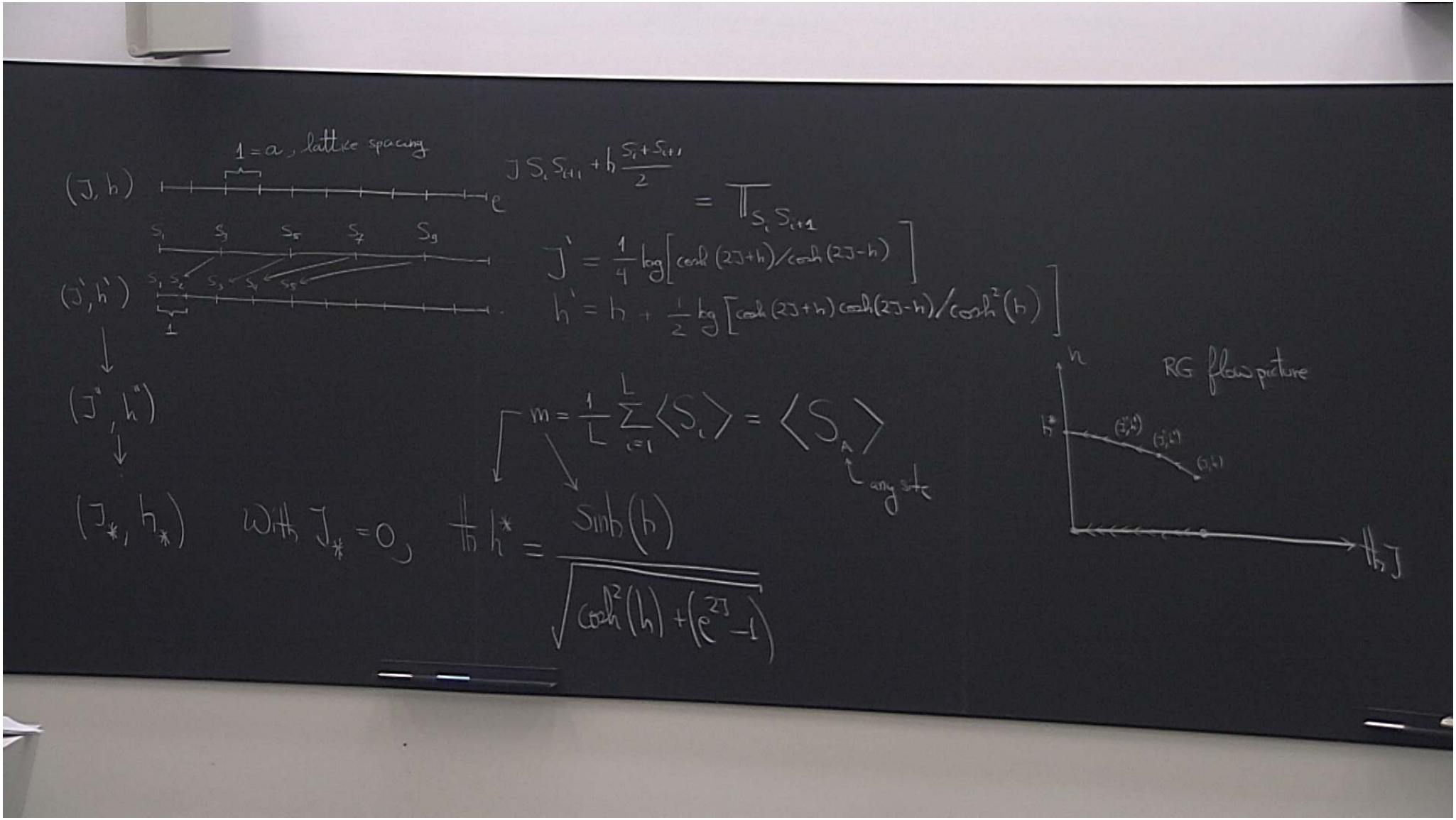
Title: PSI 2019/2020 - Statistical Mechanics (Vieira) - Lecture 3

Speakers: Pedro Vieira

Collection: PSI 2019/2020 - Statistical Mechanics (Vieira)

Date: October 08, 2019 - 10:45 AM

URL: http://pirsa.org/19100025


1 = a, lattice spacing
(J, h)
(J', h')
(J*, h*)
(J*, h*) With J* = 0, tanh h* = sinh(h) / sqrt(cosh²(h) + (e^{2J} − 1))
J S_i S_{i+1} + h (S_i + S_{i+1})/2 = T_{S_i S_{i+1}}
J' = (1/4) log[cosh(2J+h)/cosh(2J−h)]
h' = h + (1/2) log[cosh(2J+h) cosh(2J−h)/cosh²(h)]
m = (1/L) Σ_{i=1}^{L} ⟨S_i⟩ = ⟨S_A⟩
any site
RG flow picture
h
h*
h, J

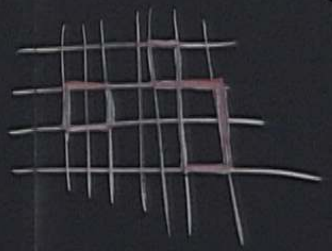
$$Z_{\text{Ising}} = \sum_{\{S_i = \pm 1\}} \prod_{\langle i,j \rangle} \left(1 + v\, S_i S_j\right)$$

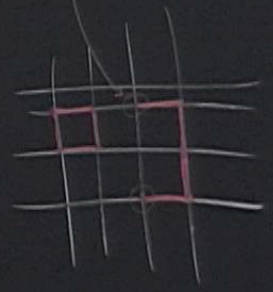
$$= \sum_{\text{sticks}} 1 \quad \text{if } \nexists \text{ open ends}$$

(0 otherwise)

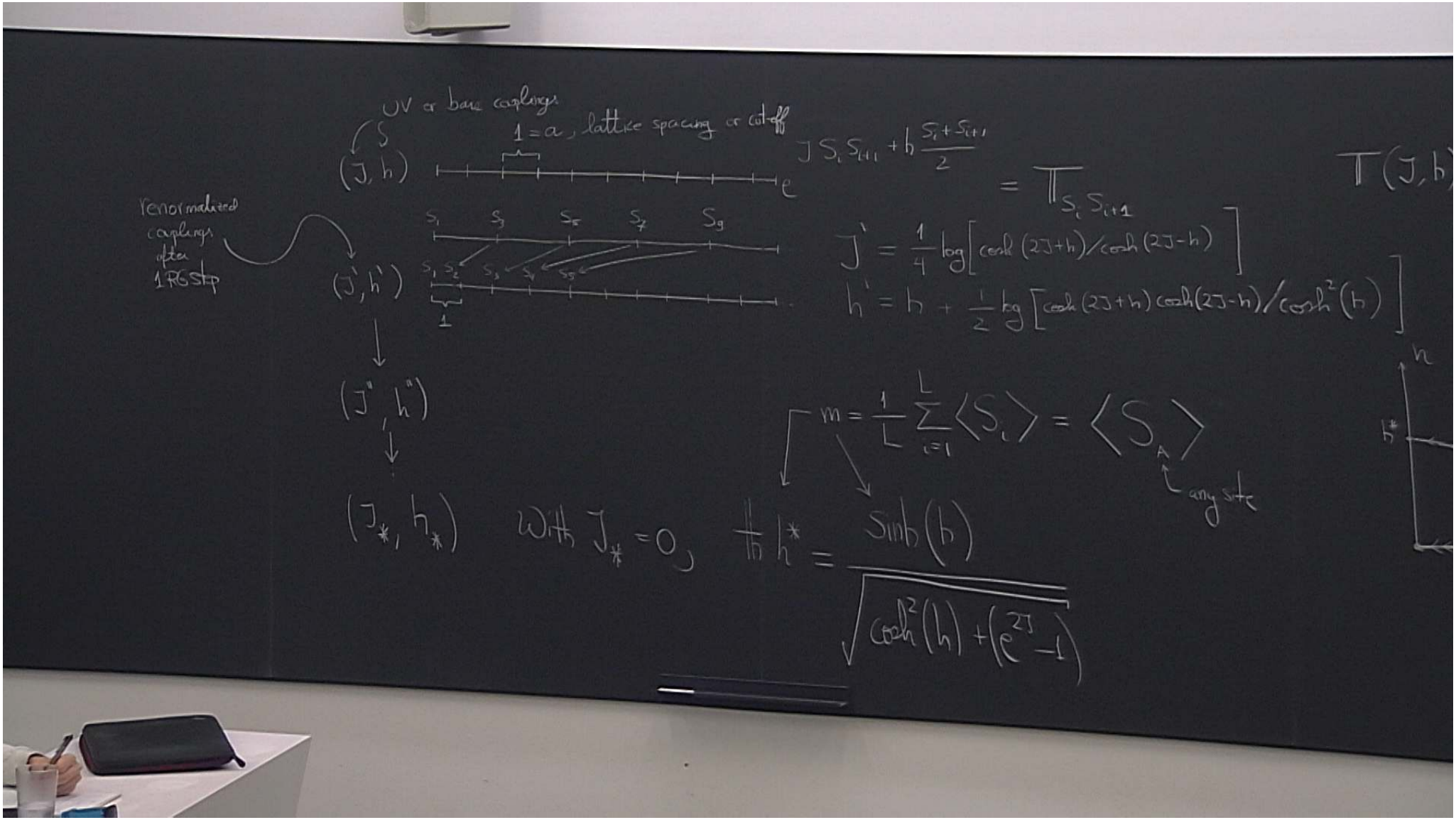
$\Sigma S_a = 0$

(— or — @ each link)





$$= Z_{\text{loops with orientations}}$$


UV or bare couplings
1 = a, lattice spacing or cut-off
(J, h)
renormalized couplings after 1 RG step
(J', h')
1
(J'', h'')
(J*, h*) With J* = 0,
$J S_i S_{i+1} + h \frac{S_i + S_{i+1}}{2} = T_{S_i S_{i+1}}$
$T(J, h)$
$J' = \frac{1}{4} \log\left[\cosh(2J+h)/\cosh(2J-h)\right]$
$h' = h + \frac{1}{2} \log\left[\cosh(2J+h)\cosh(2J-h)/\cosh^2(h)\right]$
$m = \frac{1}{L}\sum_{i=1}^{L}\langle S_i\rangle = \langle S_A\rangle$
any site
$h^* = \frac{\sinh(h)}{\sqrt{\cosh^2(h) + (e^{2J} - 1)}}$
h
h*

$$\frac{S_i + S_{i+1}}{2} = T_{S_i S_{i+1}}$$

$$\left[T(J,h) \cdot T(J,h) = r\, T(J',h') \right]$$

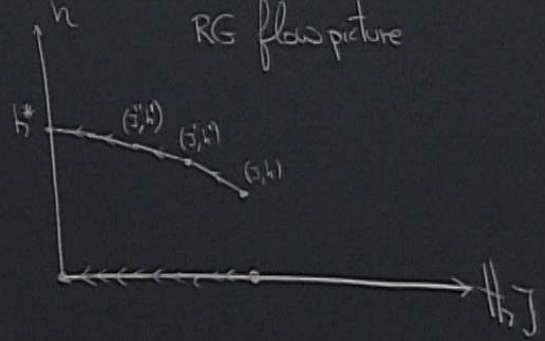
$$\frac{1}{4}\log\left[\cosh(2J+h)/\cosh(2J-h)\right]$$

$$+ \frac{1}{2}\log\left[\cosh(2J+h)\cosh(2J-h)/\cosh^2(h)\right]$$

$$\sum_{i=1}^{L}\langle S_i\rangle = \langle S_A\rangle$$

A: any site

RG flow picture

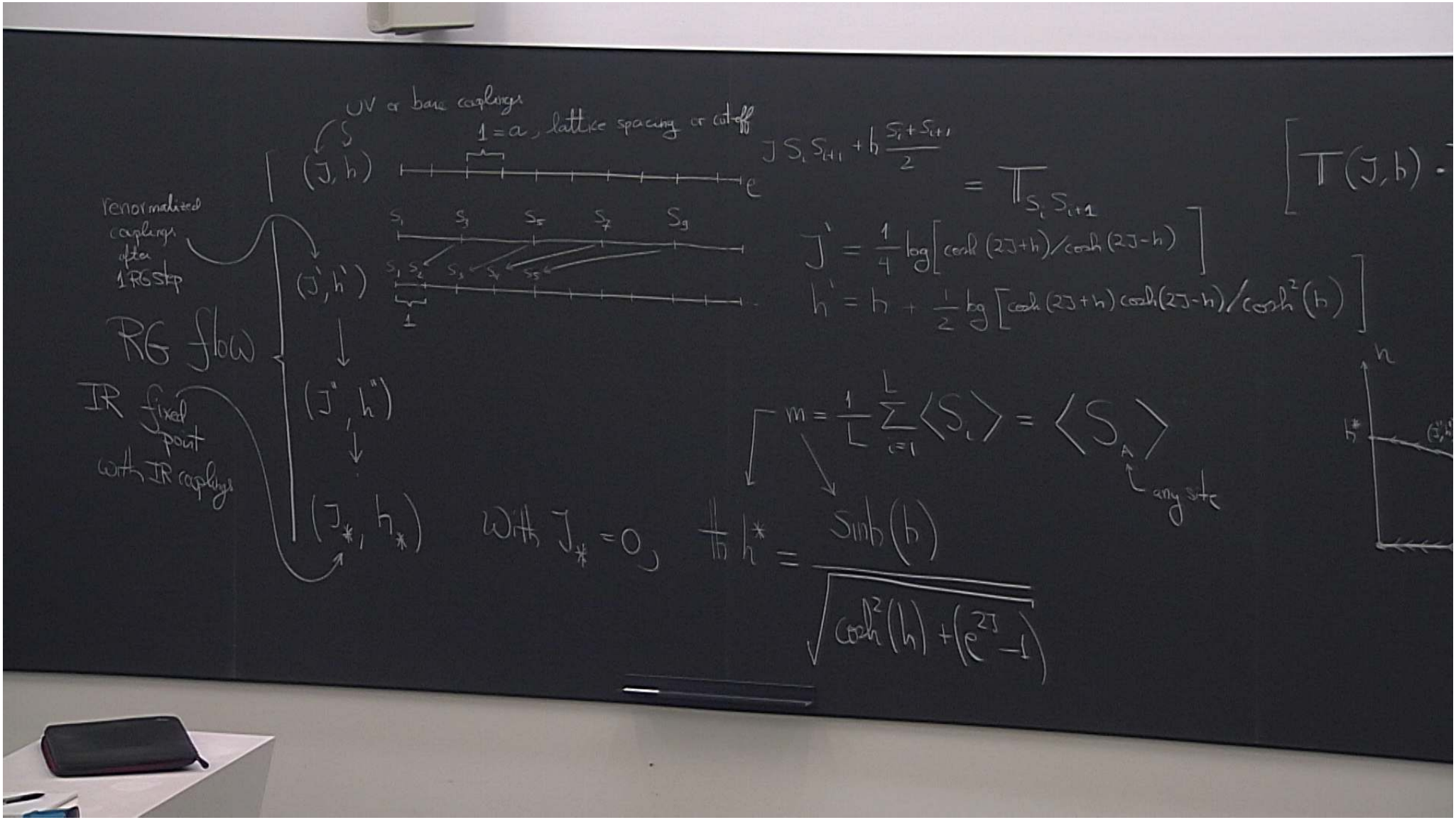


$$Z_{\text{Ising}} = \sum_{\{S_i = \pm 1\}} \prod_{\langle i,j\rangle} (1$$

$$= \sum_{\text{sticks}} 1 \quad \text{if } \nexists \text{ open e}$$

(— or — @ each link)

$$= \sum_{\text{loops with o}}$$


UV or bare couplings
1 = a, lattice spacing or cut-off
renormalized couplings after 1 RG step
RG flow
IR fixed point with IR couplings
(J, h)
(J', h')
(J'', h'')
(J_*, h_*)
$J S_i S_{i+1} + h \frac{S_i + S_{i+1}}{2} = T_{S_i S_{i+1}}$
$J' = \frac{1}{4} \log\left[\cosh(2J+h)/\cosh(2J-h)\right]$
$h' = h + \frac{1}{2} \log\left[\cosh(2J+h)\cosh(2J-h)/\cosh^2(h)\right]$
$m = \frac{1}{L}\sum_{i=1}^{L} \langle S_i \rangle = \langle S_A \rangle$
any site
With $J_* = 0$, $h^* =$
$\frac{\sinh(h)}{\sqrt{\cosh^2(h) + (e^{2J} - 1)}}$
h
h*

$J S_i S_{i+1} + h \frac{S_i + S_{i+1}}{2} = T_{S_i S_{i+1}}$

$\left[T(J,h) \cdot T(J,h) = \mathcal{N} \, T(J',h') \right]$

$J' = \frac{1}{4} \log \left[\cosh(2J+h) / \cosh(2J-h) \right]$

$h' = h + \frac{1}{2} \log \left[\cosh(2J+h) \cosh(2J-h) / \cosh^2(h) \right]$

$Z_{\text{Ising}} = \sum_{\{S_i = \pm 1\}} T$

$= \sum_{\text{sticks}} 1$

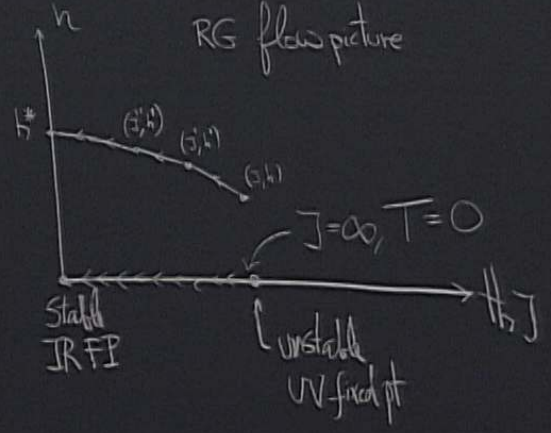
(– or — e

$= \sum_{\text{loops}}$

$m = \frac{1}{L} \sum_{i=1}^{L} \langle S_i \rangle = \langle S_A \rangle$

A ← any site

$h^* = \frac{\sinh(h)}{\sqrt{\cosh^2(h) + (e^{2J} - 1)}}$



$$Z_{\text{Ising}} = \sum_{\{S_i = \pm 1\}} \prod_{\langle i,j \rangle} \left(1 + v\, S_i S_j\right)$$

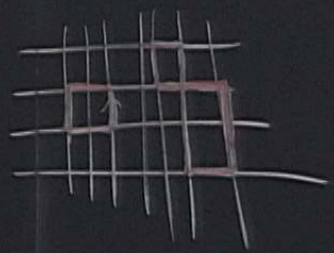
v: $\tanh J$

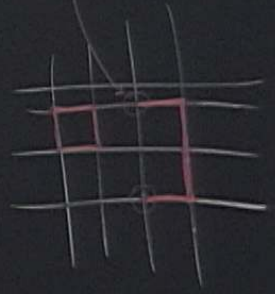
$$= \sum_{\text{sticks}} 1 \quad \text{if } \nexists \text{ open ends}$$

(0 otherwise)

$\sum S_a = 0$

(– or — @ each link)

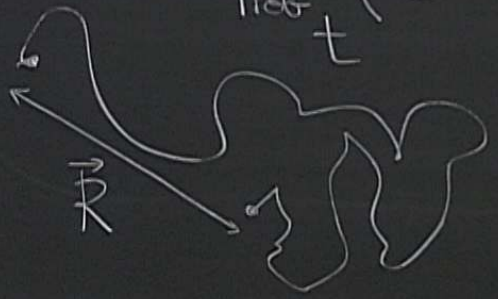




$$= Z_{\text{loops with orientations}}$$

Today Random Walks

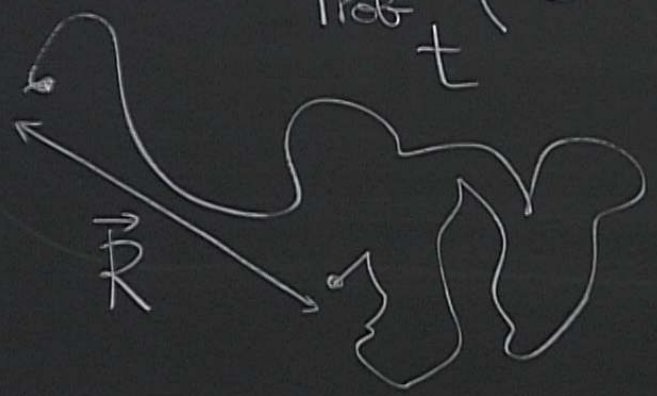
① $\text{Prob}_t(\vec{0} \to \vec{R}) = ?$



② RG perspective on RW

Today Random Walks [2nd Ref RG part ch1, ID Pn etc]

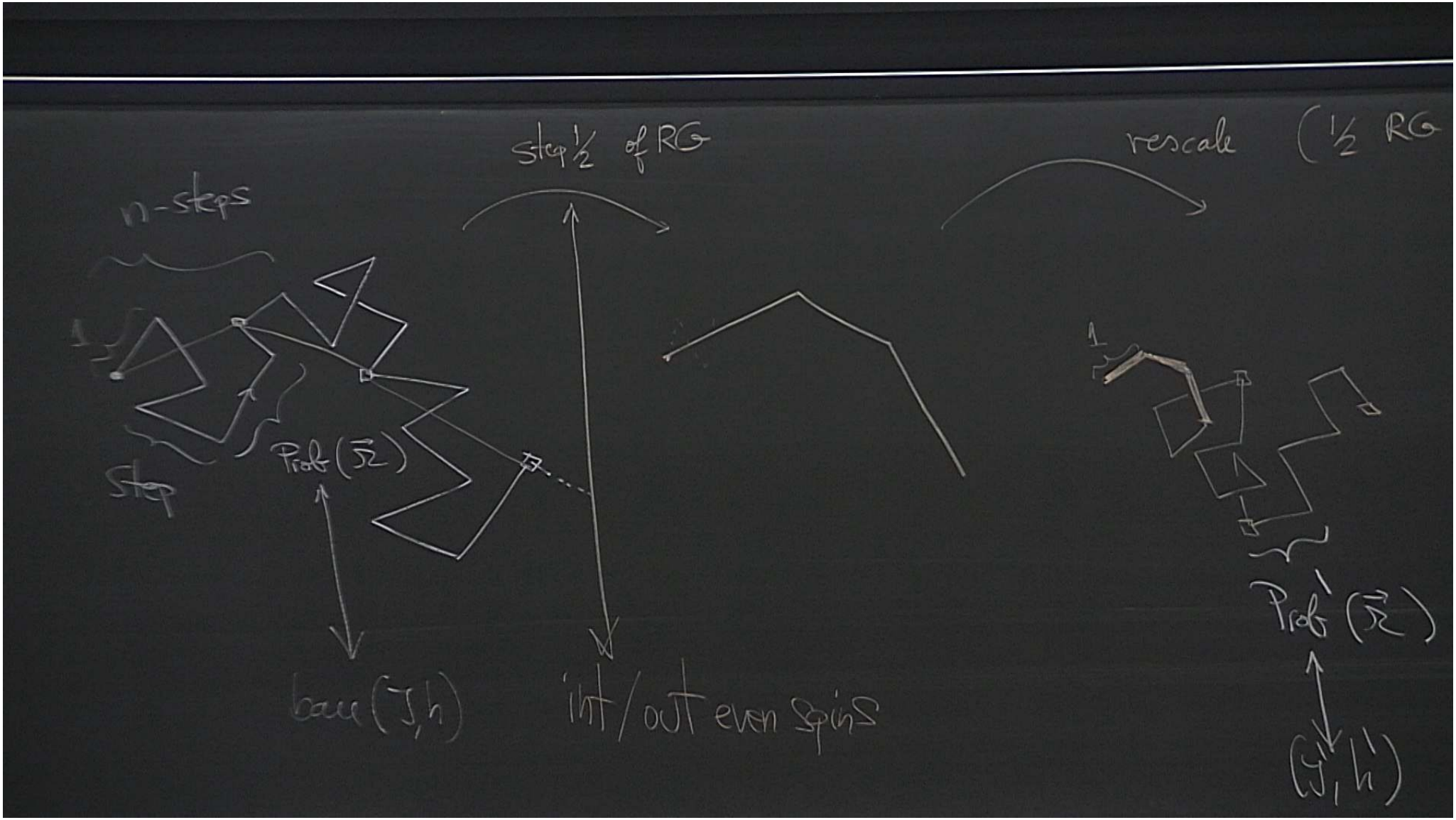
① $\text{Prob}_t(\vec{0} \to \vec{R}) = ?$

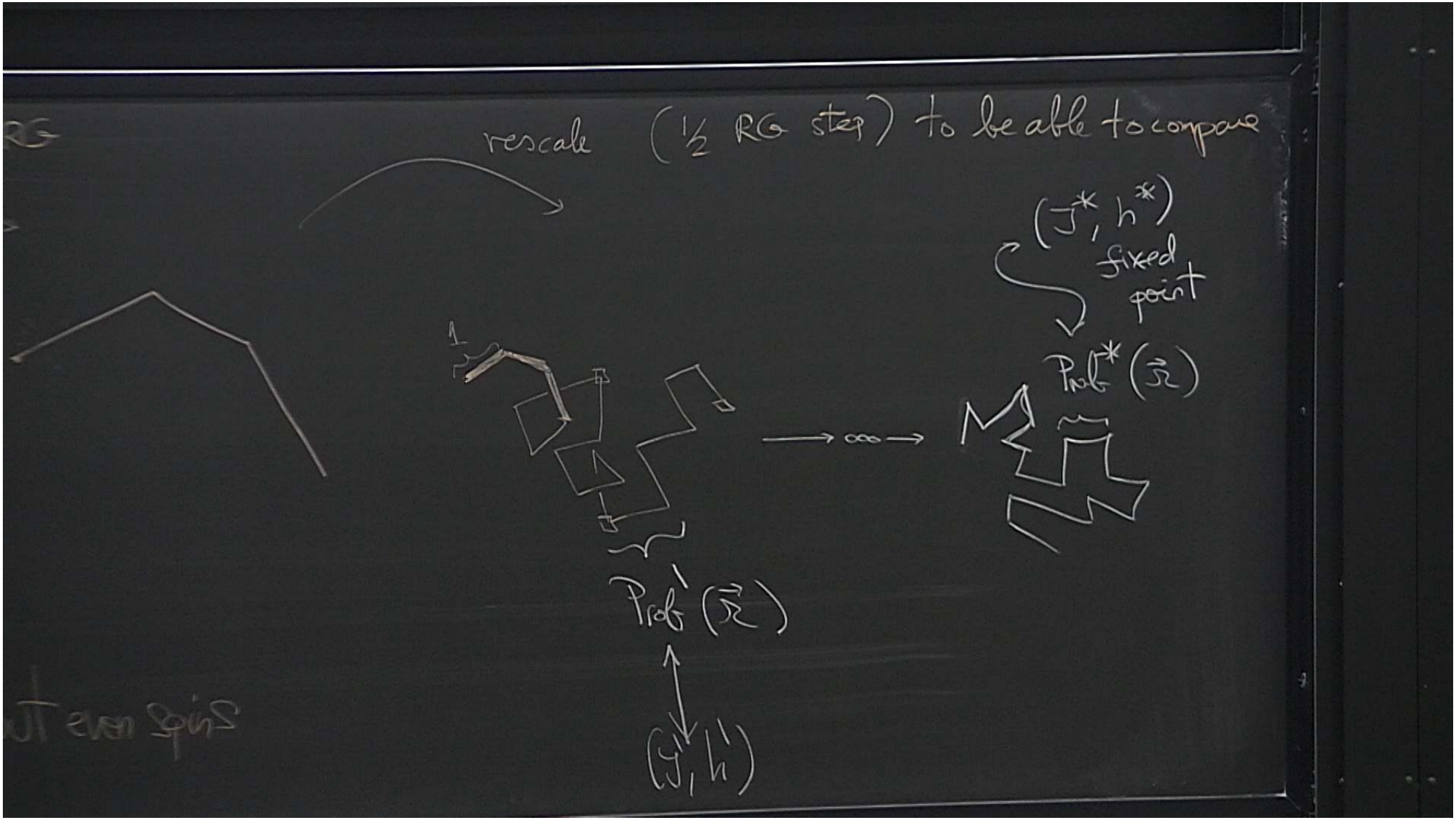


② RG perspective on RW

$t = 0, 1, 2, \ldots$

Step


n-steps
step
Prob(σ)
step ½ of RG
rescale
(½ RG
Prob'(σ')
(J, h)
int/out even spins
(J', h')


RG
rescale (½ RG step) to be able to compare
(J^*, h^*)
fixed point
$Prob^*(\vec{s})$
$Prob'(\vec{s})$
(J', h')
even spins

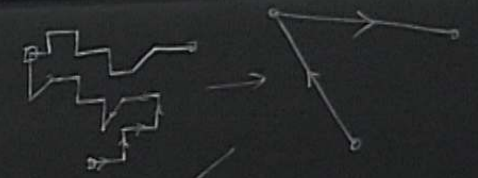
Examples:

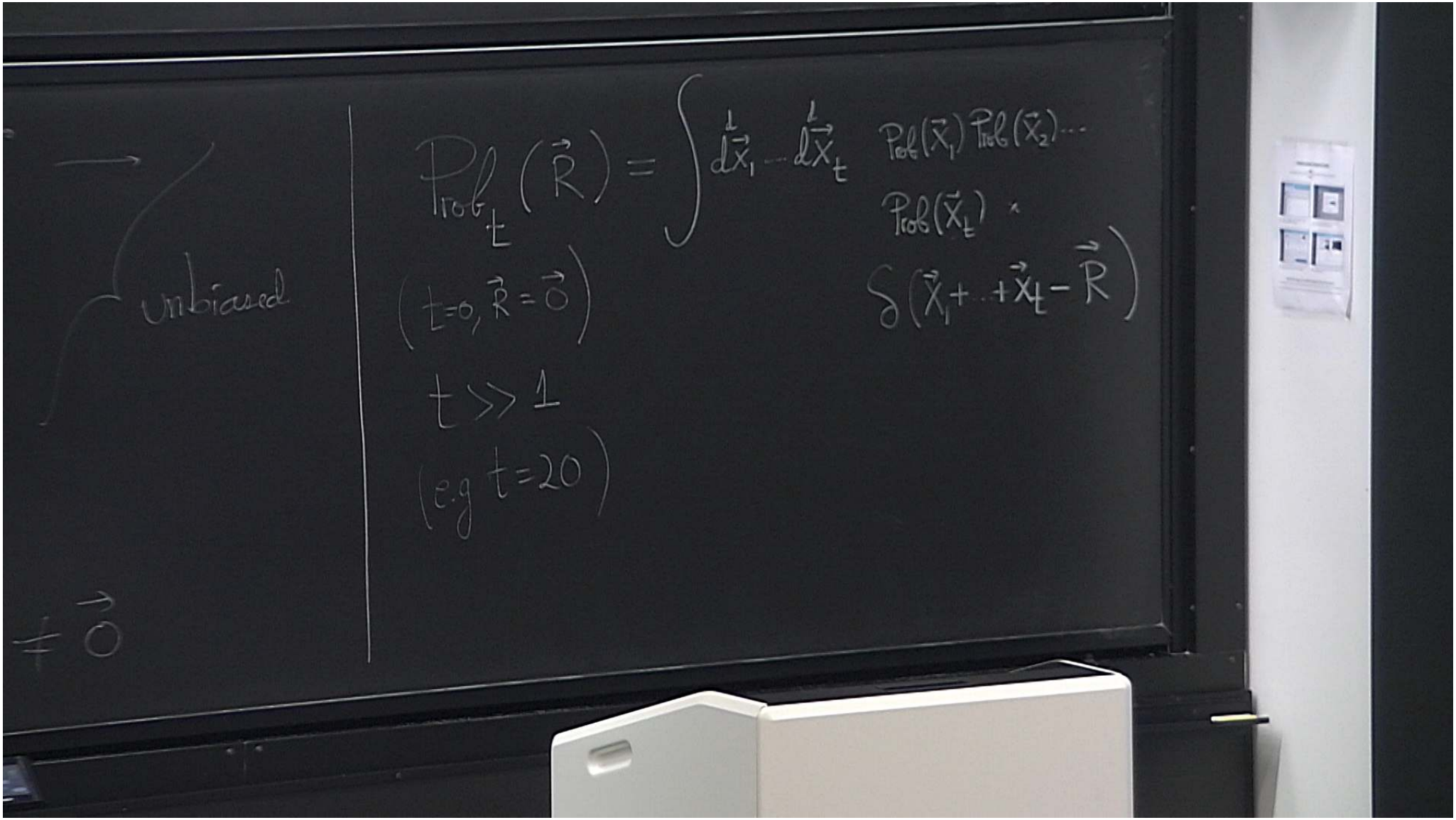
$$P_{\text{init}}(\vec{x}) = \begin{cases} \frac{1}{2d} & \text{, } 0 \text{ otherwise.} \\ \delta(|\vec{x}| - a) \quad \left(\text{Area} = 4\pi a^2 \text{ in } 2d\right) & \\ e^{-(\vec{x}-\vec{x}_0)^2/2\sigma_0^2} & \leftarrow \text{biased walk} \end{cases}$$

$\langle \vec{x} \rangle = \vec{0}$ — unbiased

$\langle \vec{x} \rangle = \vec{x}_0 \neq \vec{0}$

$\vec{x}_0$ — bias (wind)

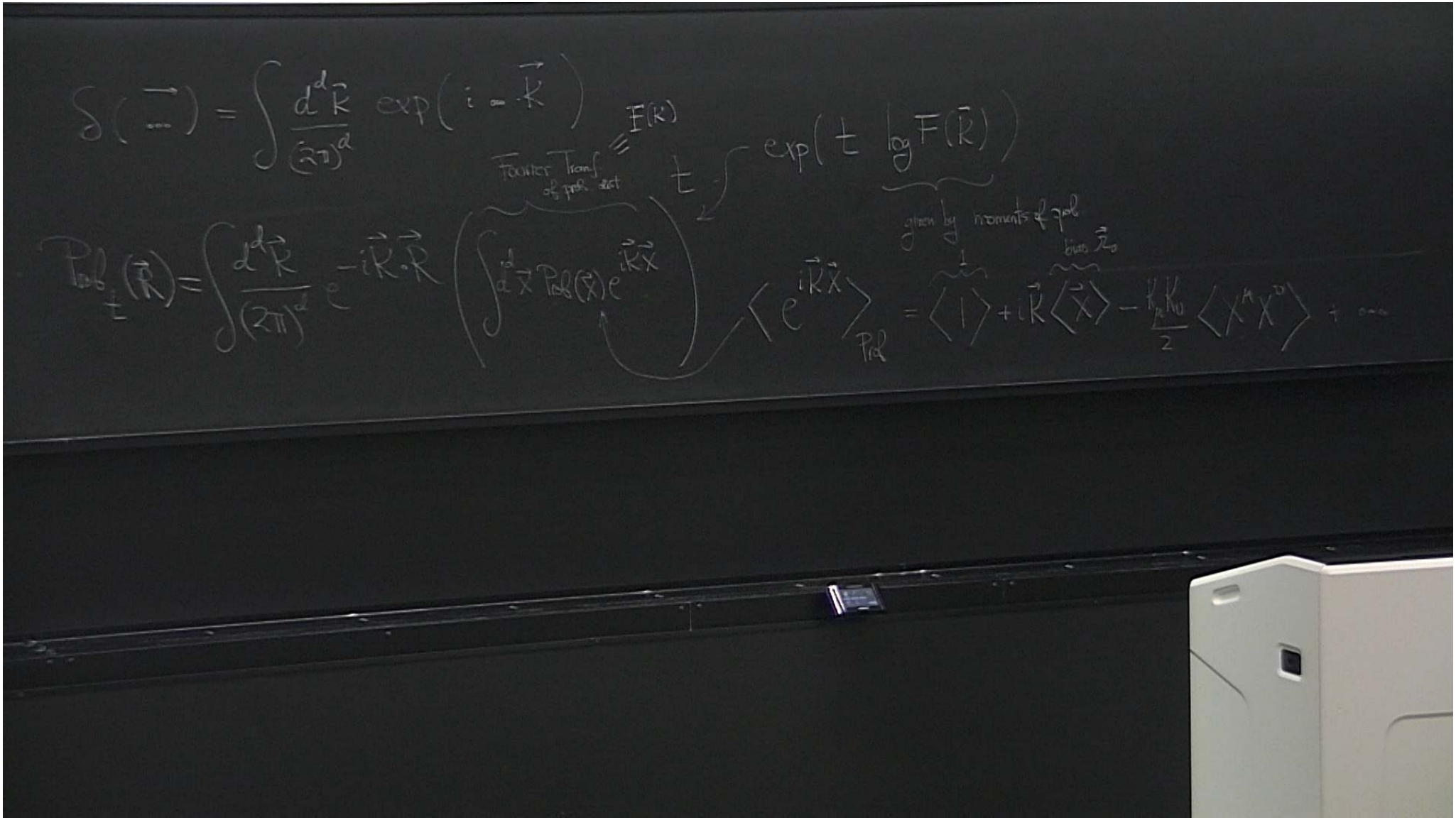


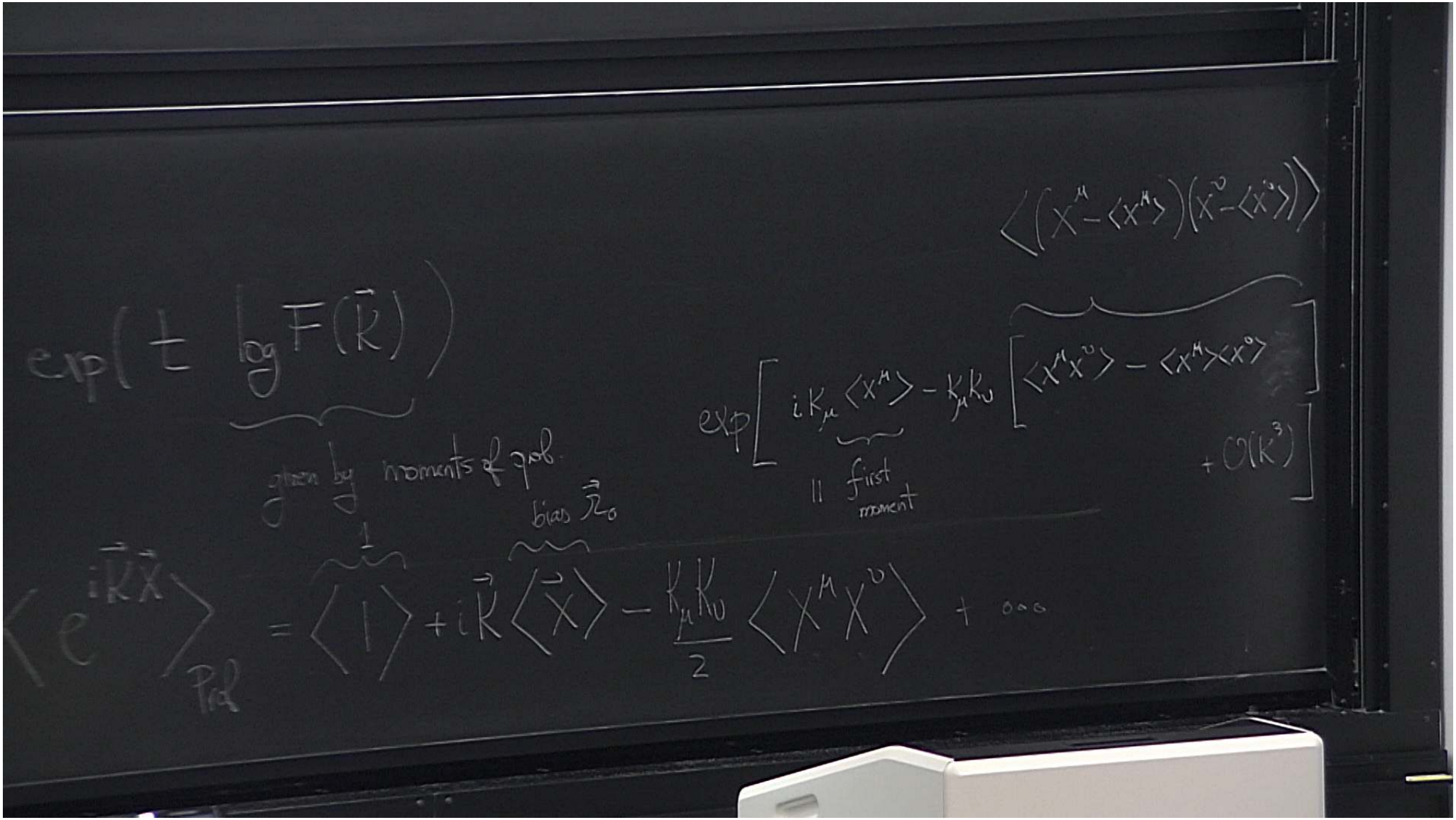

unbiased
$\neq \vec{0}$
$\text{Prob}_t(\vec{R}) = \int d\vec{x}_1 \cdots d\vec{x}_t \; \text{Prob}(\vec{x}_1)\,\text{Prob}(\vec{x}_2)\cdots$
$\text{Prob}(\vec{x}_t) \times$
$\delta(\vec{x}_1 + \cdots + \vec{x}_t - \vec{R})$
$(t=0, \vec{R}=\vec{0})$
$t \gg 1$
(e.g $t=20$)

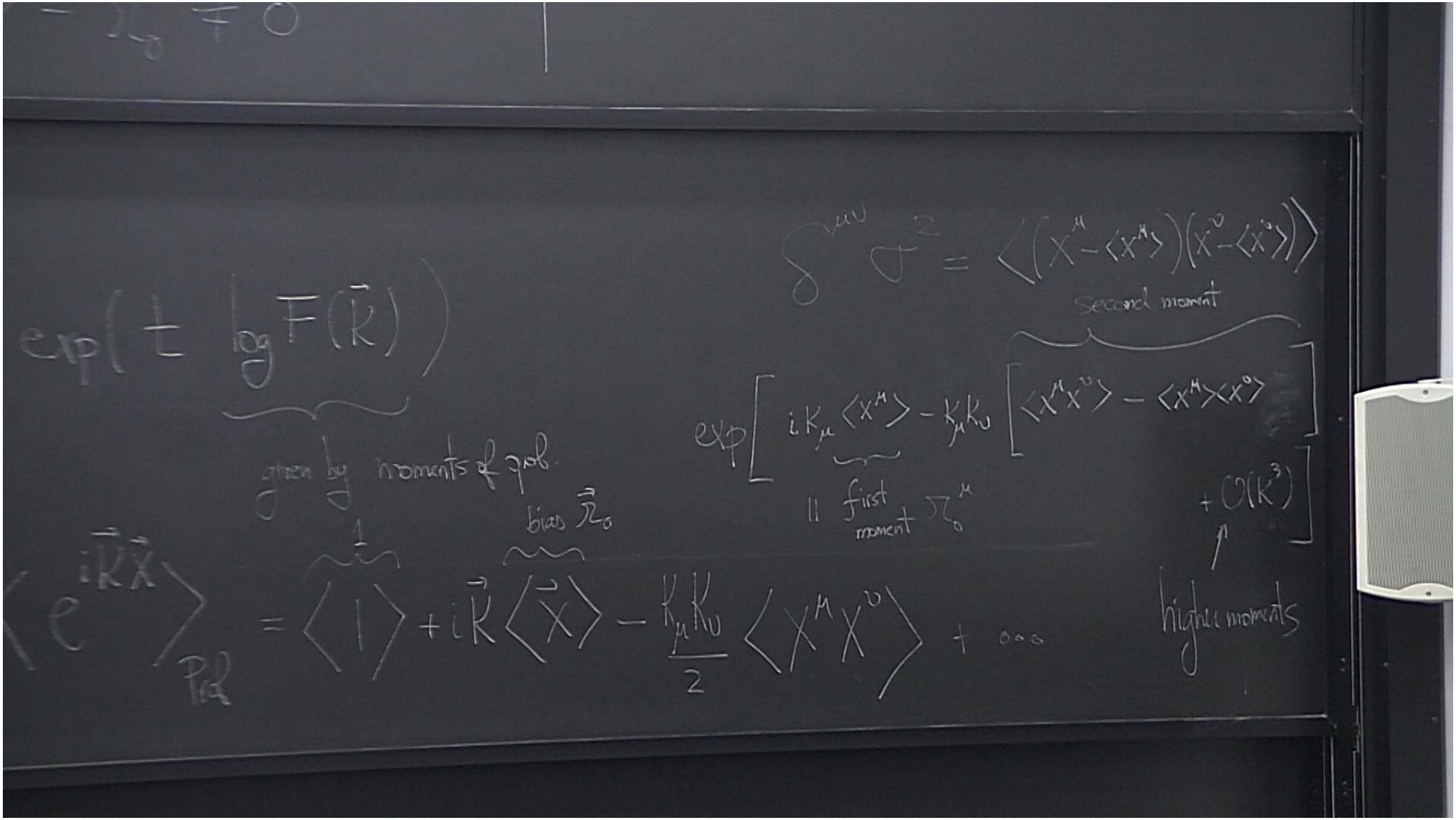
$$\delta(\vec{\cdots}) = \int \frac{d^d\vec{k}}{(2\pi)^d} \exp\left(i \cdots \cdot \vec{k}\right)$$

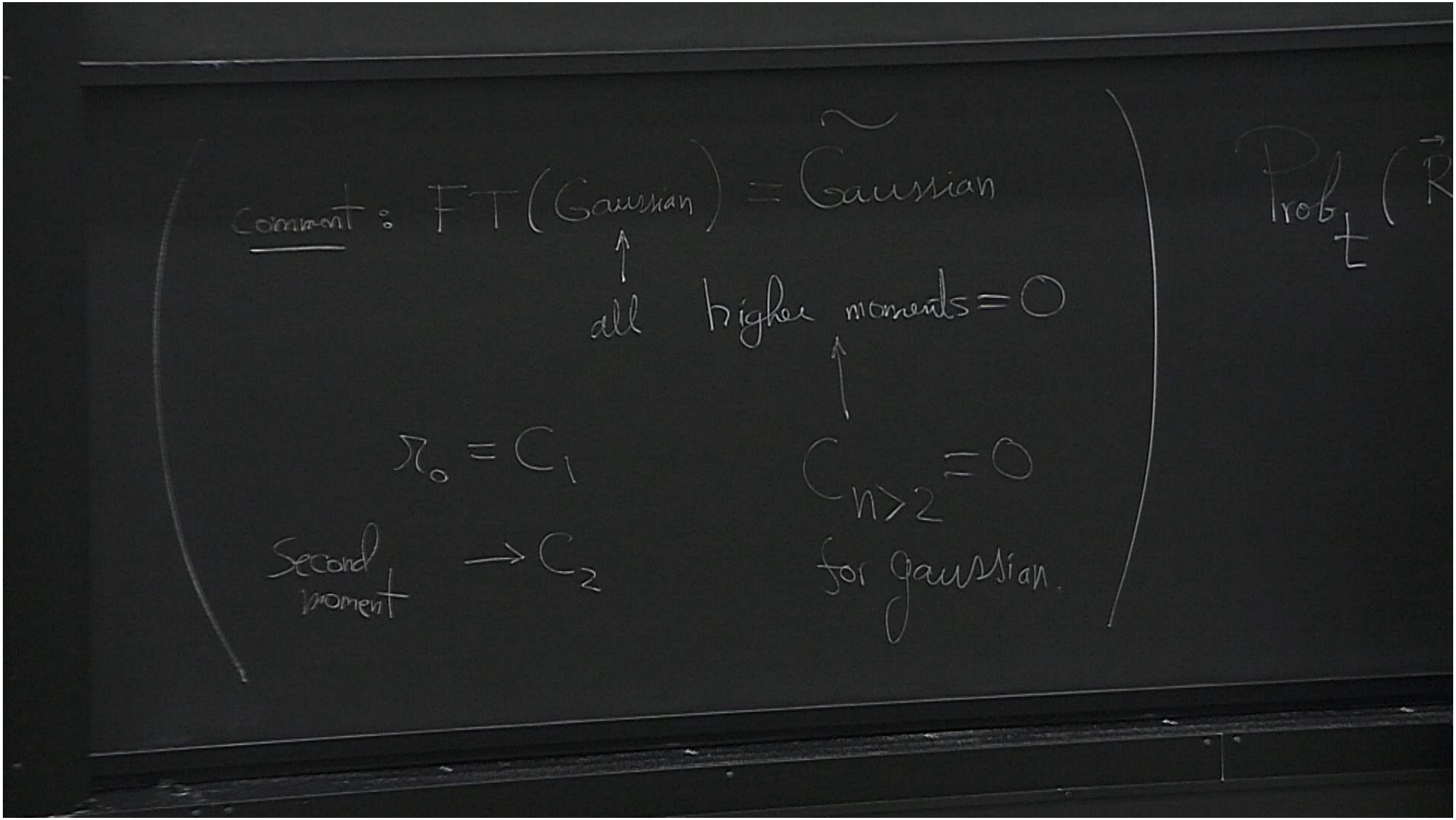
$$\equiv F(k)$$

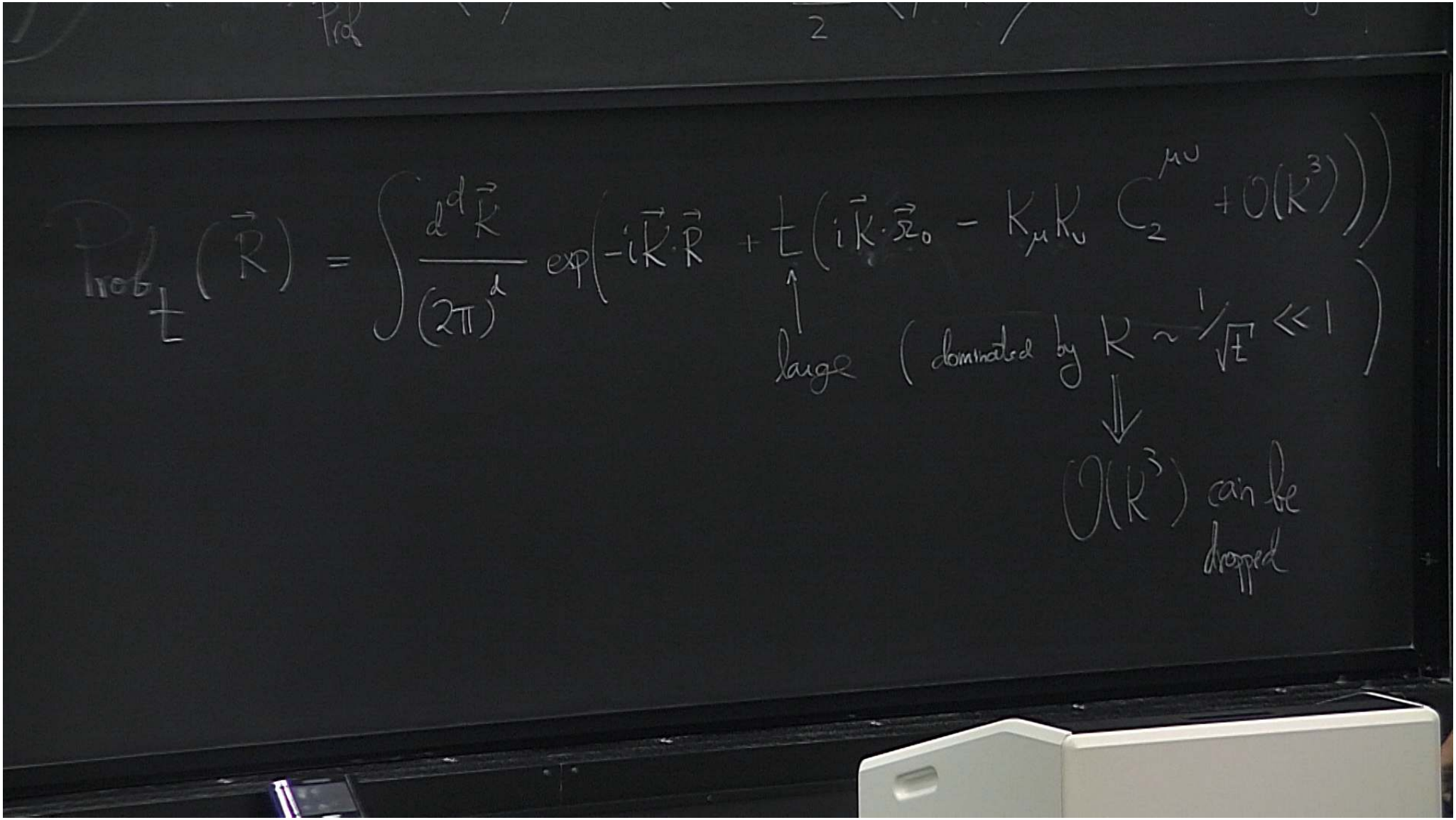
$$\text{Prob}_t(\vec{R}) = \int \frac{d^d\vec{k}}{(2\pi)^d} e^{-i\vec{k}\cdot\vec{R}} \left(\overbrace{\int d^d\vec{x}\, \text{Prob}(\vec{x}) e^{i\vec{k}\cdot\vec{x}}}^{\text{Fourier Transf. of prob. dist}} \right)_t$$

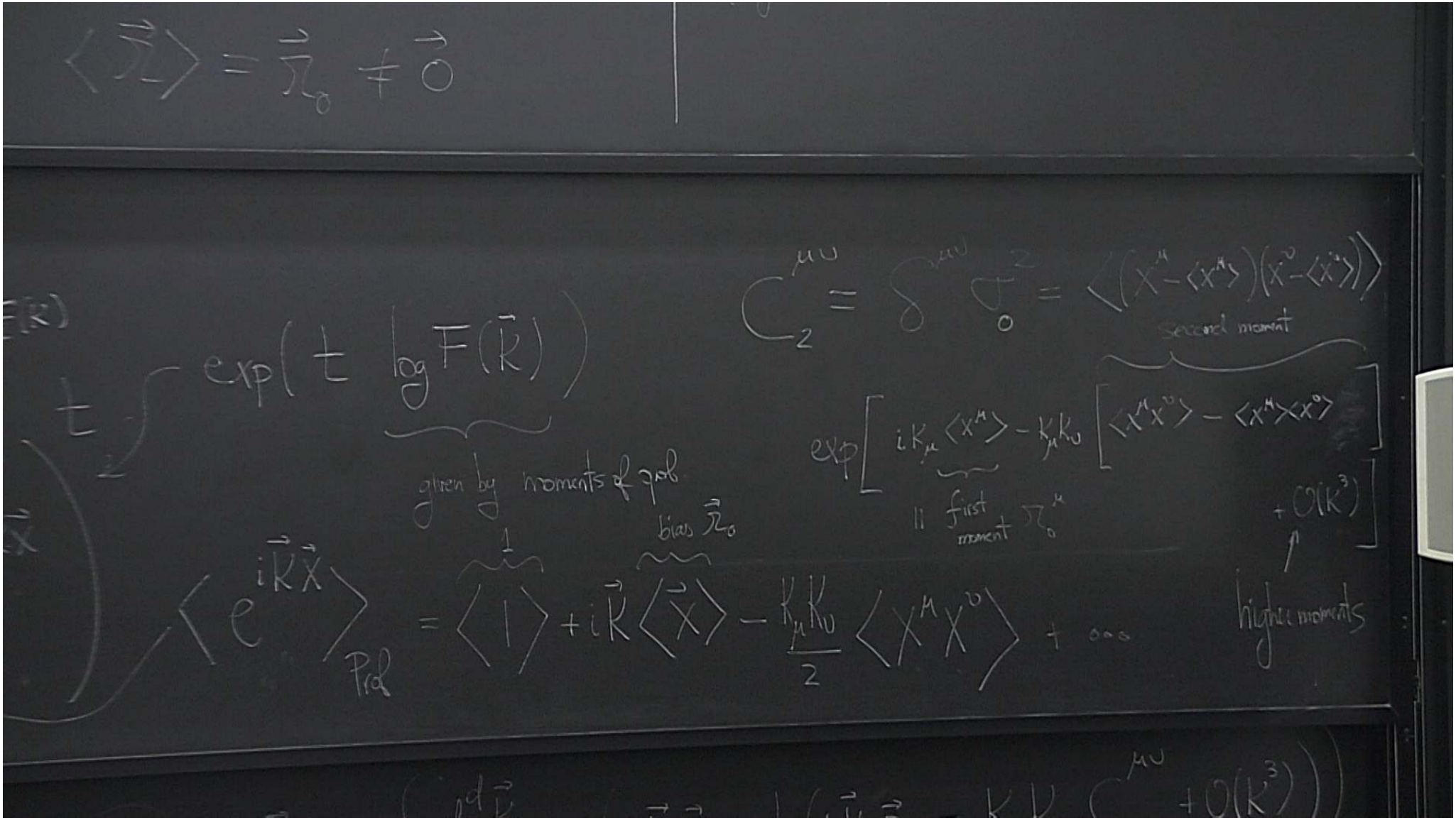

Fourier Transf of prob. dist
given by moments of prob
bias $\vec{x}_0$


exp(± log F(k))
given by moments of prob.
bias $\vec{x}_0$
first moment
$\langle e^{i\vec{k}\vec{x}} \rangle_{Pr} = \langle 1 \rangle + i\vec{k}\langle \vec{x} \rangle - \frac{k_\mu k_\nu}{2} \langle x^\mu x^\nu \rangle + \dots$
$\exp\left[i k_\mu \langle x^\mu \rangle - k_\mu k_\nu \left[\langle x^\mu x^\nu \rangle - \langle x^\mu \rangle \langle x^\nu \rangle \right] + O(k^3) \right]$
$\langle (x^\mu - \langle x^\mu \rangle)(x^\nu - \langle x^\nu \rangle) \rangle$


$\exp\left(i\vec{K}\cdot\vec{\lambda} + \log F(\vec{K})\right)$
given by moments of prob.
$\delta^{\mu\nu}\sigma^2 = \langle (X^\mu - \langle X^\mu\rangle)(X^\nu - \langle X^\nu\rangle)\rangle$
second moment
$\exp\left[iK_\mu \langle X^\mu\rangle - K_\mu K_\nu \left[\langle X^\mu X^\nu\rangle - \langle X^\mu\rangle\langle X^\nu\rangle \right] + O(K^3) \right]$
$iK_\mu\langle X^\mu\rangle$ = first moment λ_0^μ
$\langle e^{i\vec{K}\vec{X}}\rangle_{P(\vec{X})} = \langle 1\rangle + i\vec{K}\langle\vec{X}\rangle - \frac{K_\mu K_\nu}{2}\langle X^\mu X^\nu\rangle + \dots$
bias $\vec{\lambda}_0$
higher moments


Comment: FT(Gaussian) = Gaussian
all higher moments = 0
$\mu_0 = C_1$
$C_{n>2} = 0$
Second moment $\rightarrow C_2$
for gaussian.
Prob


$\mathrm{Prob}_t(\vec{R}) = \int \frac{d^d\vec{K}}{(2\pi)^d} \exp\left(-i\vec{K}\cdot\vec{R} + t\left(i\vec{K}\cdot\vec{\Omega}_0 - K_\mu K_\nu C_2^{\mu\nu} + O(K^3)\right)\right)$
large
(dominated by $K \sim \frac{1}{\sqrt{t}} \ll 1$)
$O(K^3)$ can be dropped


$\langle \vec{x} \rangle = \vec{x}_0 \neq \vec{0}$
$C_2^{\mu\nu} = \delta^{\mu\nu} \sigma_0^2 = \langle (x^\mu - \langle x^\mu \rangle)(x^\nu - \langle x^\nu \rangle) \rangle$
second moment
$\exp(\pm \log F(\vec{k}))$
given by moments of prob
$\exp\left[iK_\mu \langle x^\mu \rangle - K_\mu K_\nu \left[\langle x^\mu x^\nu \rangle - \langle x^\mu \rangle \langle x^\nu \rangle \right] + O(K^3) \right]$
first moment x_0^μ
bias $\vec{x}_0$
$\langle e^{i\vec{K}\vec{x}} \rangle_{Prob} = \langle 1 \rangle + i\vec{K} \langle \vec{x} \rangle - \frac{K_\mu K_\nu}{2} \langle x^\mu x^\nu \rangle + \dots$
higher moments

$$\langle \vec{z} \rangle = \vec{z}_0 \neq \vec{0}$$

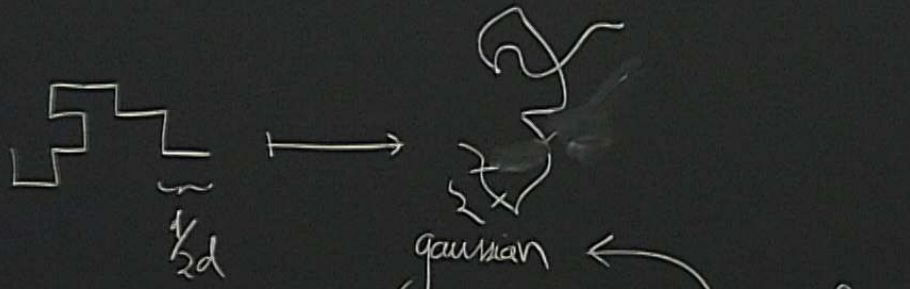
$$\text{no}\,G_t(\vec{R}) = \int \frac{d^d\vec{k}}{(2\pi)^d} \exp\left(-i\vec{k}\cdot\vec{R} + t\left(i\vec{k}\cdot\vec{z}_0 - k_\mu k_\nu C_2^{\mu\nu} + O(k^3)\right)\right)$$

t large (dominated by $k \sim 1/\sqrt{t} \ll 1$)

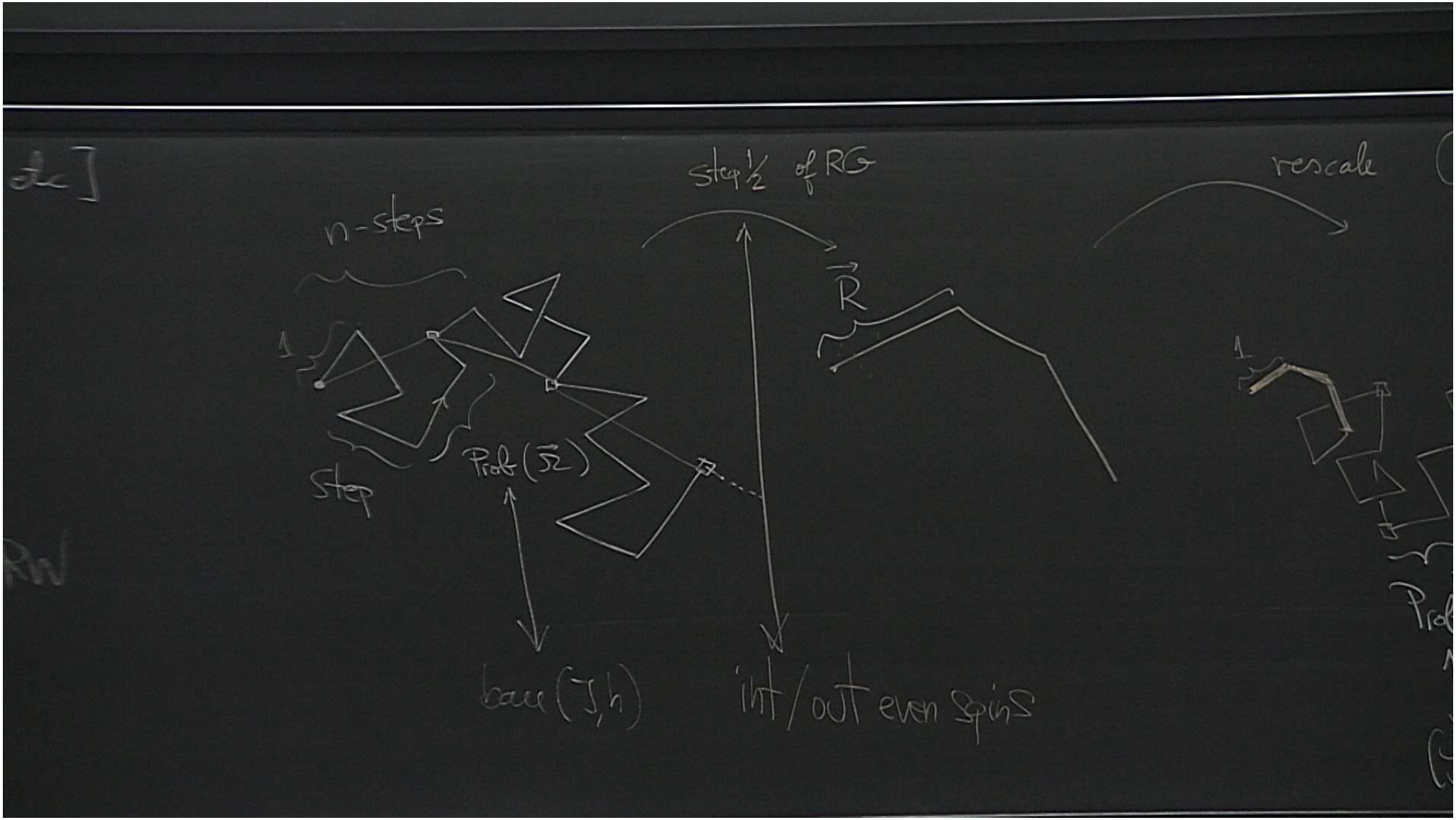
$$= (2\pi t \sigma_0)^{d/2} \exp\left(-\frac{1}{2}\frac{(\vec{R} - t\vec{z}_0)^2}{t\sigma_0^2}\right)$$

$O(k^3)$ can be dropped

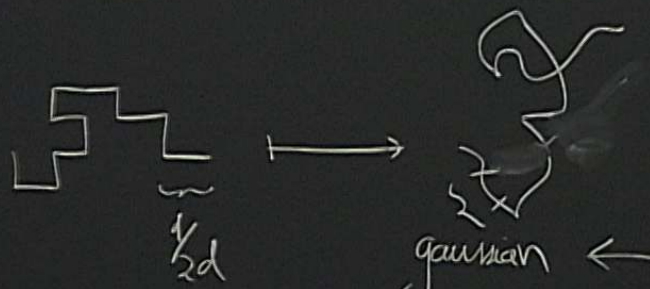
Gaussian = fixed point



start with
bare pot w/ finite moments


n-steps
step ½ of RG
rescale
$\vec{R}$
Prob($\vec{s}$)
step
bare ($\vec{J}$,h)
int/out even spins

Gaussian = fixed point



start with
bare pdf w/ finite moments

RG flow

$$\sqrt{\sigma_0^2} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\text{prob}_t(\vec{R}) = \int \frac{d^d\vec{k}}{(2\pi)^d} \exp\left(-i\vec{k}\cdot\vec{R} + t\left(i\vec{k}\cdot\vec{z}_0 - k_\mu k_\nu C_2^{\mu\nu} + O(k^3)\right)\right)$$

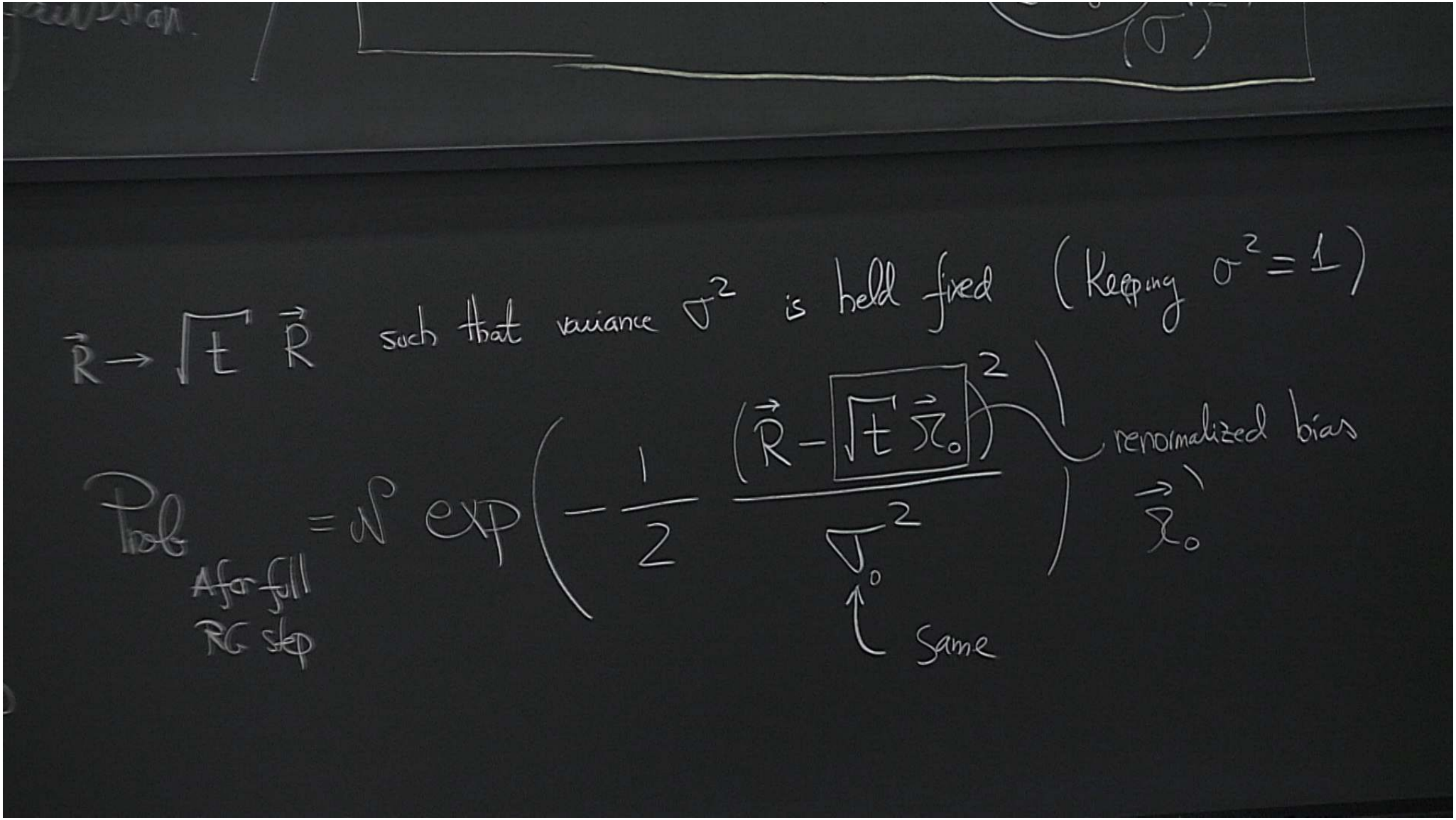
t large (dominated by $k \sim 1/\sqrt{t} \ll 1$)

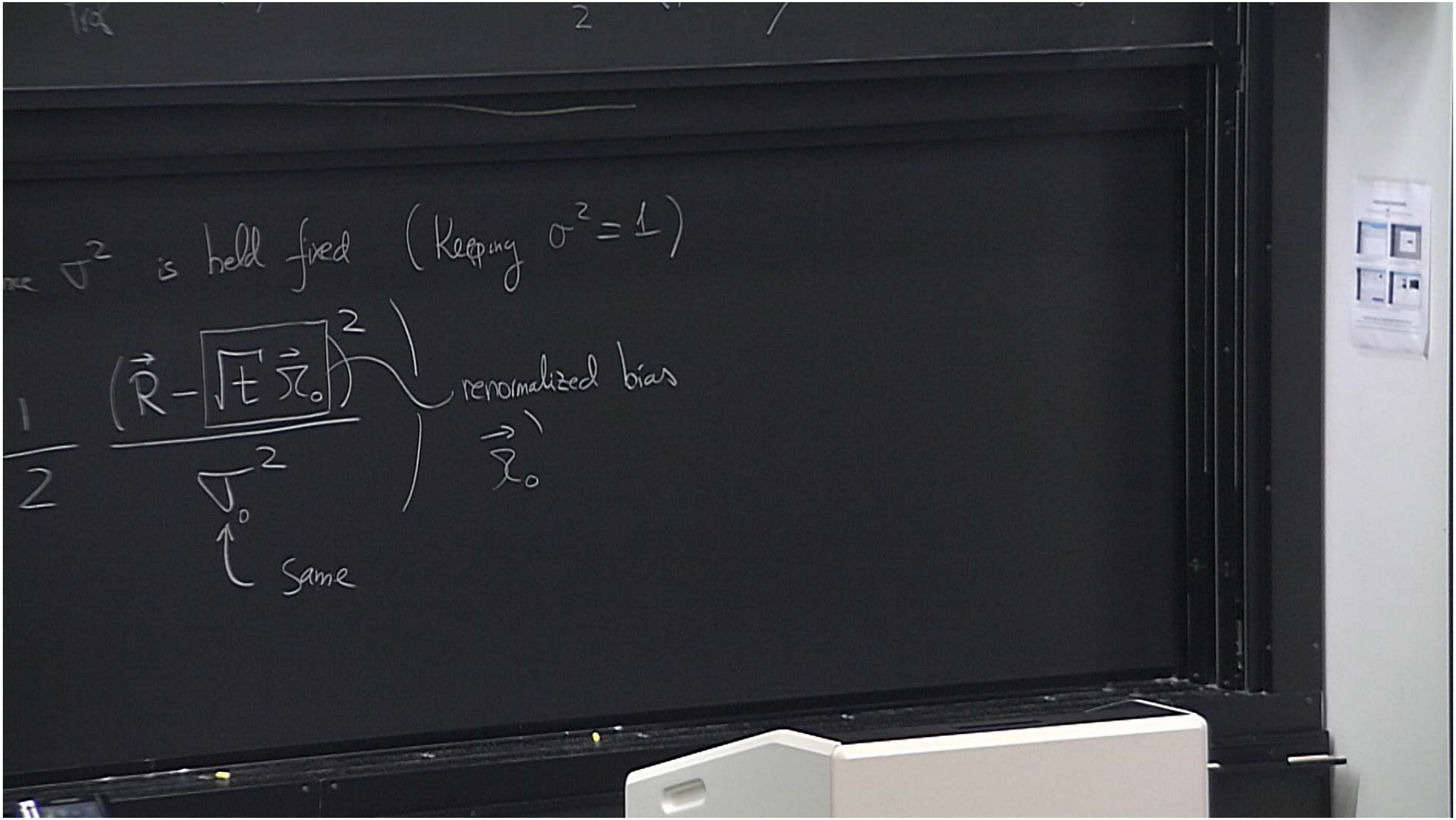
$$= (2\pi t \sigma_0)^{d/2} \exp\left(-\frac{1}{2}\frac{(\vec{R}-t\vec{z}_0)^2}{t\sigma_0^2}\right)$$

$(\sigma)^2$

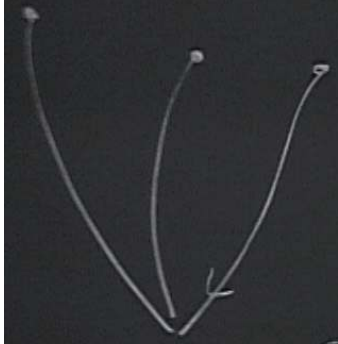
$\Downarrow$

$O(k^3)$ can be dropped


$\vec{R} \to \sqrt{t}\,\vec{R}$ such that variance σ^2 is held fixed (Keeping $\sigma^2 = 1$)
Prob
= $\mathcal{N}$ exp $\left(-\frac{1}{2} \frac{(\vec{R} - \sqrt{t}\,\vec{x}_0)^2}{\sigma_0^2} \right)$
renormalized bias $\vec{x}_0'$
After full RG step
Same


σ^2 is held fixed (Keeping $\sigma^2 = 1$)
$\frac{1}{2}\frac{(\vec{R} - \sqrt{t}\,\vec{x}_0)^2}{\sigma_0^2}$
renormalized bias
$\vec{x}_0'$
Same

Step



{ ≠ base prob w/ same $\sigma^2 = C_2$, $\Sigma_0 = C_1$ but ≠ $C_3, C_4, \ldots$

σ^2: 2nd moment, Σ_0: 1st moment

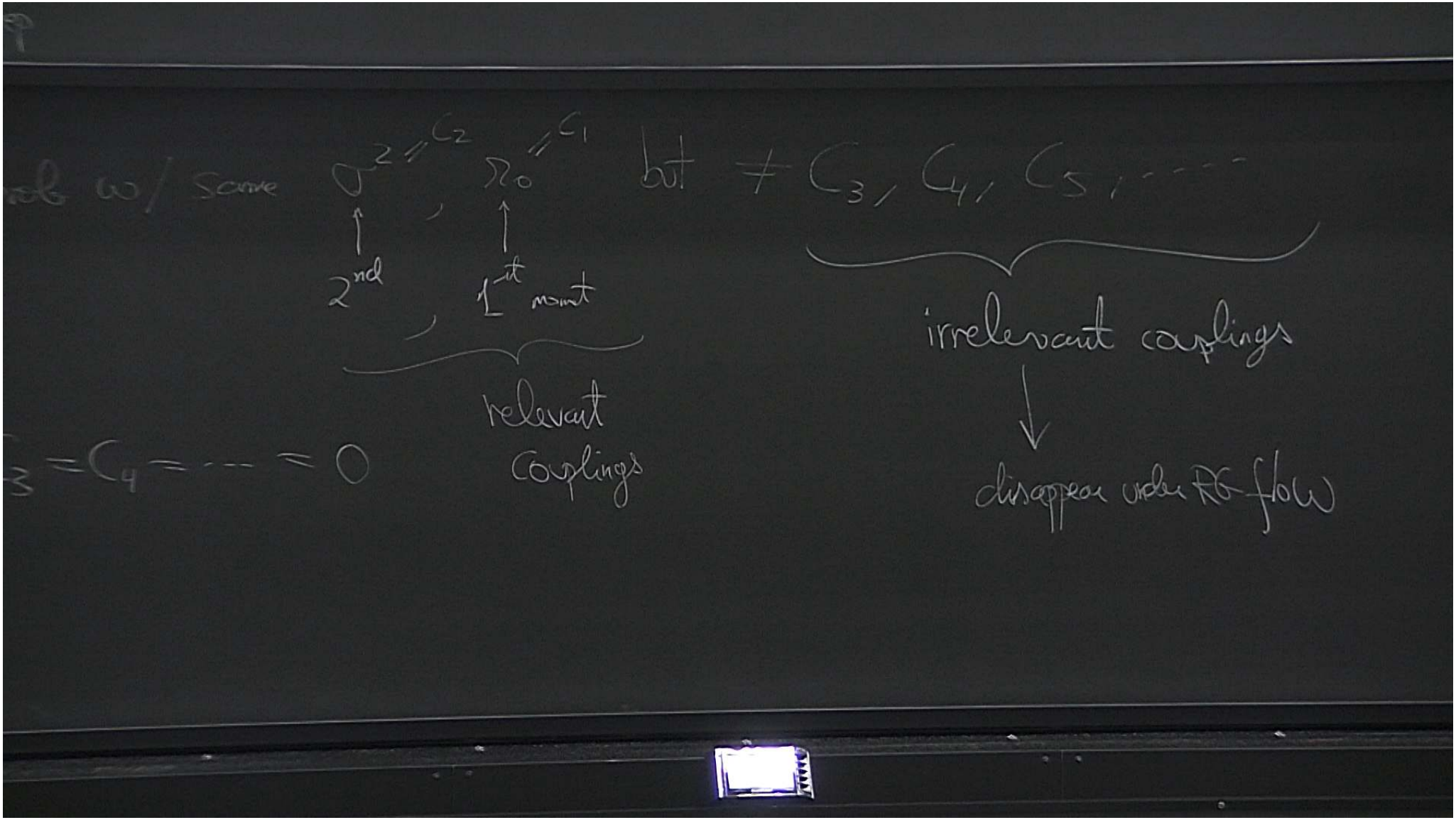
relevant couplings

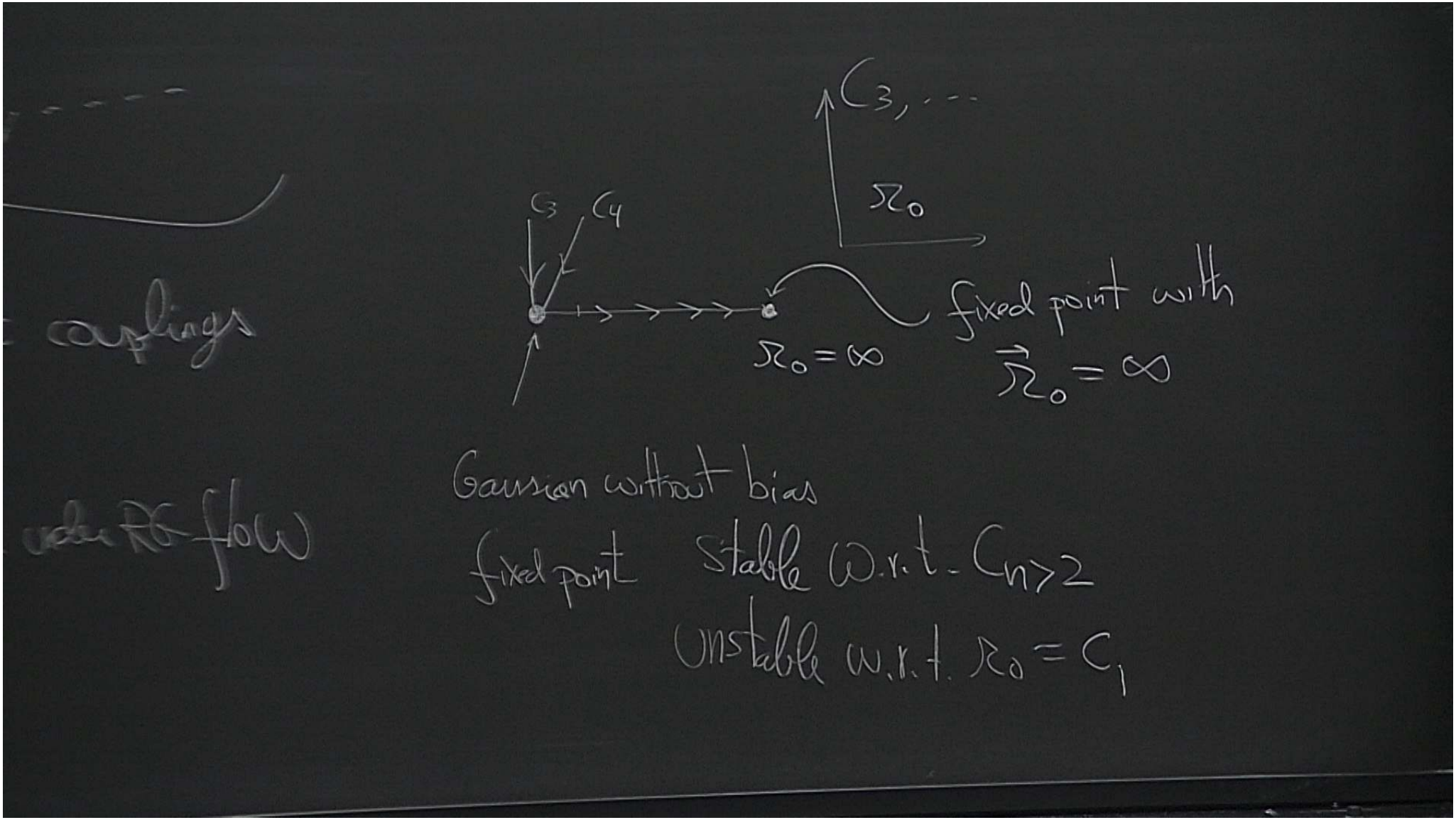
irrele

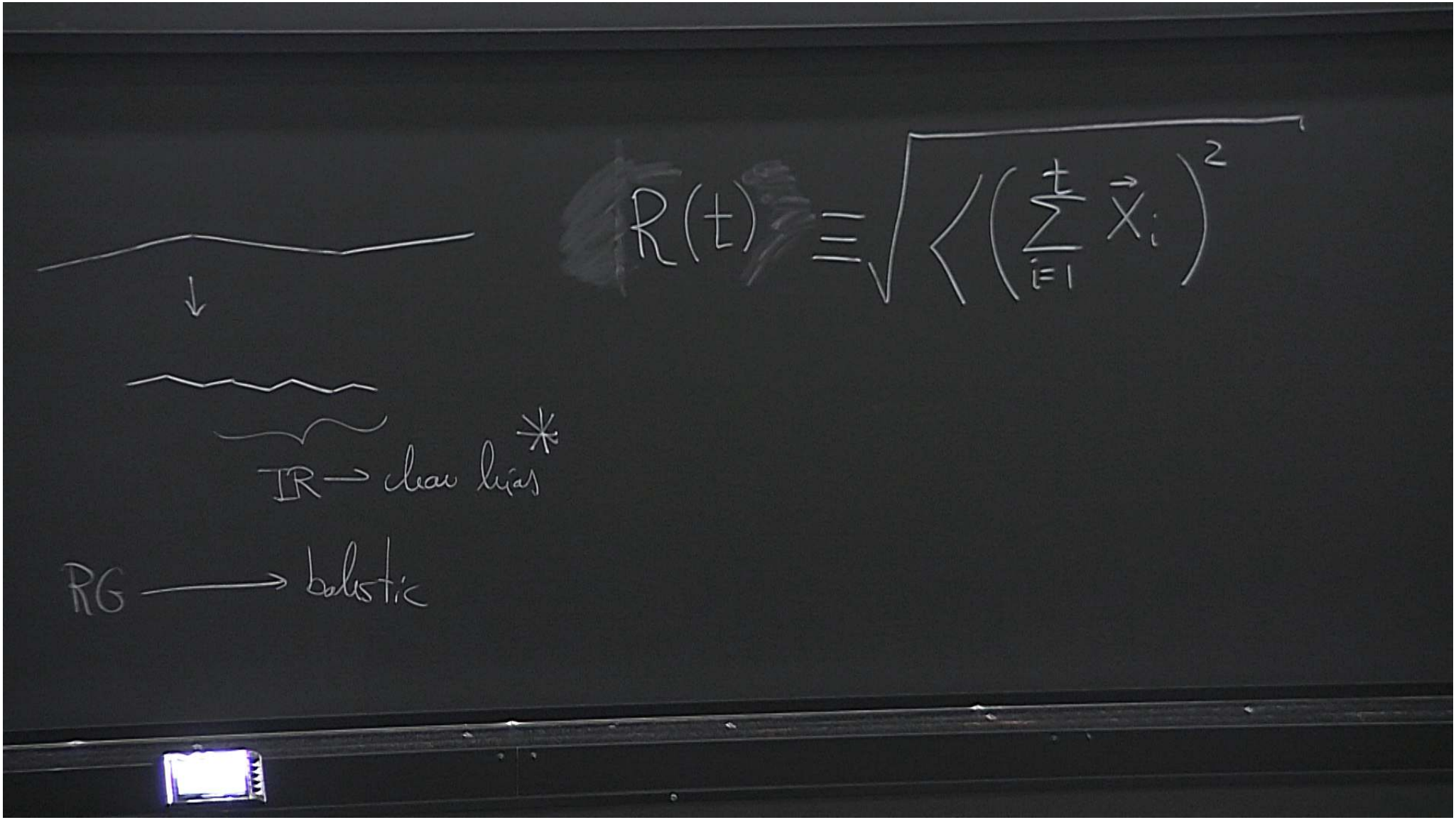
dis

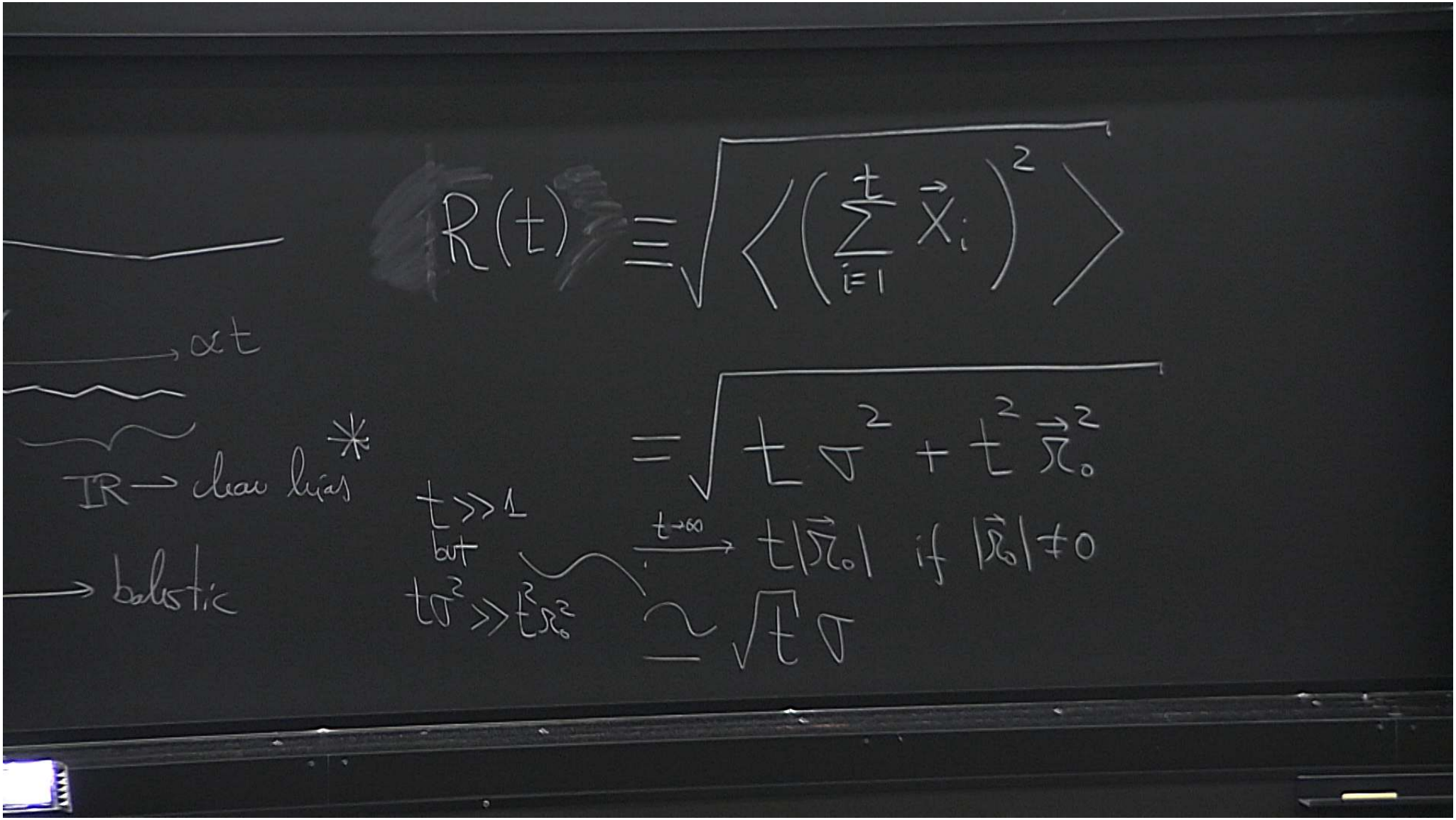
Gaussian fixed pt with $C_3 = C_4 = \cdots = 0$

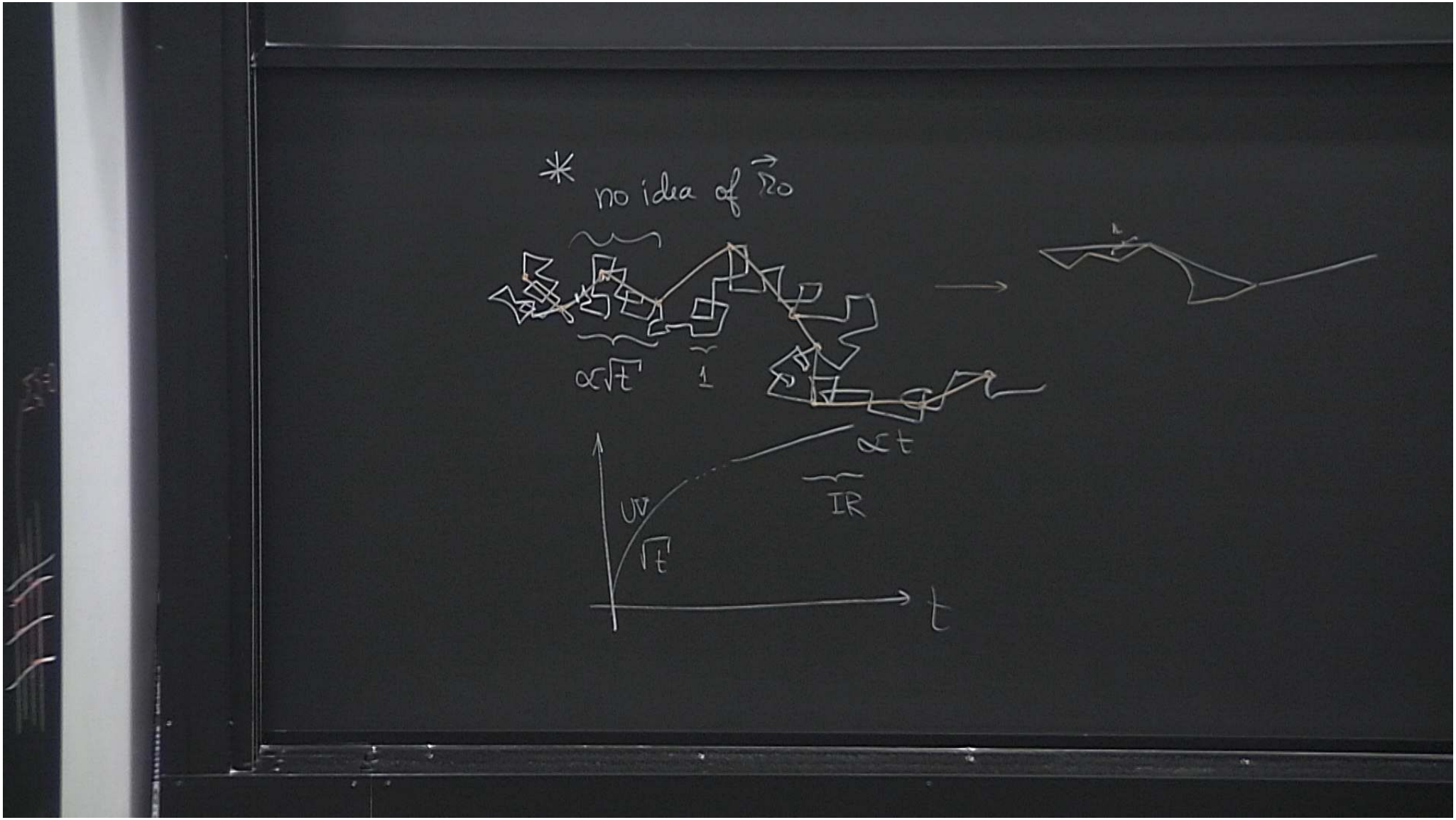
Universality!

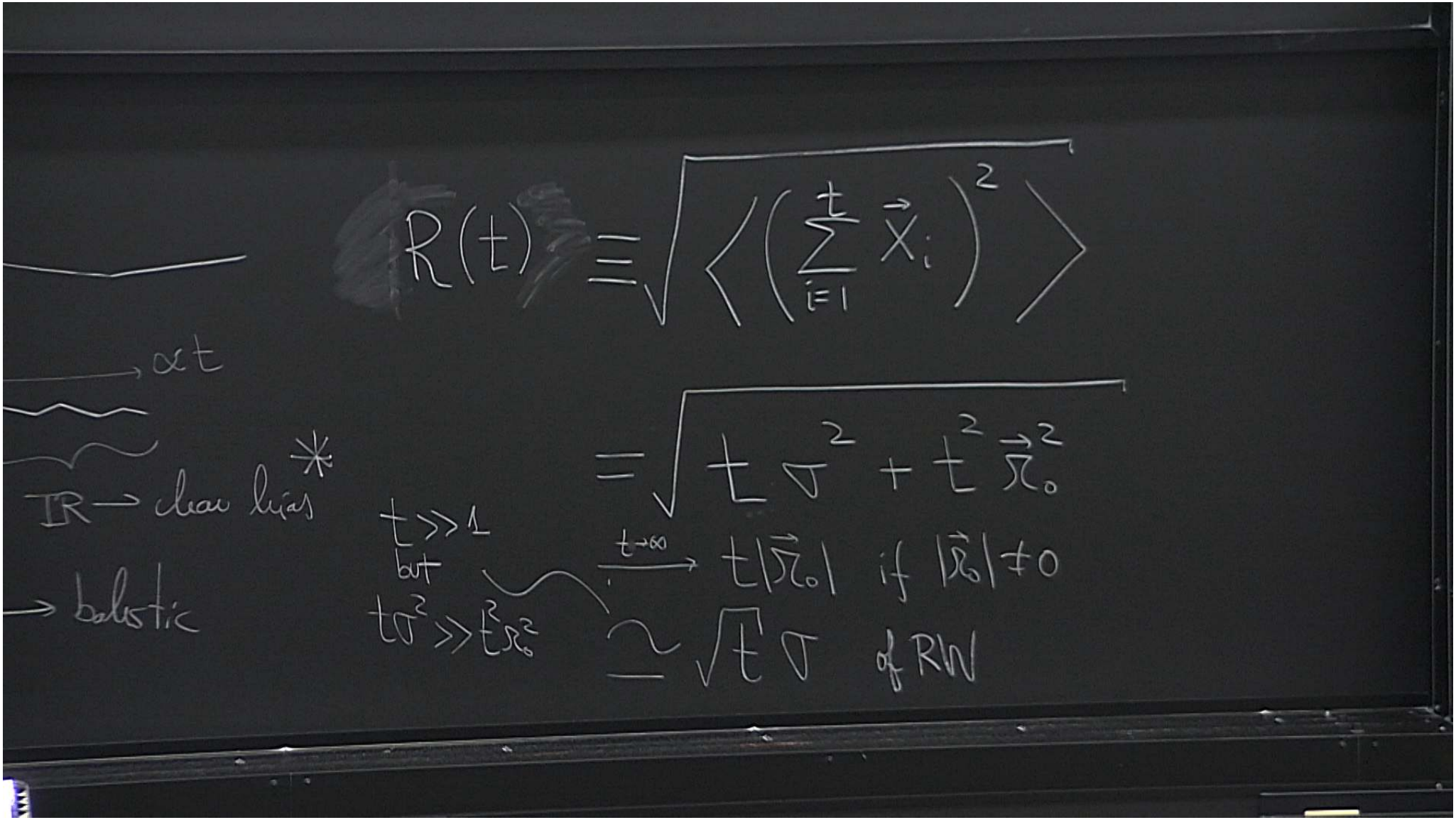

$\sigma^2 = C_2$
$\pi_0 = C_1$
but $\neq C_3, C_4, C_5, \cdots$
2nd
1st moment
relevant couplings
irrelevant couplings
disappear under RG flow
$C_3 = C_4 = \cdots = 0$


$C_3, \ldots$
$\mathcal{R}_0$
C_3 C_4
$\mathcal{R}_0 = \infty$
fixed point with
$\vec{\mathcal{R}}_0 = \infty$
Gaussian without bias
fixed point stable w.r.t. $C_{n>2}$
unstable w.r.t. $\mathcal{R}_0 = C_1$


$R(t) \equiv \sqrt{\left\langle \left(\sum_{i=1}^{t} \vec{x}_i \right)^2 \right\rangle}$
IR → clean limit *
RG → ballistic


$R(t) \equiv \sqrt{\left\langle \left(\sum_{i=1}^{t} \vec{X}_i \right)^2 \right\rangle}$
$= \sqrt{t\sigma^2 + t^2 \vec{x}_0^2}$
αt
IR → clear bias *
ballistic
$t \gg 1$
but
$t\sigma^2 \gg t^2 x_0^2$
$t \to \infty$
$t|\vec{x}_0|$ if $|\vec{x}_0| \neq 0$
$\simeq \sqrt{t}\sigma$


no idea of $\vec{r}_0$
$\propto\sqrt{t}$
1
$\propto t$
UV
IR
$\sqrt{t}$
t


$R(t) \equiv \sqrt{\left\langle \left(\sum_{i=1}^{t} \vec{x}_i \right)^2 \right\rangle}$
$= \sqrt{t\sigma^2 + t^2 \vec{x}_0^2}$
$\propto t$
IR → clear bias *
→ ballistic
$t \gg 1$
but
$t\sigma^2 \gg t^2 \vec{x}_0^2$
$t \to \infty$
$t|\vec{x}_0|$ if $|\vec{x}_0| \neq 0$
$\simeq \sqrt{t}\,\sigma$ of RW

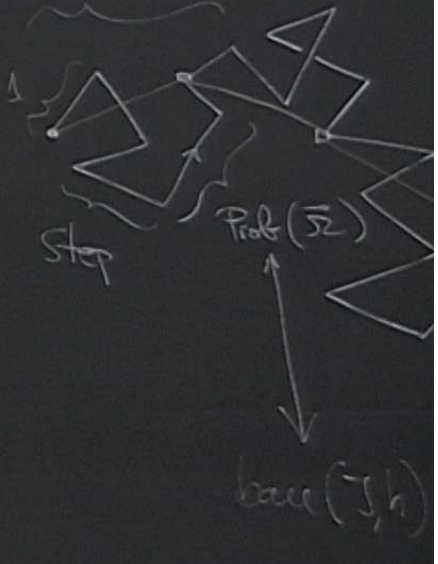
Universality !

Today Random Walks [2nd Ref RG part ch1, ID P_n etc]

① $P_{\text{Prob}_t}(\vec{0} \to \vec{R}) = ?$

Gaussian

② RG perspective on RW

$t = 0, 1, 2, \ldots$

Step