vec{q} \cdot \vec{x}_l}$$

$$t \text{ factors}$$

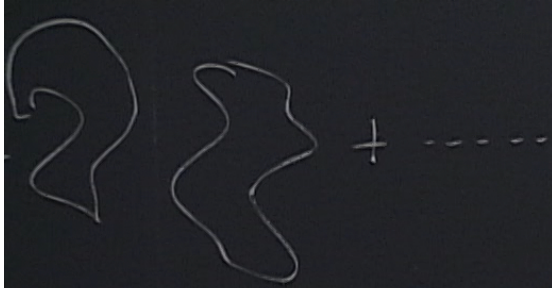
$$G = \int \frac{d^d q}{(2\pi)^d} e^{i\vec{q} \cdot \vec{x}} \underbrace{\frac{1}{1-\mu A}}_{\sum_{\mu}^t A^t} \overset{\#}{\substack{\vec{q} \rightarrow -\vec{q}}} \int \frac{d^d q}{(2\pi)^d} e^{(\sum_i \vec{x}_i - \vec{x}) \cdot \vec{q}}$$

$$= \delta_{\vec{x}, \sum \vec{x}_i}$$

Closed Paths

$$Z = 1 + \{\text{loop}\} + \frac{1}{2!} \{\text{two loops}\} + \dots$$

$$= \exp\left(\sum \{\text{loop}\}\right)$$



$$= \exp \left[\sum_{t=0}^{\infty} \sum_{\vec{r}} P_t(\vec{r} \rightarrow \vec{r}) (2d)^t \mu^t \right]$$

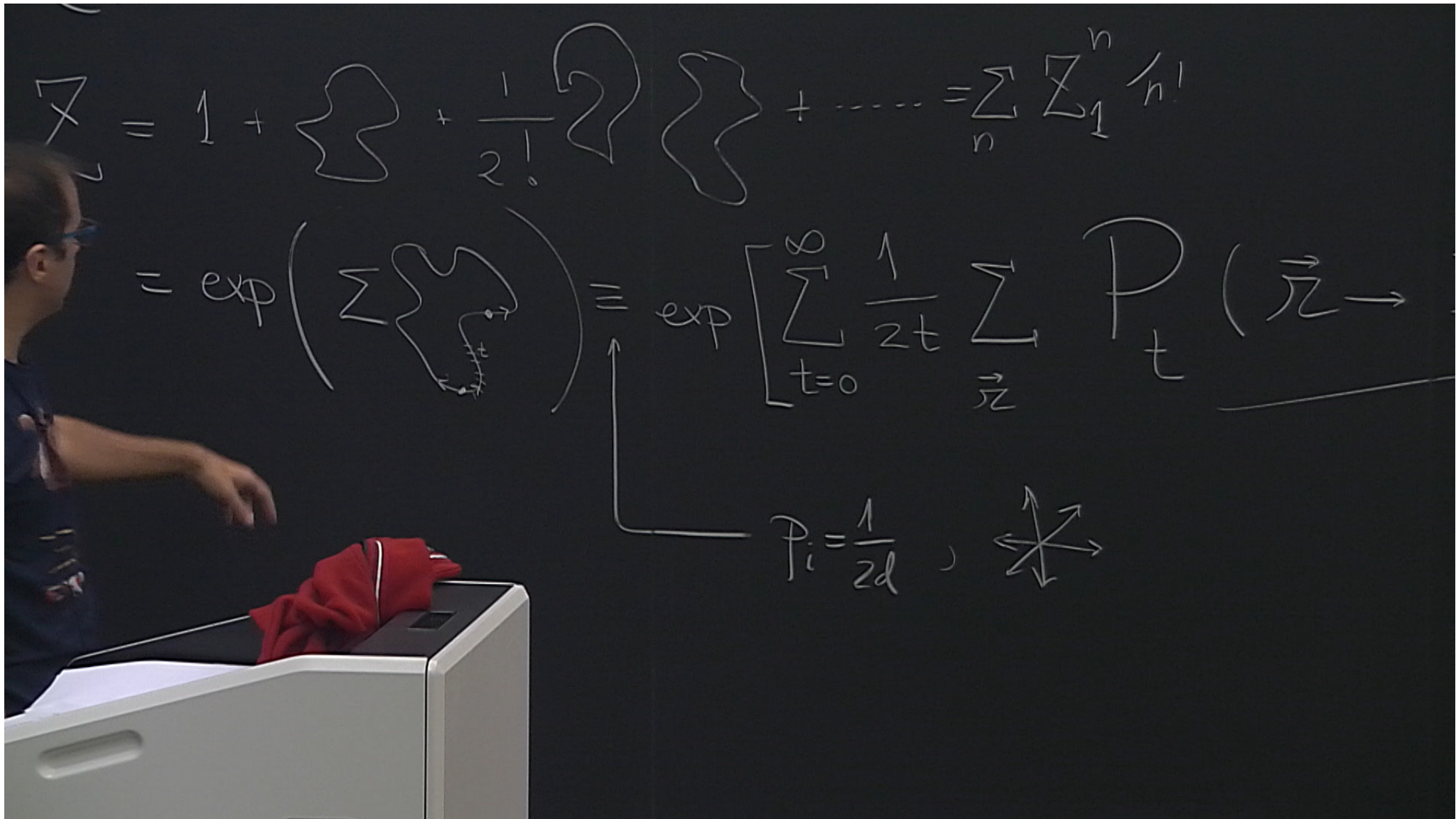
$\underbrace{\hspace{10em}}_{\Omega_t(\vec{R} \rightarrow \vec{R})}$

$P_i = \frac{1}{2d}$,

$$Z = 1 + \left\{ \text{loop} \right\} + \frac{1}{2!} \left\{ 2 \text{ loops} \right\} + \dots$$

$$= \exp \left(\sum \left\{ \text{loop with } t \text{ vertices} \right\} \right) \equiv \exp \left[\sum_{t=0}^{\infty} \frac{1}{2^t} \sum_{\vec{z}} P_t(\vec{z} \rightarrow \dots) \right]$$

$$P_i = \frac{1}{2d}, \quad \text{with a star graph diagram}$$



$$G = \int \frac{d^d q}{(2\pi)^d} e^{-i \vec{q} \cdot \vec{r}} \underbrace{\frac{1}{1 - \mu A}}_{\sum_{\mu}^t A^t} = \# \int \frac{d^d q}{(2\pi)^d} e^{-i \vec{q} \cdot \vec{r}} \sum_{\mu}^t A^t$$

$$= \int \frac{d^d q}{(2\pi)^d} e^{-i \vec{q} \cdot \vec{r}} \left(\sum_i \vec{r}_i - \vec{r} \right) \vec{q}$$

$$= \int \frac{d^d q}{(2\pi)^d} e^{-i \vec{q} \cdot \vec{r}} \sum_i \vec{r}_i$$

Closed Paths

$$Z = 1 + \left\{ \frac{1}{2!} \right\} + \dots$$

$$= \exp \left(\sum_i \vec{r}_i \right)$$

$$\frac{F}{V} = \frac{1}{2} \int \frac{d^d \vec{q}}{(2\pi)^d} \log \left(1 - \overbrace{2d\mu \sum_{\nu, \pm} \frac{1}{2d} e^{\pm i q_\nu}}^{\#R} \right)$$

FREE ENERGY FOR CLOSED LOOP PARTITION FUNCTION

$$\log \left(1 - 2\mu \sum_{\nu=x,y,z} \cos q_\nu \right) \quad d=3$$

$$\log \left[1 - 2\mu [\cos q_x + \cos q_y + \cos q_z] \right]$$