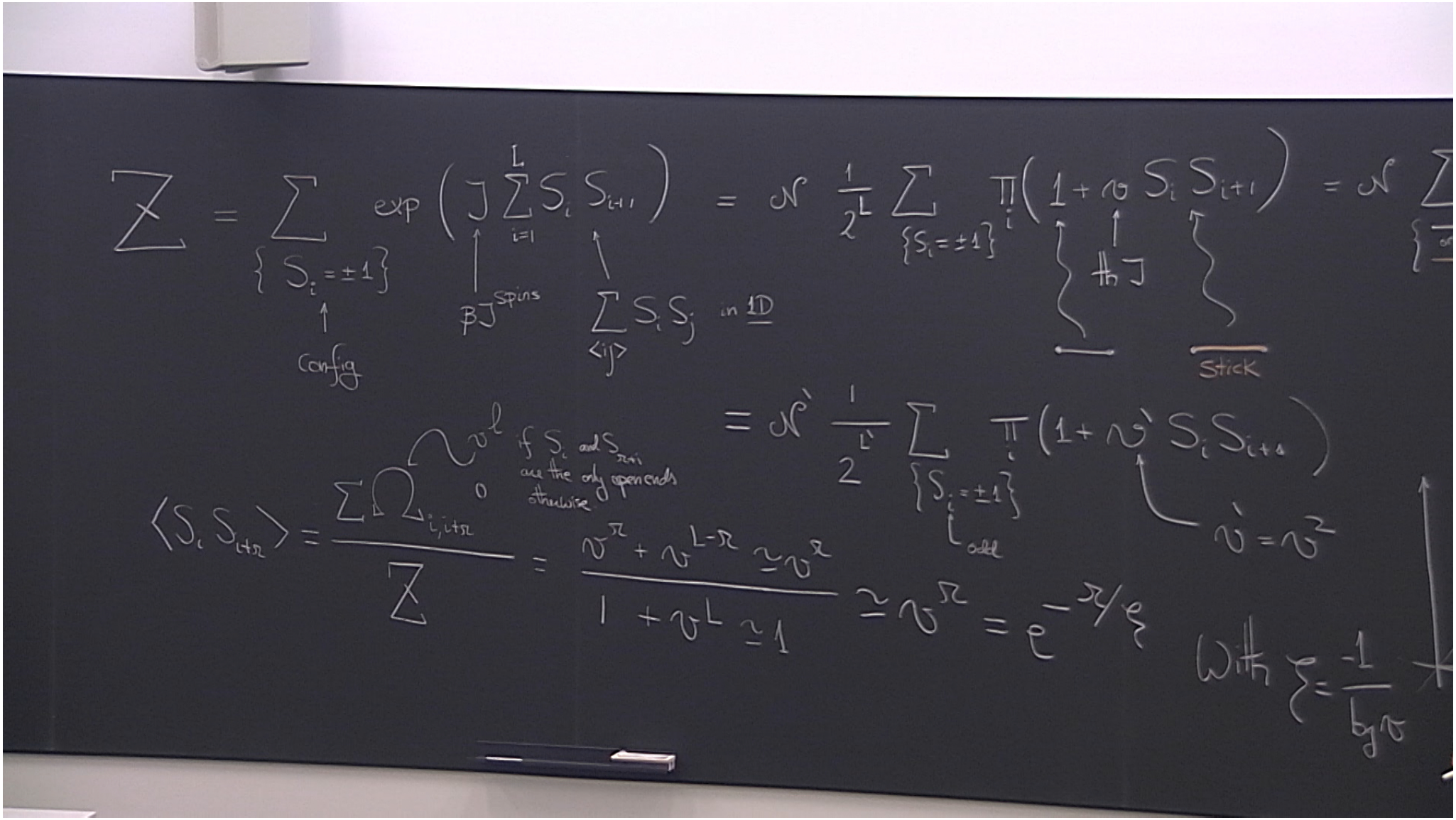
Title: PSI 2019/2020 - Statistical Mechanics (Vieira) - Lecture 2

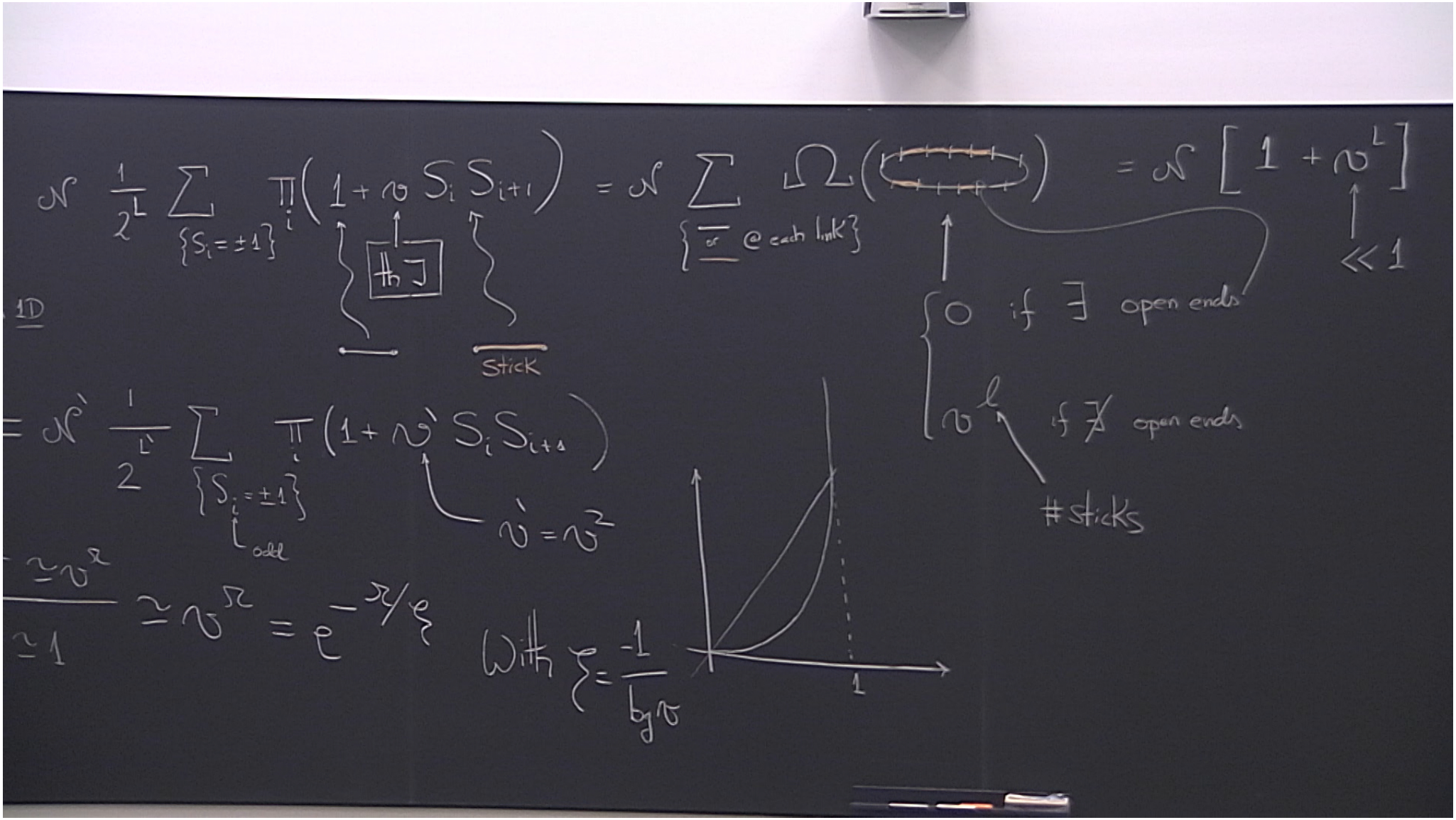
Speakers: Pedro Vieira

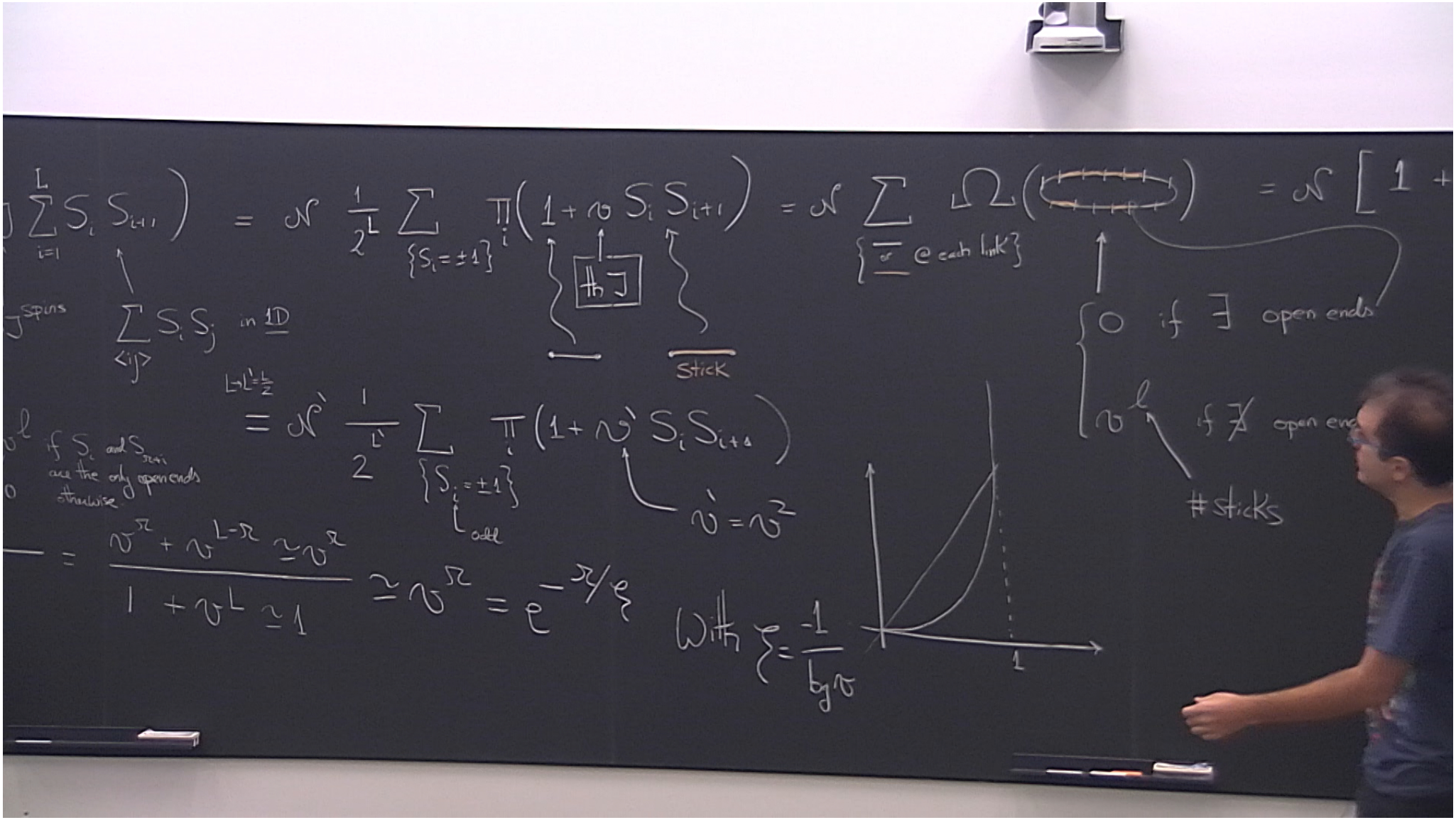
Collection: PSI 2019/2020 - Statistical Mechanics (Vieira)

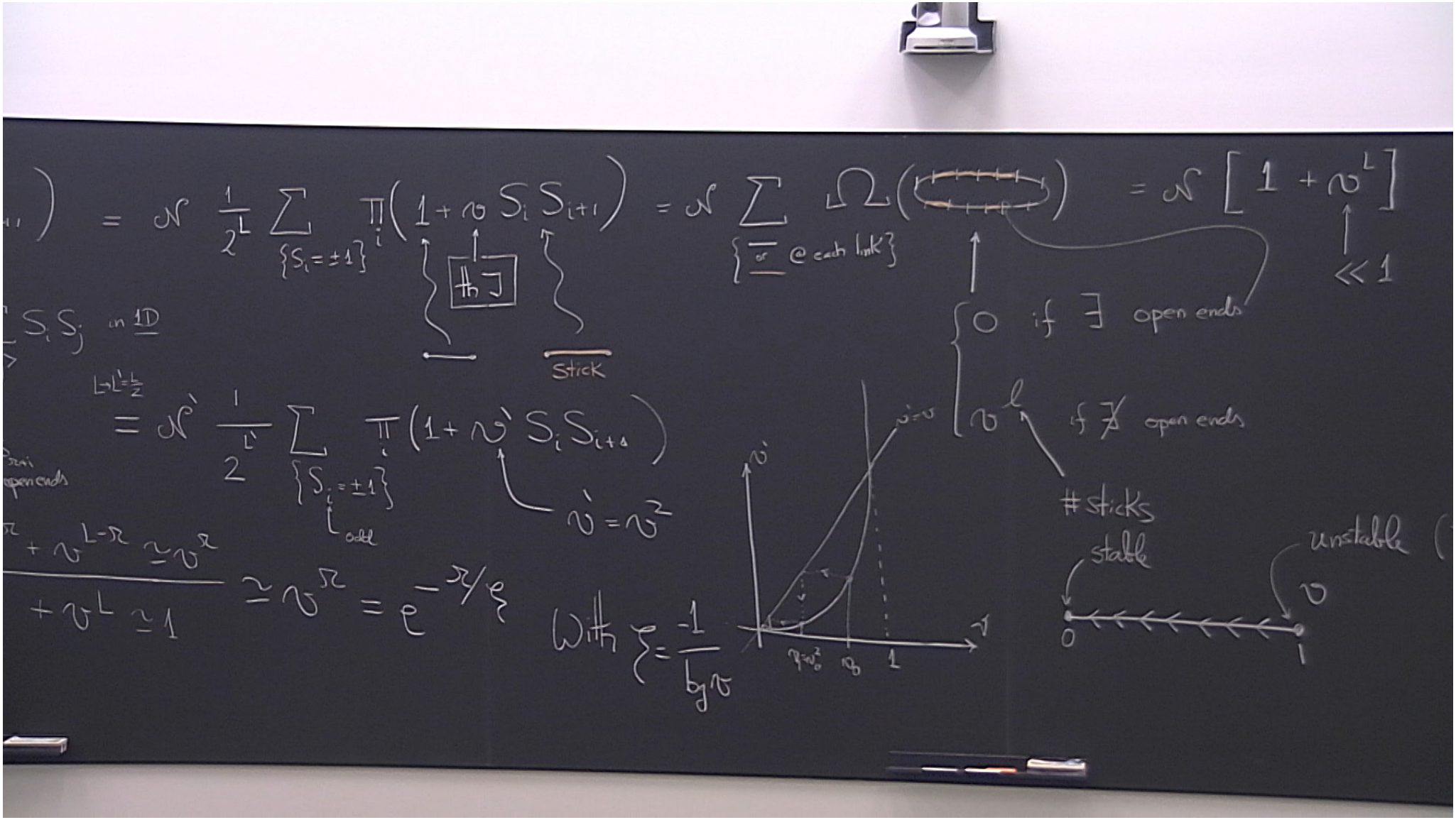
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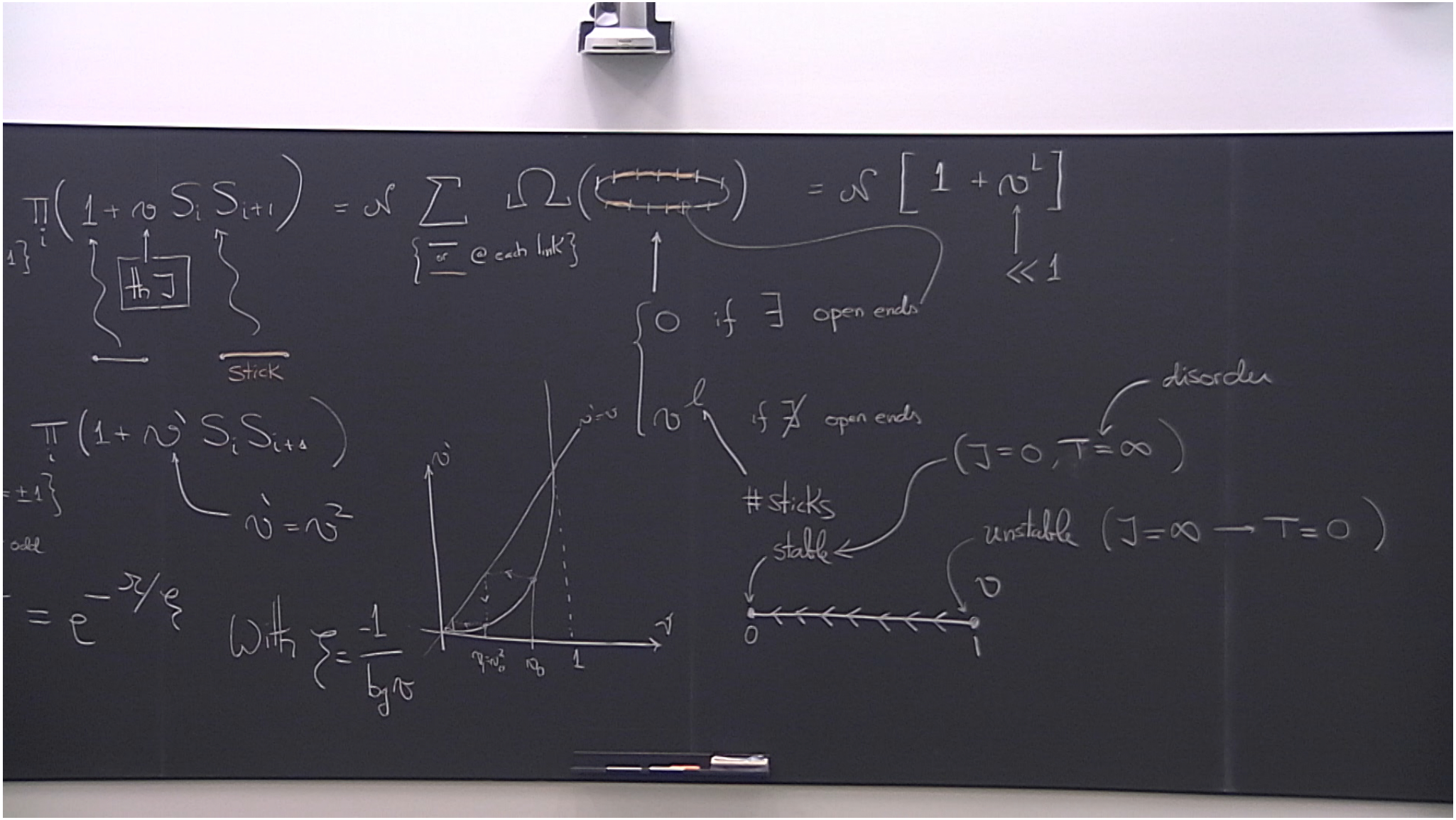
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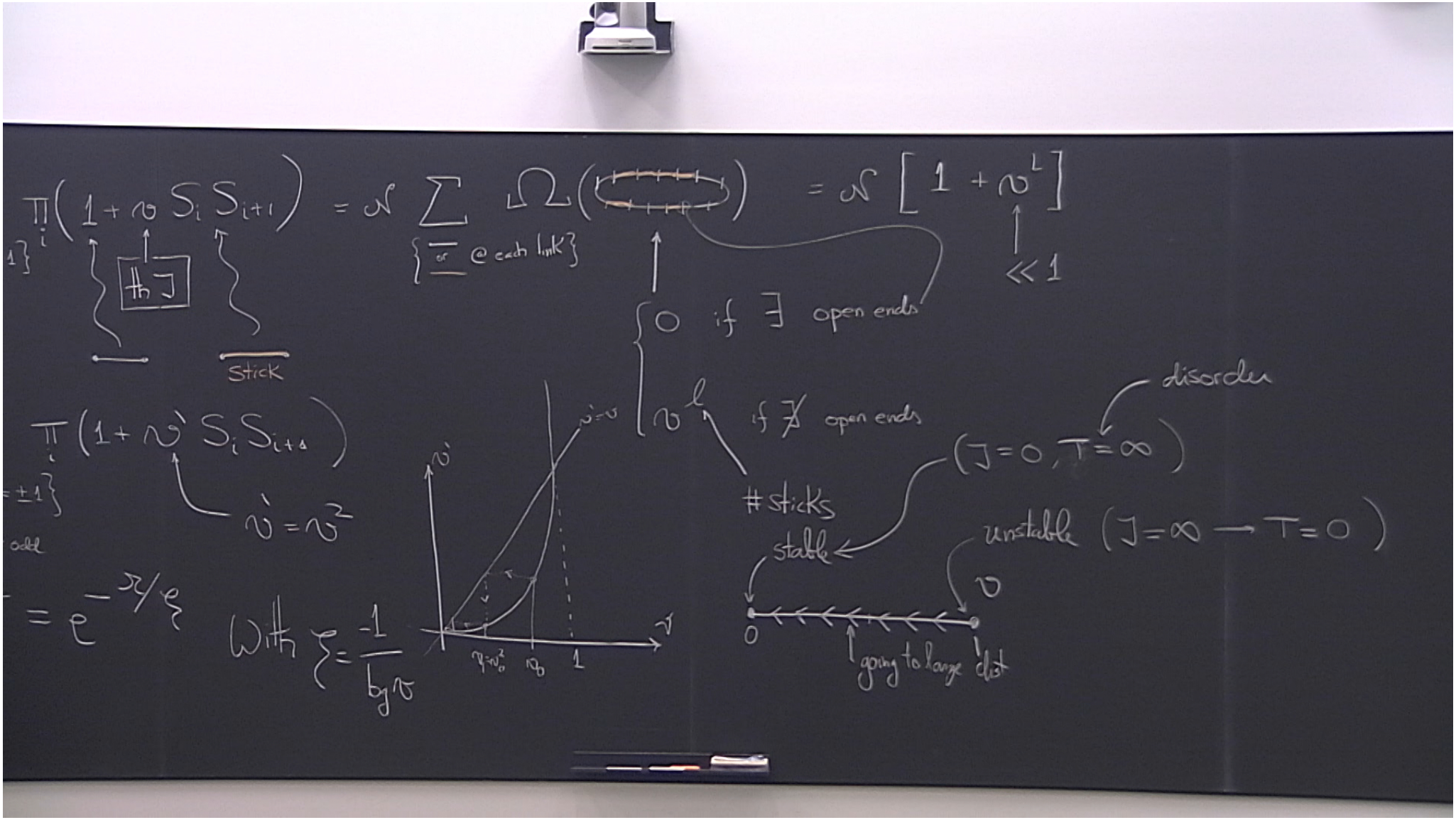

Z = Σ_{S_i = ±1} exp(J Σ_{i=1}^{L} S_i S_{i+1}) = N 1/2^L Σ_{S_i = ±1} Π_i (1 + v S_i S_{i+1})
βJ
spins
Σ_{<ij>} S_i S_j in 1D
config
#h J
stick
= N' 1/2^L Σ_{S_i = ±1} Π_i (1 + v' S_i S_{i+1})
i odd
v' = v^2
⟨S_i S_{i+r}⟩ = Σ Ω_{i,i+r} v^l / Z = (v^r + v^{L-r}) / (1 + v^L) ≃ v^r = e^{-r/ξ}
0 if S_i and S_{i+r} are the only open ends otherwise
With ξ = -1/ln v

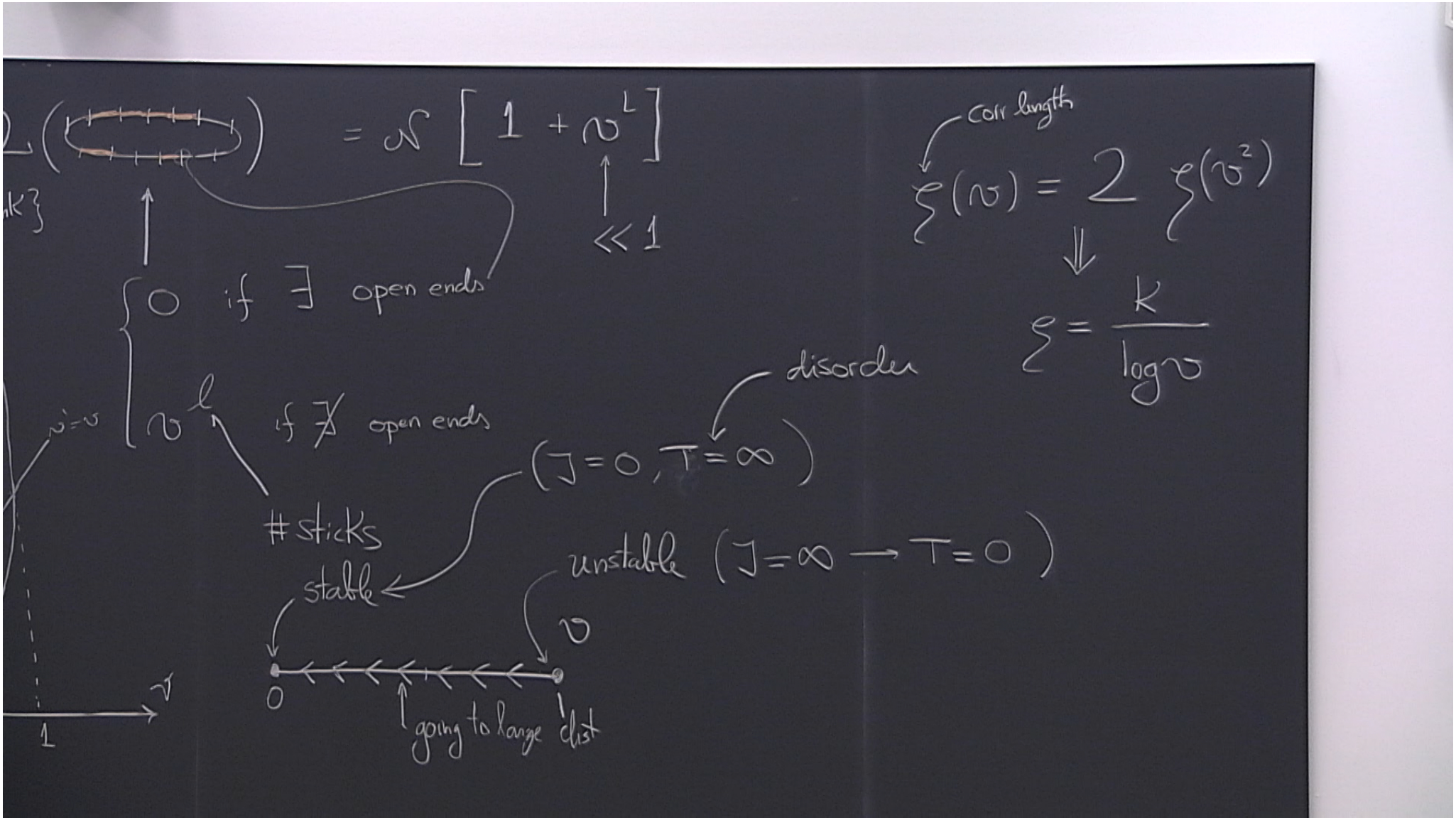

1D
Stick
open ends
sticks
With

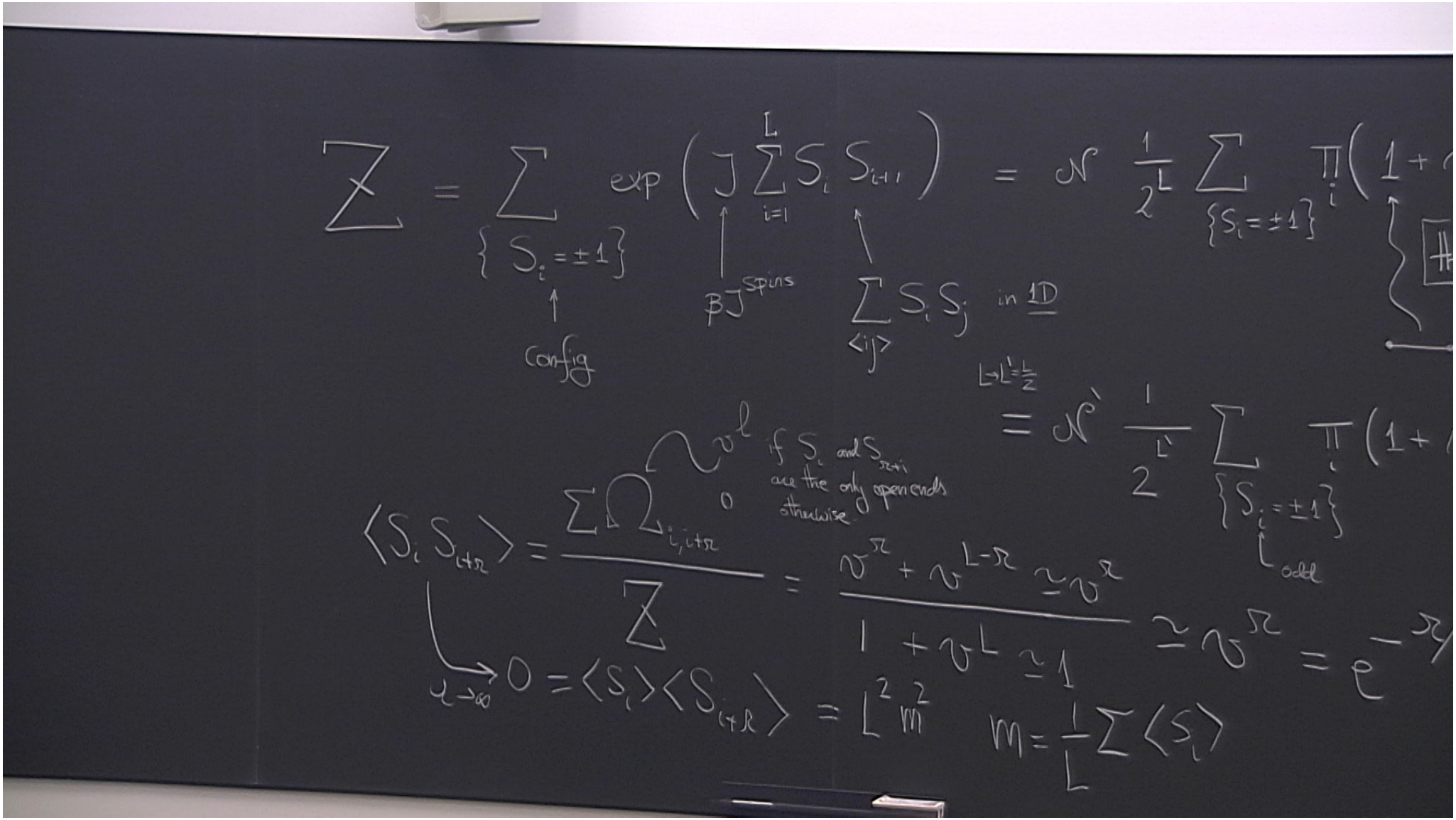

$\sum_{i=1}^{L} S_i S_{i+1}$
$= \mathcal{N} \frac{1}{2^L} \sum_{\{S_i=\pm 1\}} \prod_i \left(1 + v\, S_i S_{i+1}\right)$
$= \mathcal{N} \sum_{\{\text{or } \emptyset \text{ each link}\}} v^{\#\text{sticks}}$
$\sum_{\langle ij \rangle} S_i S_j$ in 1D
spins
Stick
$L \to L' = \frac{L}{2}$
$= \mathcal{N}' \frac{1}{2^{L'}} \sum_{\{S_i=\pm 1\}} \prod_i \left(1 + v' S_i S_{i+1}\right)$
$v' = v^2$
if S_i and S_{i+1} are the only open ends
otherwise
$= \frac{v^x + v^{L-x}}{1 + v^L} \simeq v^x = e^{-x/\xi}$
With $\xi = \frac{-1}{\log v}$
1
0 if $\exists$ open ends
v^{ℓ} if $\nexists$ open ends
#sticks

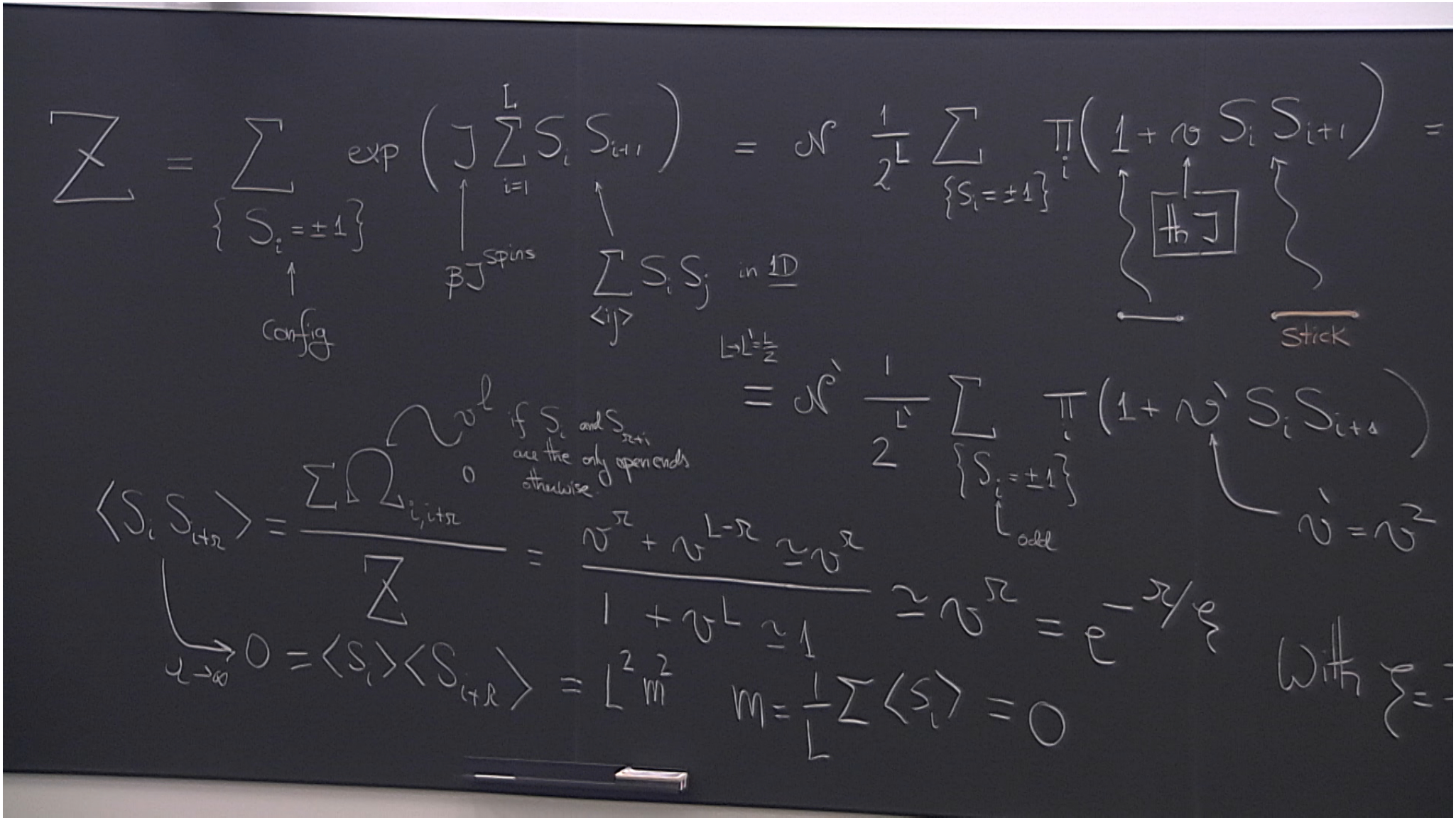

stick
stable
unstable
sticks
open ends
@ each link
in 1D

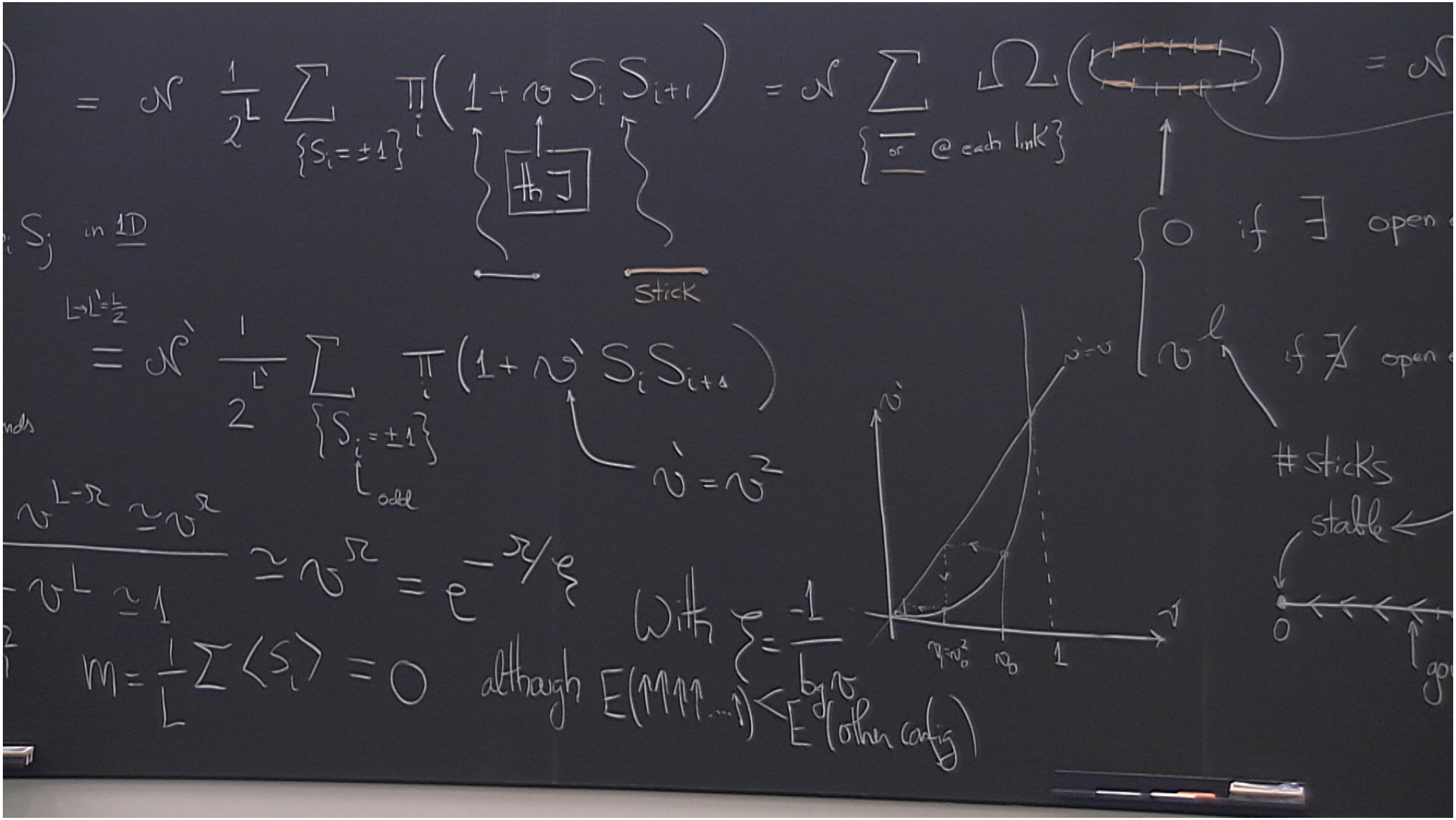

stick
disorder
(J=0, T=∞)
unstable (J=∞ → T=0)
stable
sticks
open ends
open ends
odd

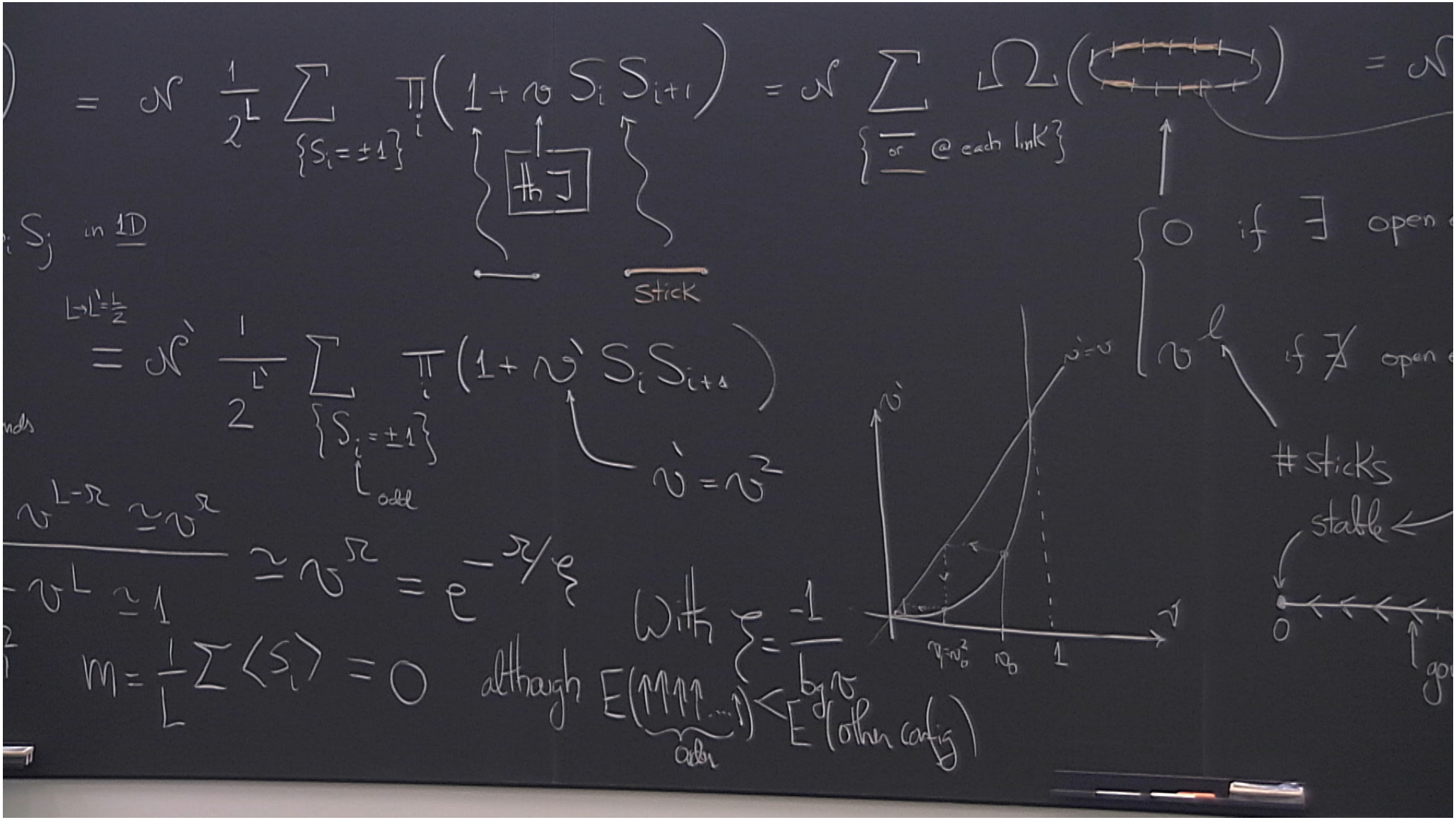

stick
disorder
(J=0, T=∞)
unstable (J=∞ → T=0)
stable
sticks
going to large dist
With ξ = -1 / ln v

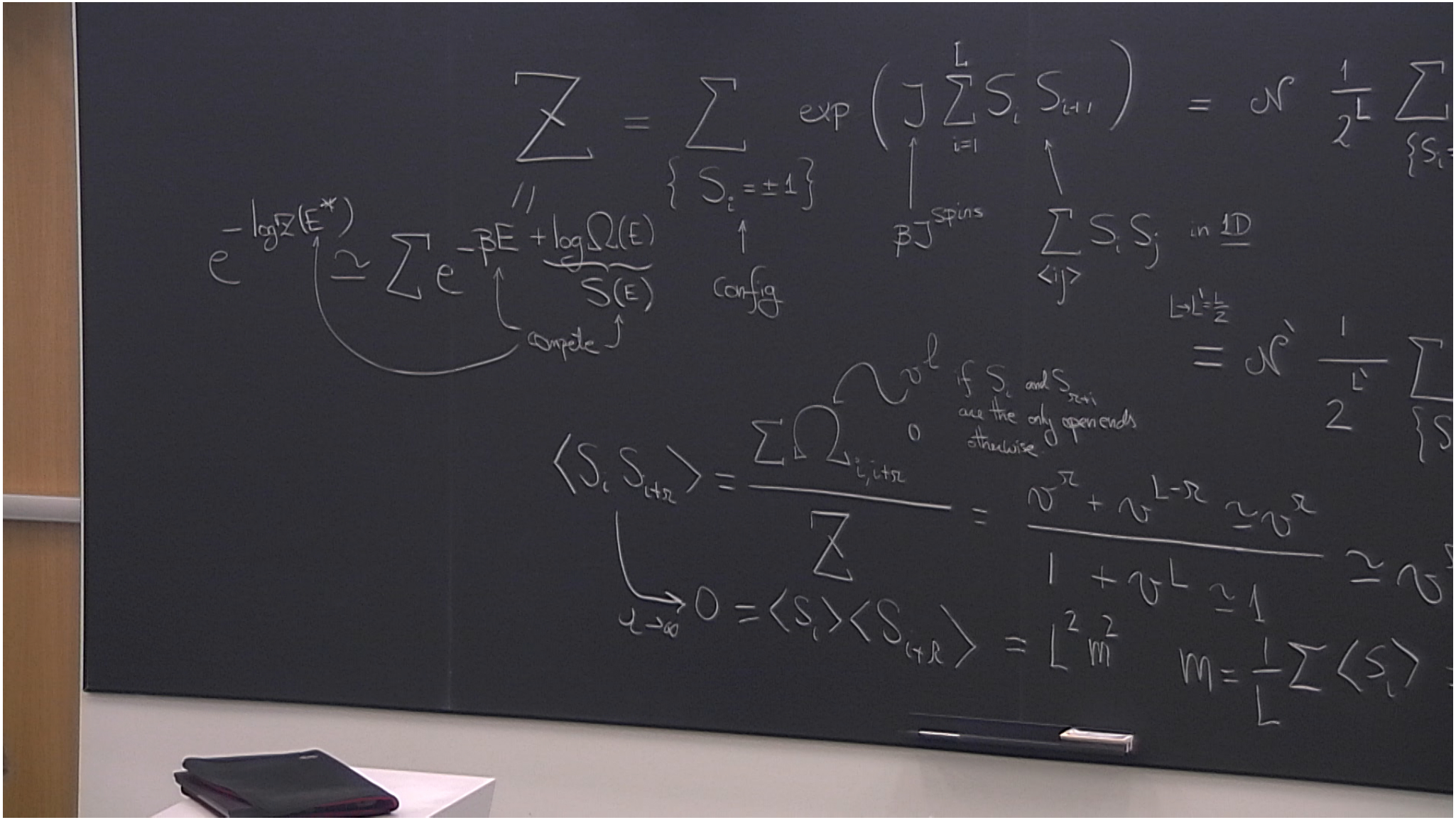

$= v \left[1 + v^L \right]$
$\ll 1$
corr length
$\xi(v) = 2\,\xi(v^2)$
$\Downarrow$
$\xi = \frac{k}{\log v}$
$\begin{cases} 0 & \text{if } \exists \text{ open ends} \\ v^L & \text{if } \nexists \text{ open ends} \end{cases}$
disorder
$(J=0, T=\infty)$
sticks
stable
unstable $(J=\infty \rightarrow T=0)$
v
0
going to large dist
$v'=v$
v
1

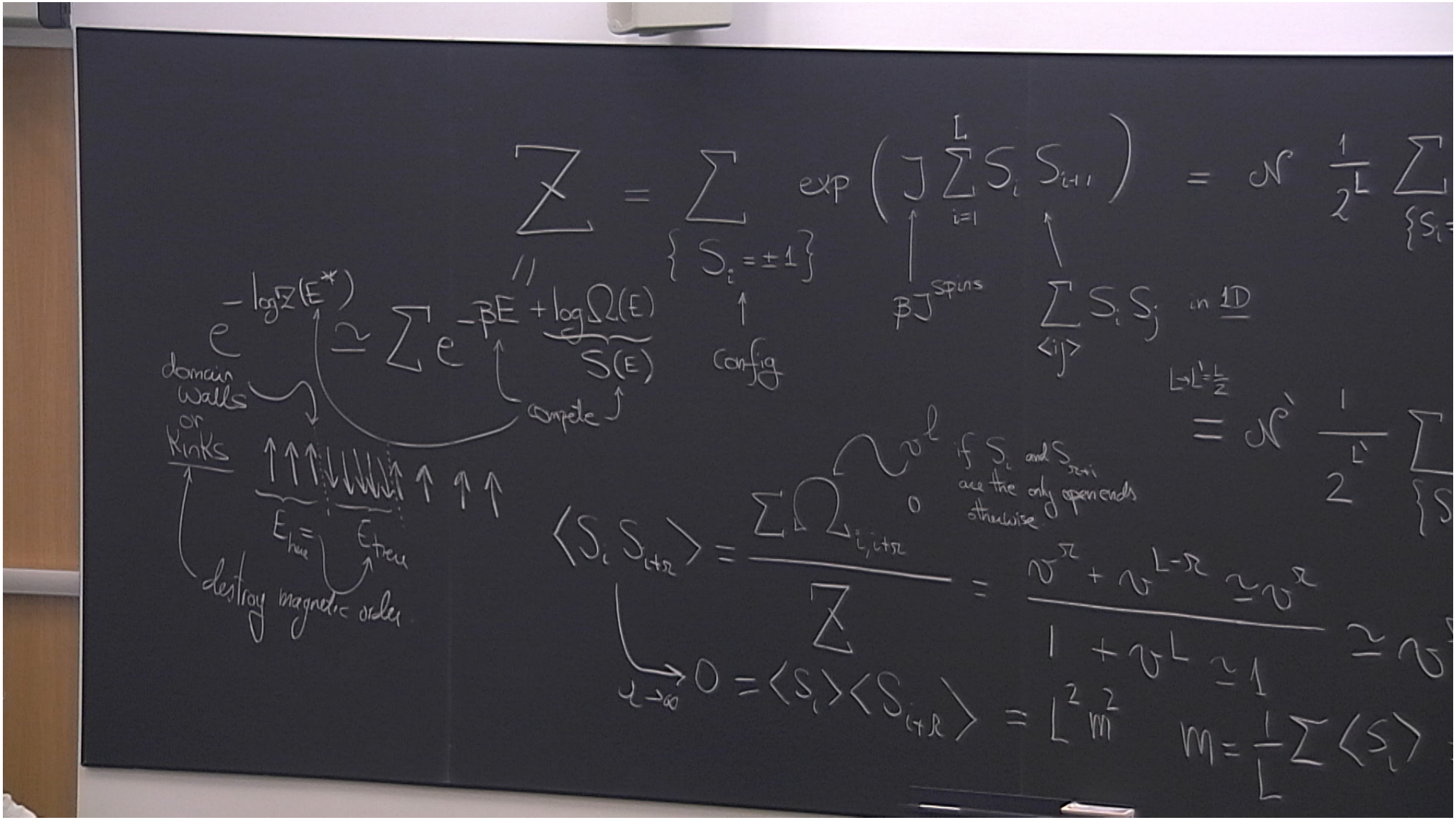

$Z = \sum_{\{S_i = \pm 1\}} \exp\left(J \sum_{i=1}^{L} S_i S_{i+1}\right) = \mathcal{N} \frac{1}{2^L} \sum_{\{S_i = \pm 1\}} \prod_i \left(1 + \right.$
config
βJ spins
$\sum_{\langle ij \rangle} S_i S_j$ in 1D
$L \to L' = \frac{L}{2}$
$= \mathcal{N}' \frac{1}{2^{L'}} \sum_{\{S_i = \pm 1\}} \prod_i (1 +$
odd
$\langle S_i S_{i+r} \rangle = \frac{\sum \Omega_{i,i+r} v^l}{Z}$
if S_i and S_{r+i} are the only open ends otherwise 0
$= \frac{v^r + v^{L-r}}{1 + v^L} \simeq v^r \simeq e^{-r/}$
$r \to \infty$
$0 = \langle S_i \rangle \langle S_{i+r} \rangle = L^2 m^2$
$m = \frac{1}{L} \sum \langle S_i \rangle$

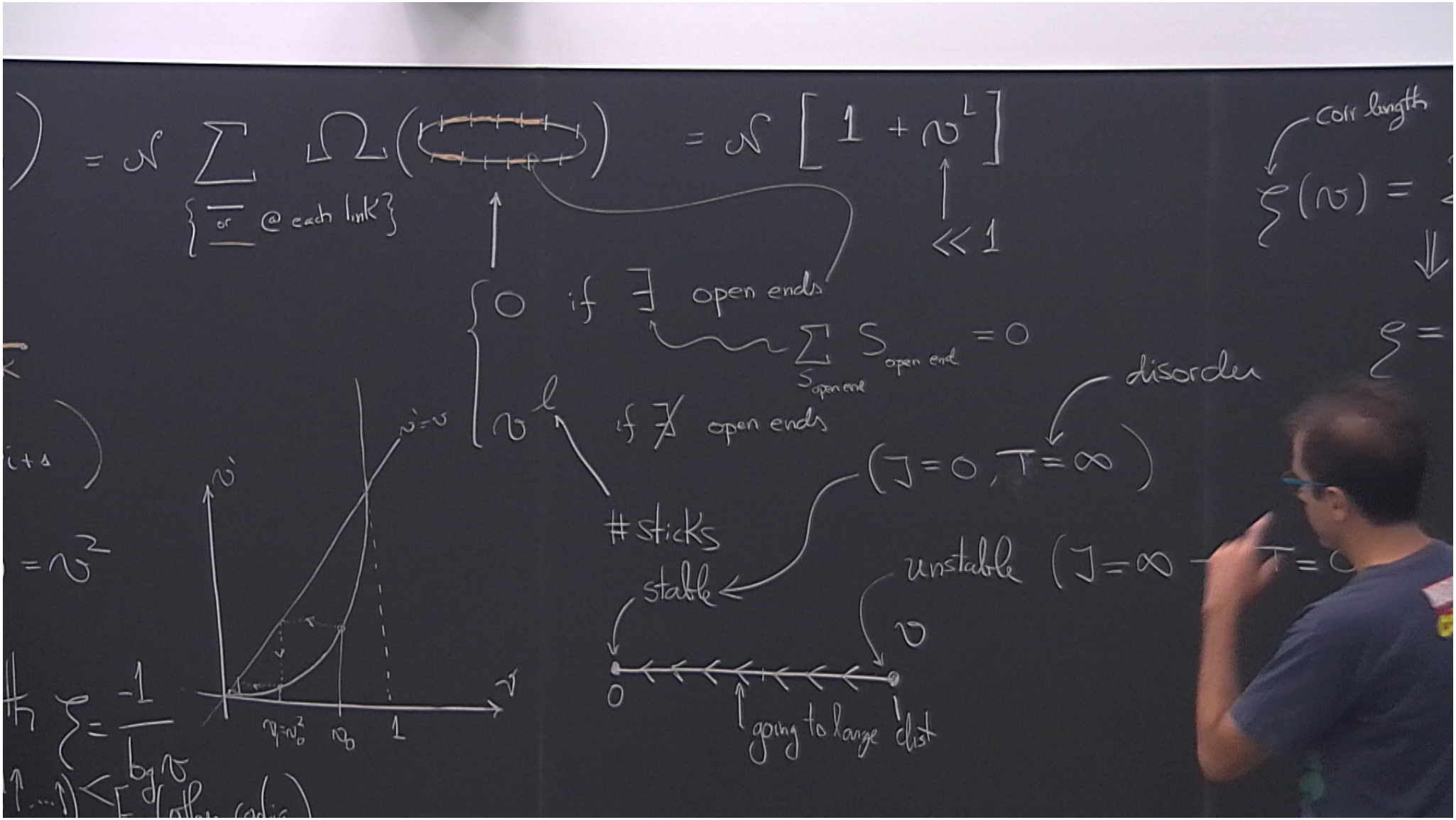

Z = Σ_{S_i = ±1} exp(J Σ_{i=1}^{L} S_i S_{i+1}) = ... Π_i (1 + v S_i S_{i+1})
βJ
spins
Σ_{<ij>} S_i S_j in 1D
config
#n J
stick
L → L' = L/2
= ... Σ_{S_i = ±1, i odd} Π_i (1 + v' S_i S_{i+2})
v' = v^2
v^l if S_i and S_{i+l} are the only open ends
0 otherwise
⟨S_i S_{i+r}⟩ = Σ Ω_{i,i+r} / Z = (v^r + v^{L-r}) / (1 + v^L) ≃ v^r = e^{-r/ξ}
with ξ =
r → ∞
0 = ⟨S_i⟩⟨S_{i+r}⟩ = L^2 m^2
m = (1/L) Σ ⟨S_i⟩ = 0


Stick
stable
sticks
with
although


Stick
sticks
stable
with
by
although
other config


Z = Σ exp (J Σ_{i=1}^{L} S_i S_{i+1}) = ν 1/2^L Σ
{S_i = ±1}
βJ
spins
Σ_{<ij>} S_i S_j in 1D
config
e^{-log Z(E*)} ≃ Σ e^{-βE + log Ω(E)}
S(E)
compute
L → L' = L/2
= ν' 1/2^{L'} Σ
if S_i and S_{i+r} are the only open ends otherwise
⟨S_i S_{i+r}⟩ = Σ Ω_{i,i+r} / Z = (v^r + v^{L-r}) / (1 + v^L) ≃ v^r
r → ∞
0 = ⟨S_i⟩⟨S_{i+r}⟩ = L^2 m^2
m = 1/L Σ ⟨S_i⟩

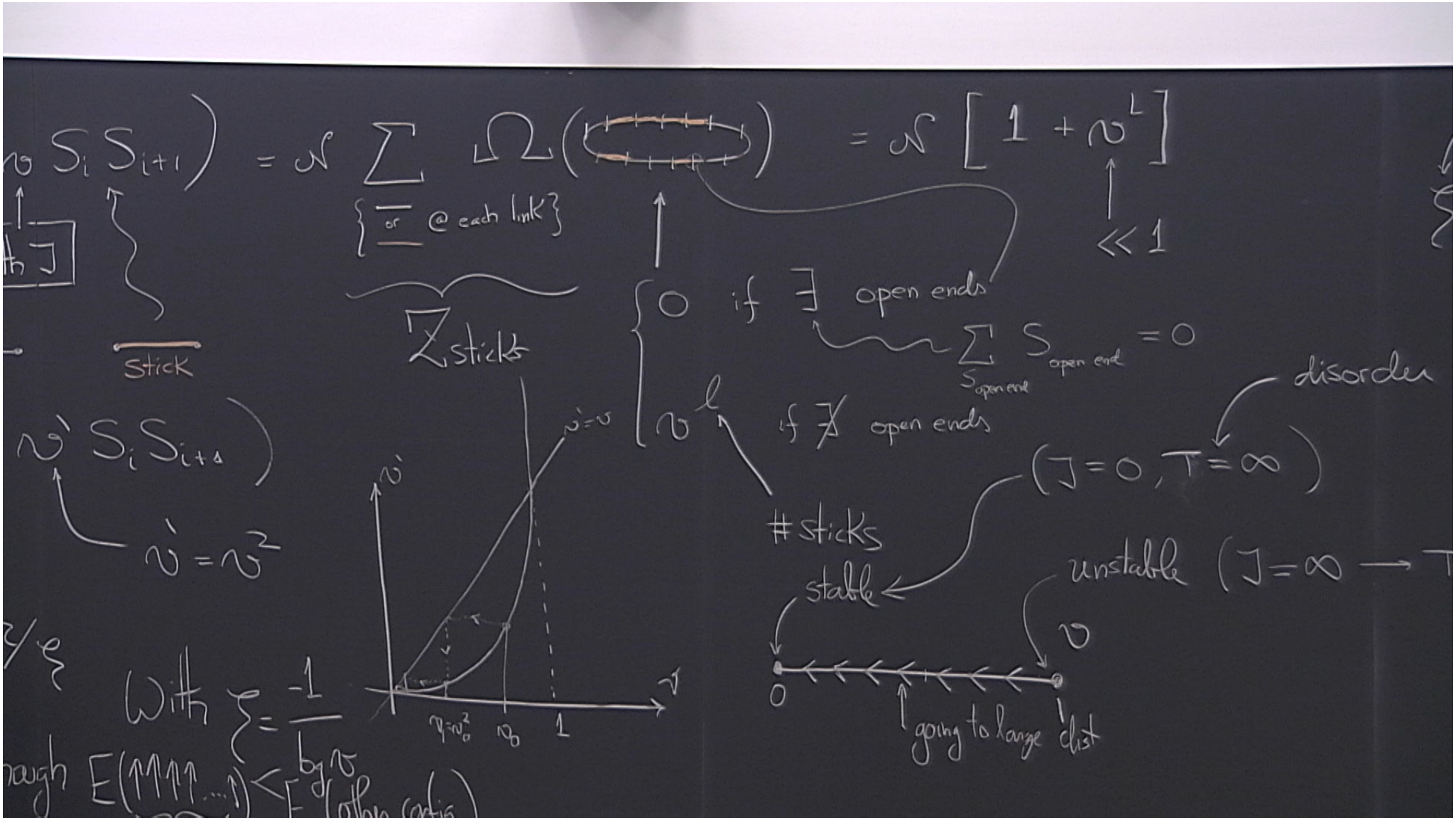

$Z = \sum_{\{S_i = \pm 1\}} \exp\left(J \sum_{i=1}^{L} S_i S_{i+1}\right)$
βJ spins
$\sum_{\langle ij \rangle} S_i S_j$ on 1D
config
$e^{-\log Z(E^*)} \simeq \sum e^{-\beta E + \log \Omega(E)}$
$S(E)$
compete
domain walls or Kinks
E_{few} E_{many}
destroy magnetic order
$\langle S_i S_{i+r} \rangle = \frac{\sum \Omega_{i,i+r}}{Z}$
v^r
if S_i and S_{i+r} are the only open ends otherwise
$= \frac{v^r + v^{L-r}}{1 + v^L} \simeq v^r$
$r \to \infty$
$0 = \langle S_i \rangle \langle S_{i+r} \rangle = L^2 m^2$
$m = \frac{1}{L} \sum \langle S_i \rangle$
$L \to L' = \frac{L}{2}$
$= \mathcal{N}' \frac{1}{2^{L'}} \sum$

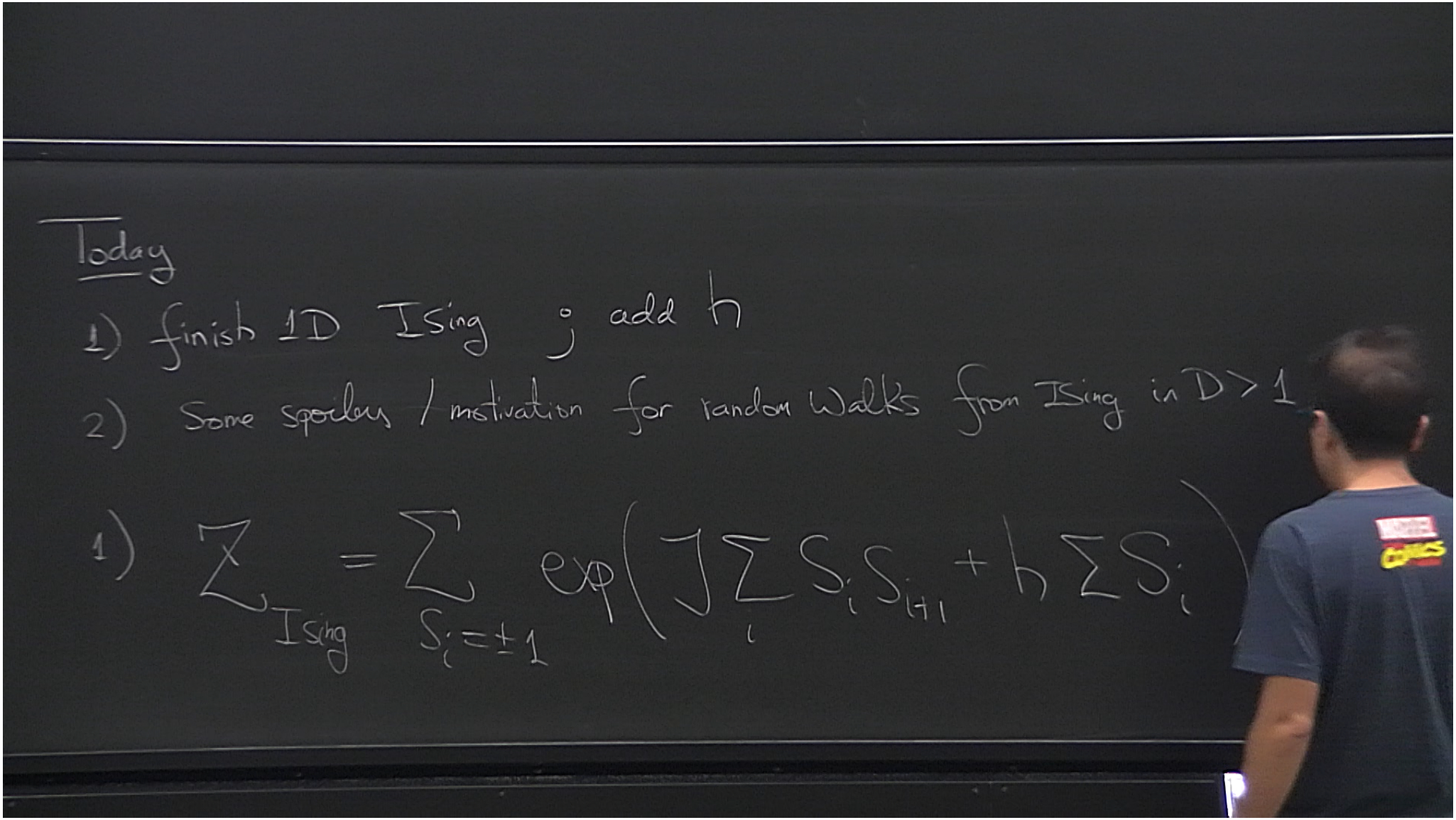

corr length
disorder
(J=0, T=∞)
unstable
stable
sticks
going to large dist
open ends
@ each link

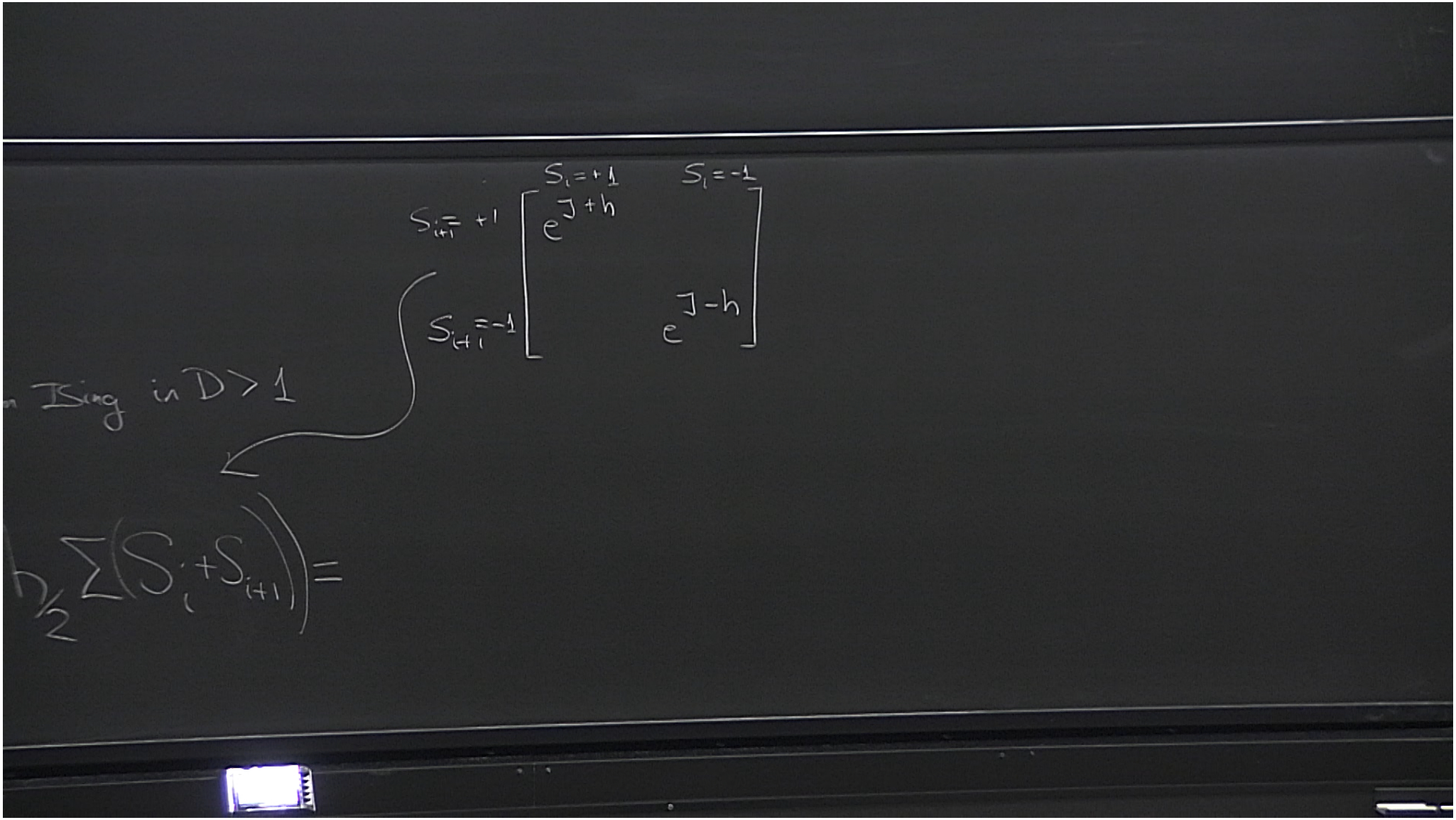
Today

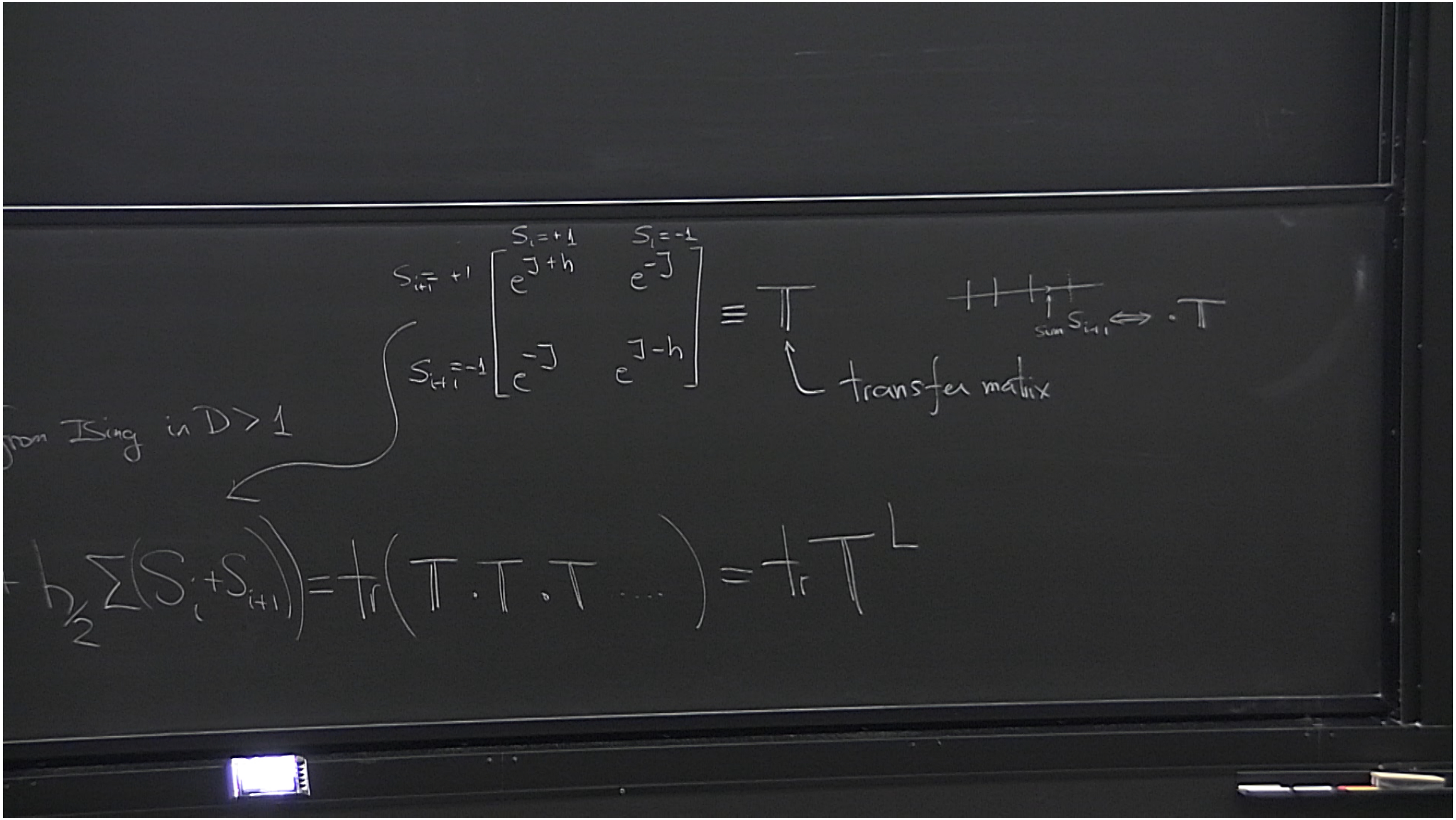
1) finish 1D Ising ; add h

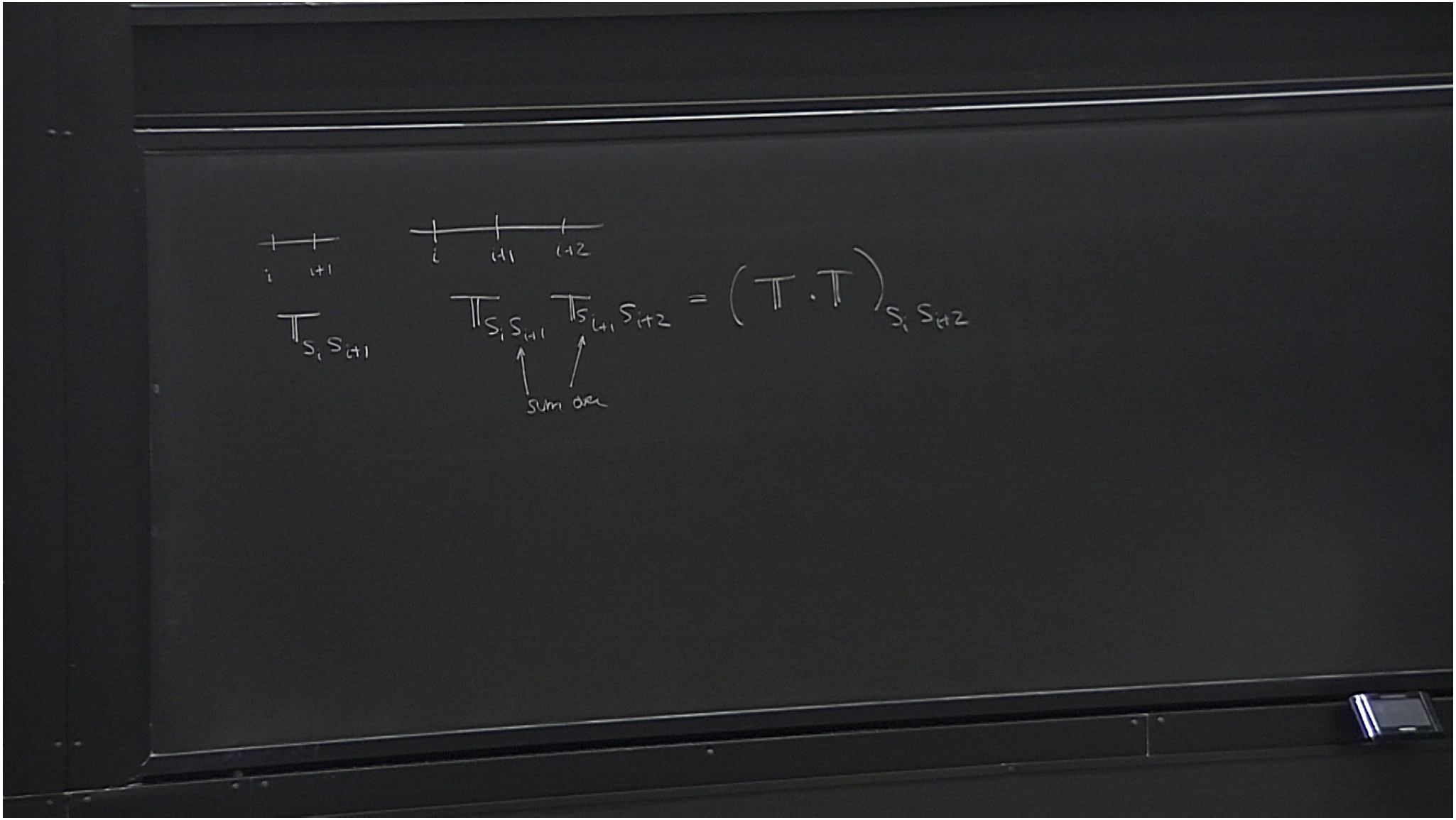
2) Some spoilers / motivation for random walks

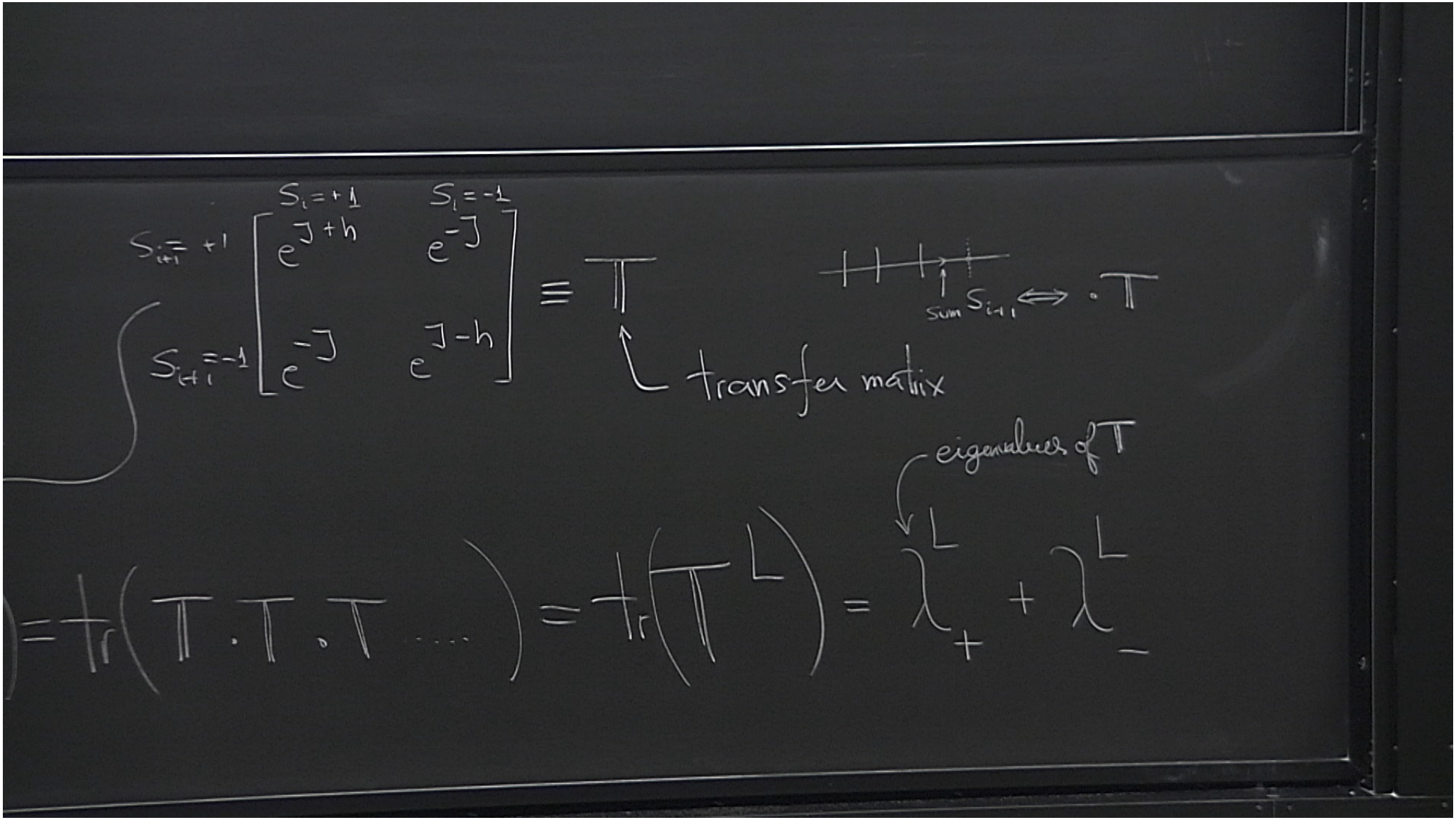

stick
open ends
disorder
stable
unstable
going to large dist

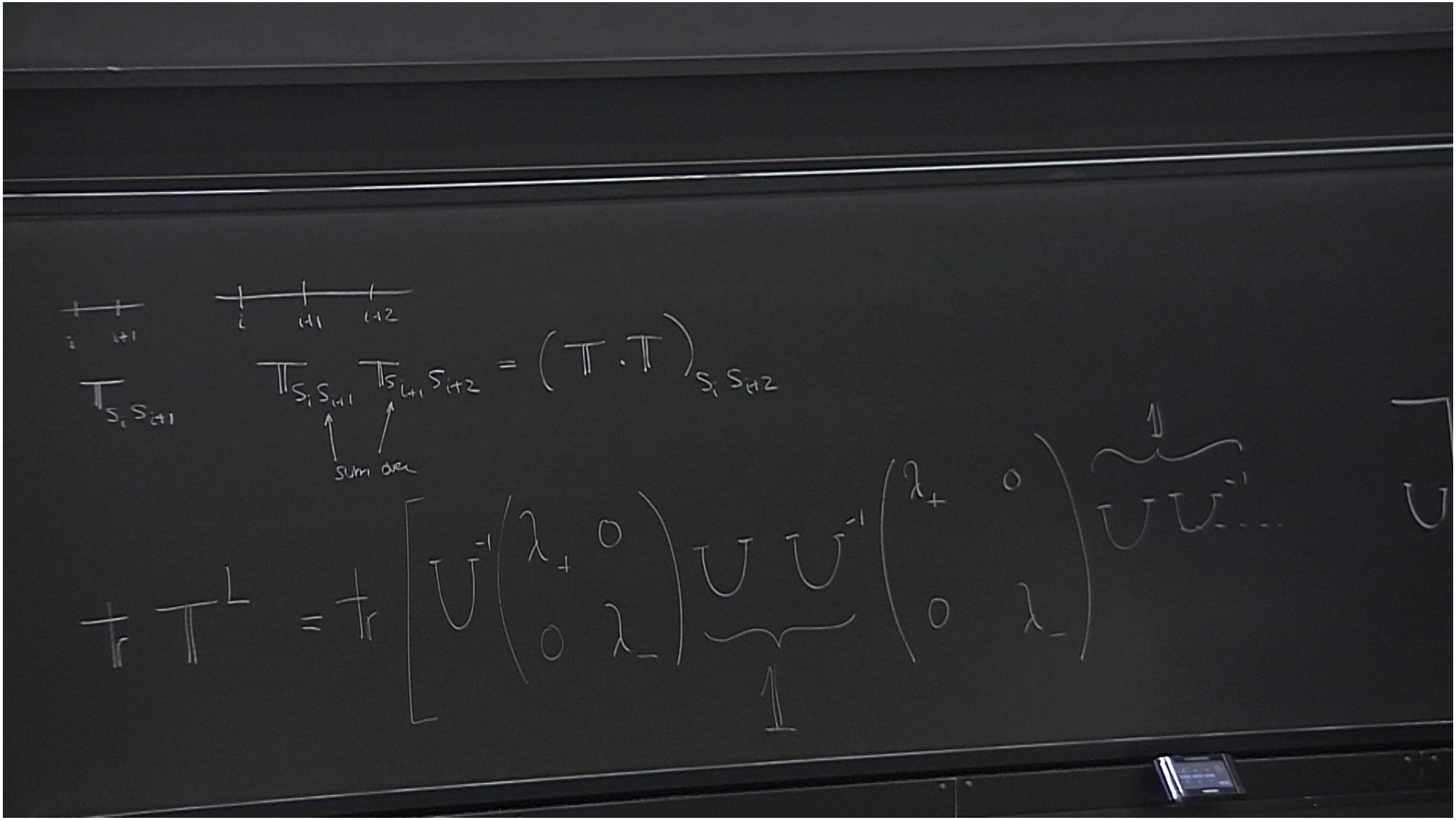

Today
1) finish 1D Ising ; add h
2) Some spoilers / motivation for random Walks from Ising in D > 1
1) $Z_{\text{Ising}} = \sum_{S_i = \pm 1} \exp\left(J \sum_i S_i S_{i+1} + h \sum S_i\right)$

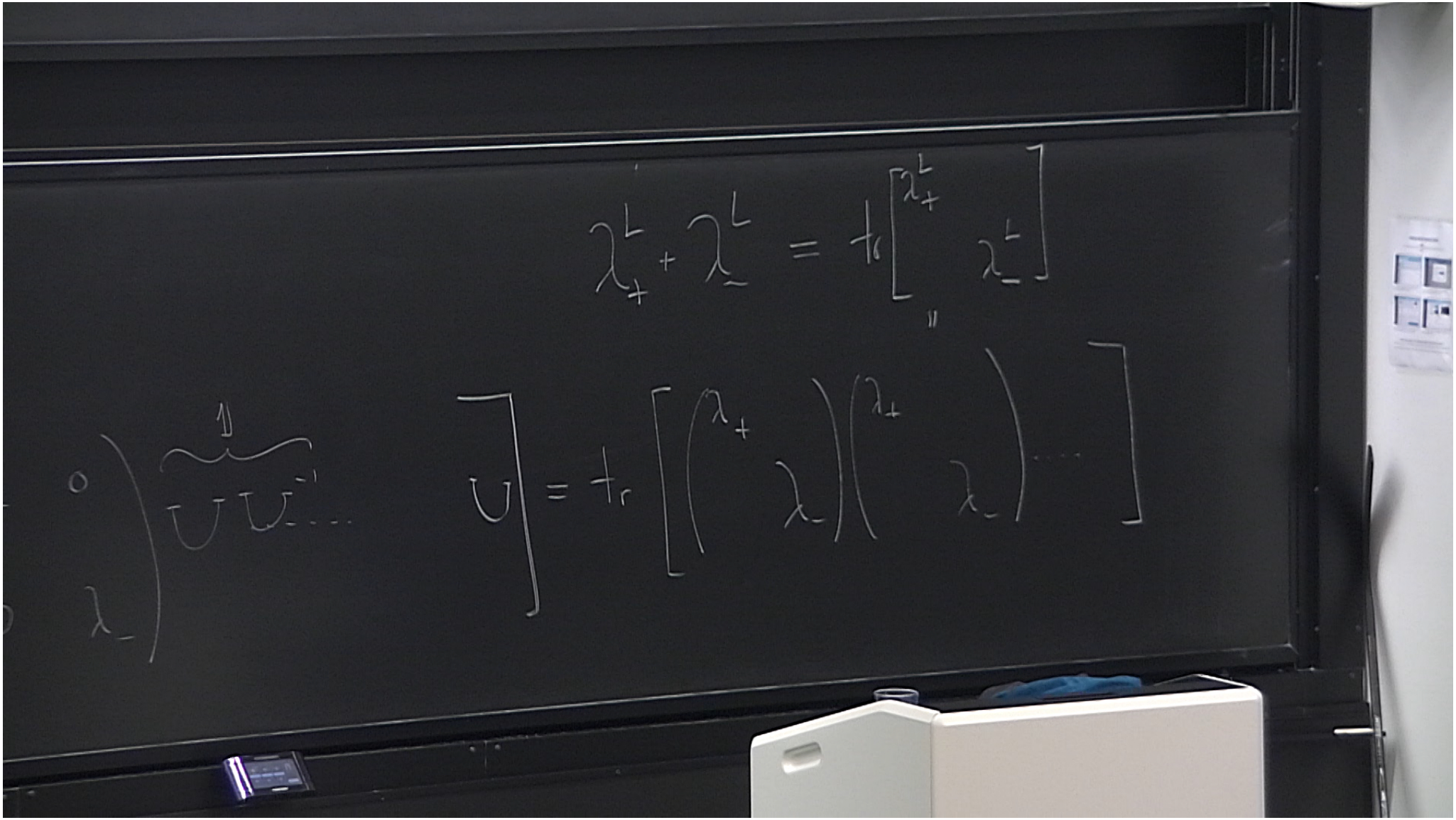

Ising in D > 1
$S_i = +1$
$S_i = -1$
$S_{i+1} = +1$
$S_{i+1} = -1$
e^{J+h}
e^{J-h}
$\frac{1}{2}\sum\left(S_i + S_{i+1}\right) =$

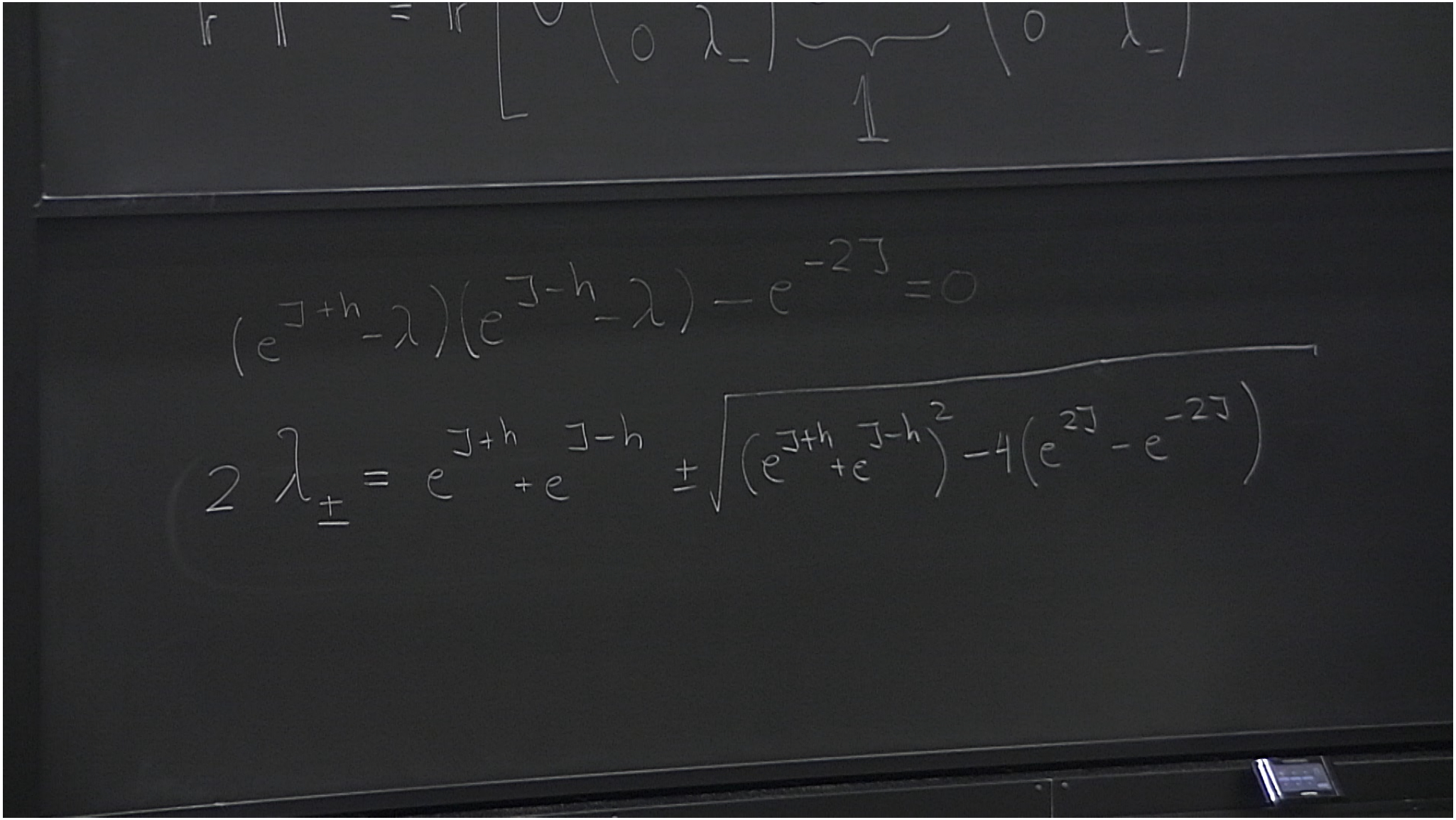

$S_i=+1$ $S_i=-1$
$S_{i+1}=+1$ e^{J+h} e^{-J}
$S_{i+1}=-1$ e^{-J} e^{J-h}
$\equiv T$
$\cdot T$
transfer matrix
$\frac{h}{2}\sum(S_i+S_{i+1})$
$= Tr(T \cdot T \cdot T \cdots) = Tr\, T^L$

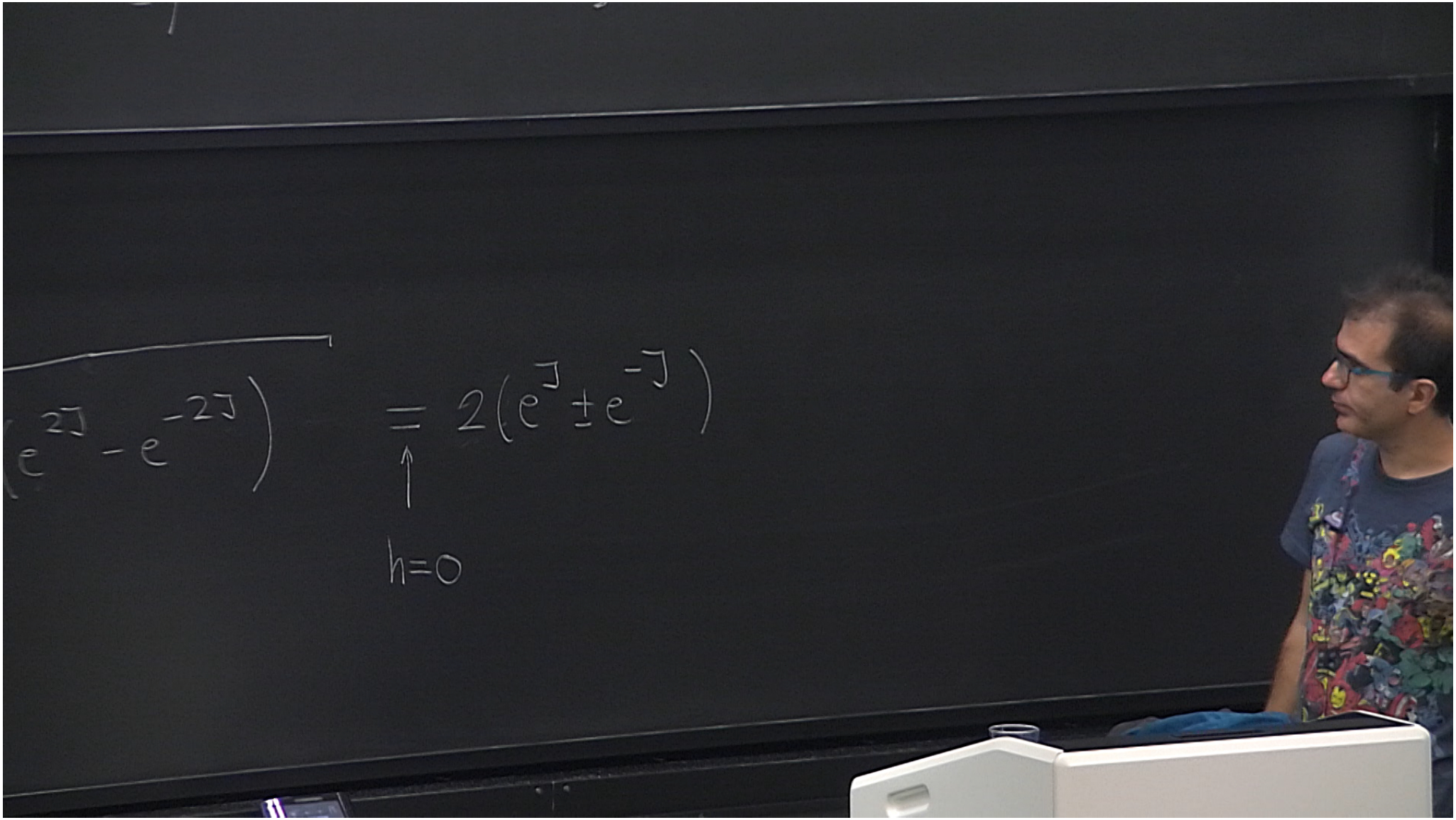

i
i+1
i
i+1
i+2
$T_{s_i s_{i+1}}$
$T_{s_i s_{i+1}} T_{s_{i+1} s_{i+2}} = (T \cdot T)_{s_i s_{i+2}}$
sum over

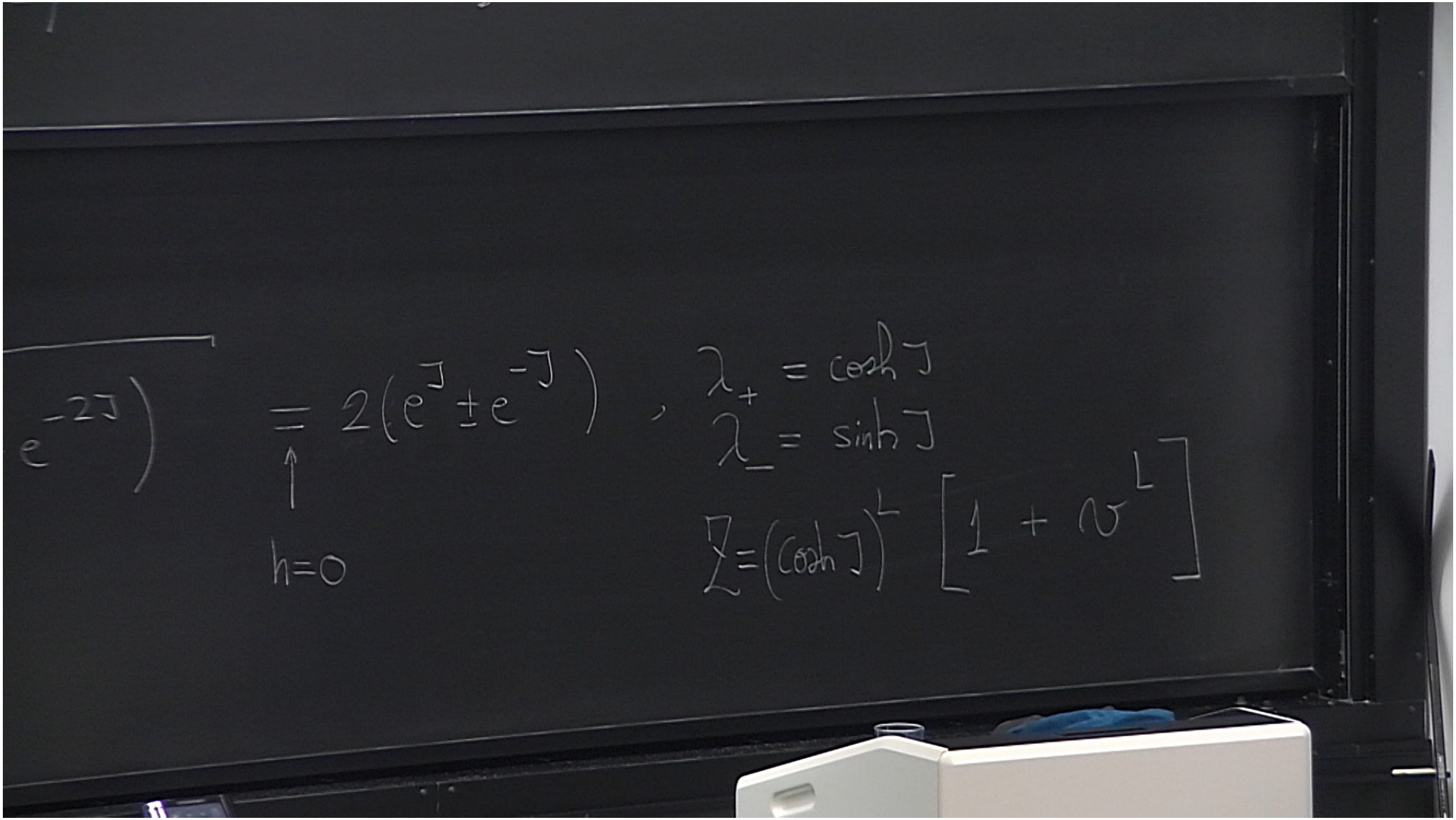

$S_i = +1$ $S_i = -1$
$S_{i+1} = +1$ e^{J+h} e^{-J}
$S_{i+1} = -1$ e^{-J} e^{J-h}
$\equiv T$
transfer matrix
sum S_{i+1} $\Leftrightarrow \cdot T$
eigenvalues of T
$= \mathrm{Tr}(T \cdot T \cdot T \cdots) = \mathrm{Tr}(T^L) = \lambda_+^L + \lambda_-^L$

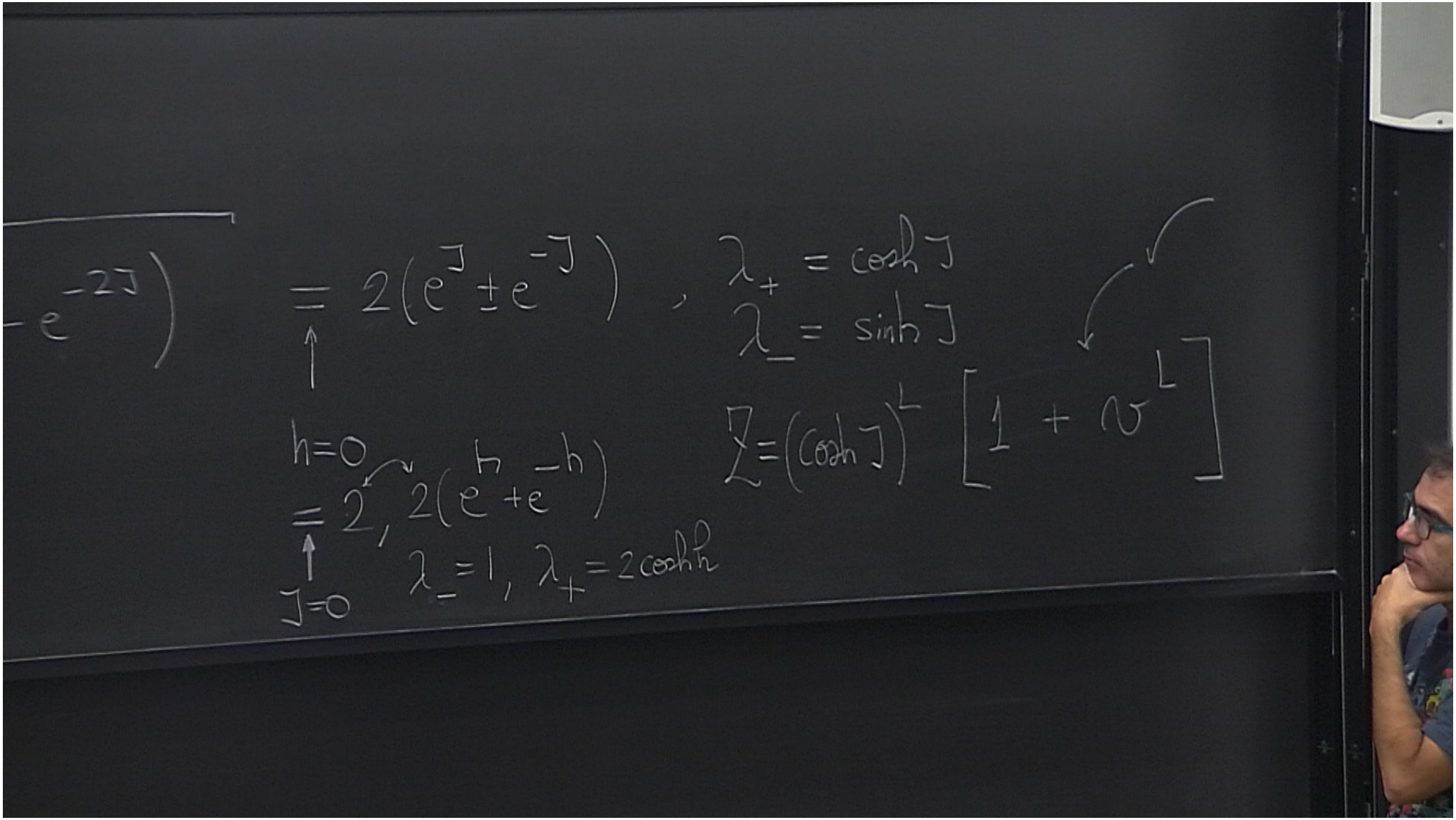

i i+1
i i+1 i+2
$T_{s_i s_{i+1}}$
$T_{s_i s_{i+1}} T_{s_{i+1} s_{i+2}} = (T \cdot T)_{s_i s_{i+2}}$
sum over
$\mathrm{Tr}\, T^L = \mathrm{Tr}\left[U^{-1} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} U \underbrace{U^{-1}}_{1} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \overbrace{U U^{-1} \cdots}^{1} U \right]$

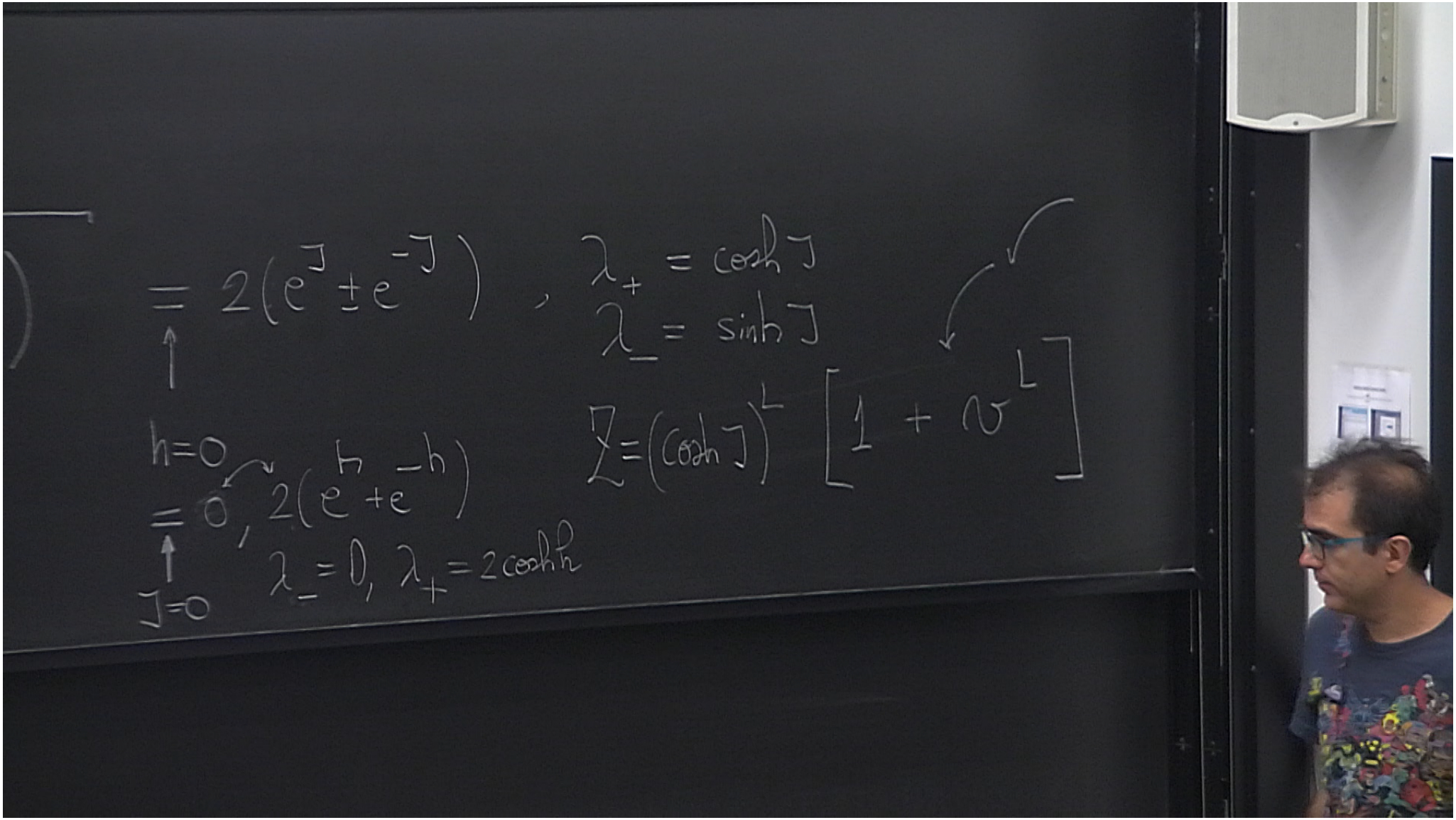

$\lambda_+^L + \lambda_-^L = tr\begin{bmatrix} \lambda_+^L & \\ & \lambda_-^L \end{bmatrix}$
$\lambda_-)\overbrace{UU^{-1}\ldots}^{1}U] = tr\left[\begin{pmatrix}\lambda_+ & \\ & \lambda_-\end{pmatrix}\begin{pmatrix}\lambda_+ & \\ & \lambda_-\end{pmatrix}\ldots\right]$

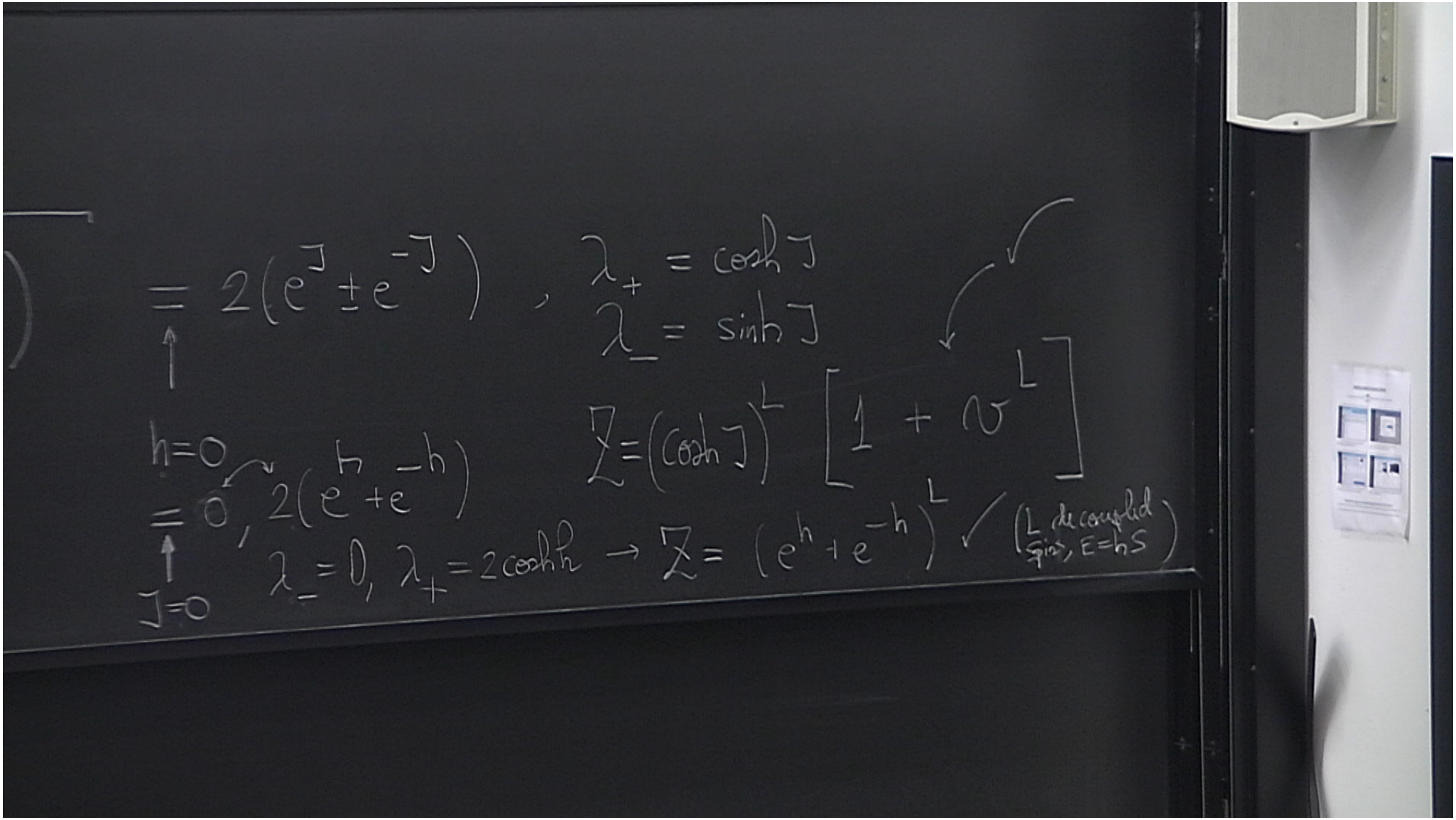

$(e^{J+h} - \lambda)(e^{J-h} - \lambda) - e^{-2J} = 0$
$2\lambda_{\pm} = e^{J+h} + e^{J-h} \pm \sqrt{(e^{J+h} + e^{J-h})^2 - 4(e^{2J} - e^{-2J})}$


$(e^{2J} - e^{-2J}) = 2(e^{J} \pm e^{-J})$
$h=0$


e^{-2J})
$= 2(e^{J} \pm e^{-J})$
$h=0$
$\lambda_+ = \cosh J$
$\lambda_- = \sinh J$
$Z = (\cosh J)^L [1 + v^L]$


$-e^{-2J}$
$= 2(e^{J} \pm e^{-J})$
$\lambda_+ = \cosh J$
$\lambda_- = \sinh J$
$h=0$
$= 2, 2(e^{h} + e^{-h})$
$J=0$
$\lambda_- = 1, \lambda_+ = 2\cosh h$
$Z = (\cosh J)^L [1 + v^L]$

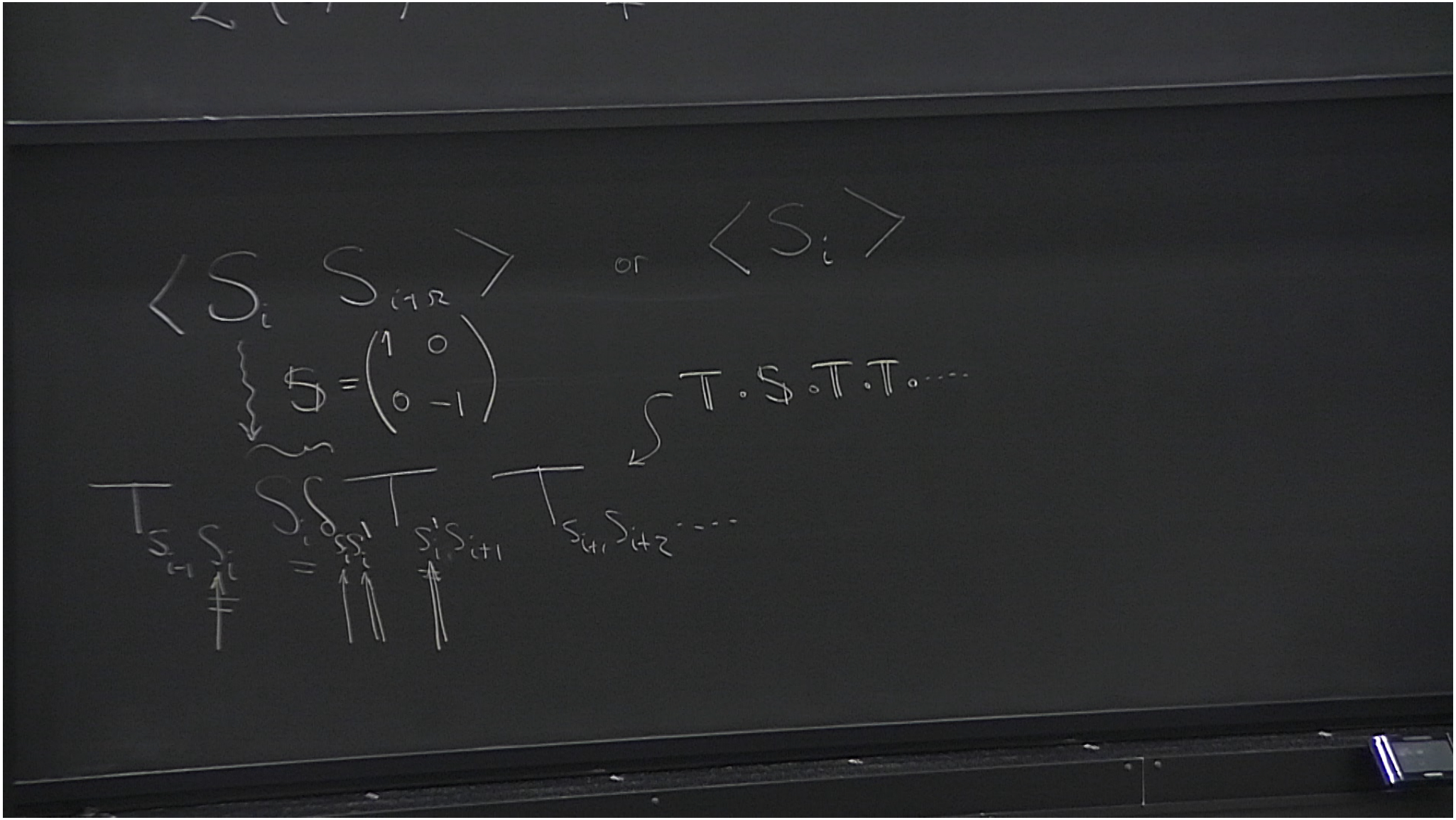

$= 2(e^{J} \pm e^{-J})$
$\lambda_+ = \cosh J$
$\lambda_- = \sinh J$
$Z = (\cosh J)^L \left[1 + v^L\right]$
$h=0$
$= 0, 2(e^{h} + e^{-h})$
$J=0$
$\lambda_- = 0, \lambda_+ = 2\cosh h$

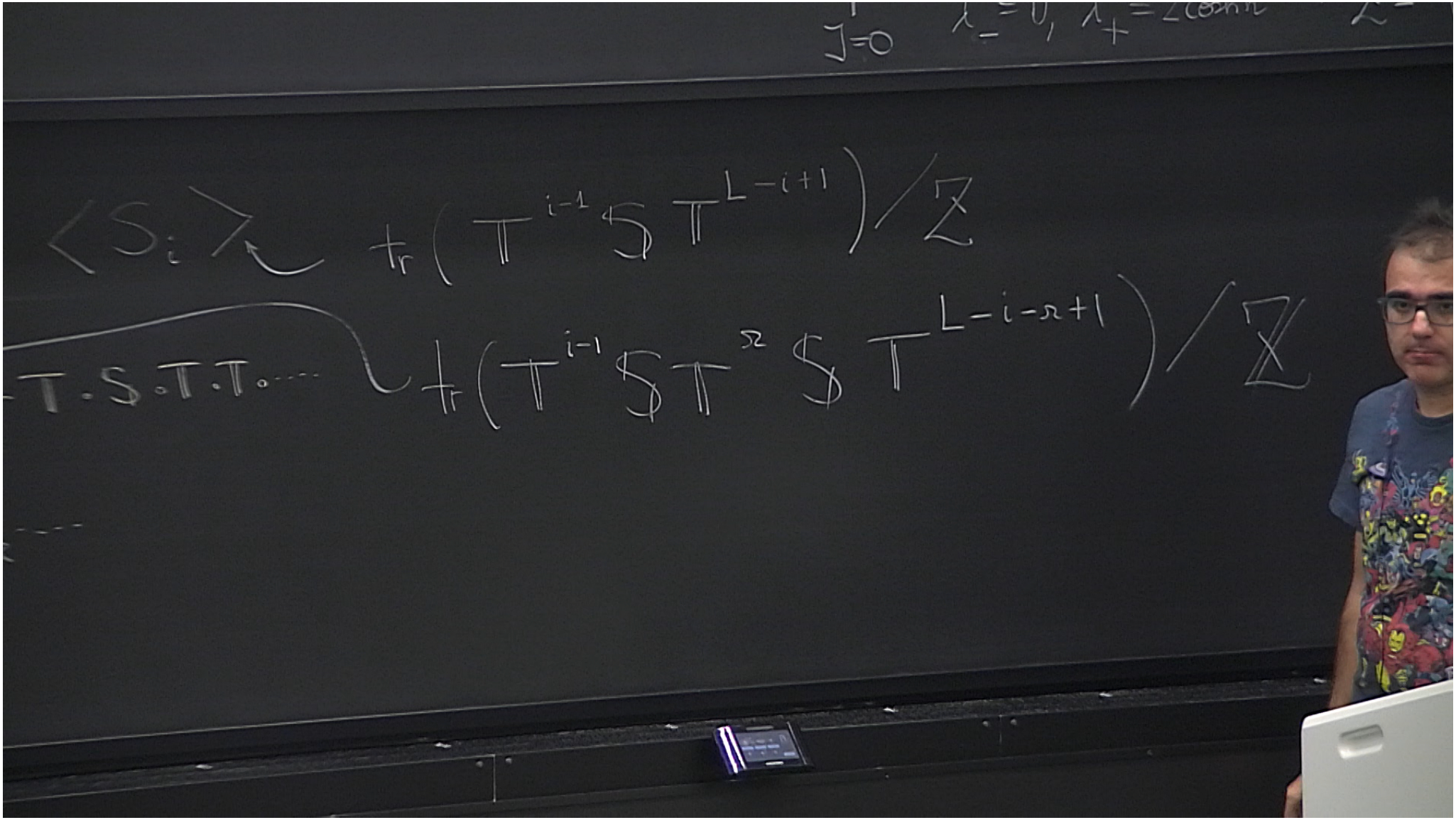

$= 2(e^{J} \pm e^{-J})$
$\lambda_+ = \cosh J$
$\lambda_- = \sinh J$
$h=0$
$= 0, 2(e^{h} + e^{-h})$
$J=0$
$\lambda_- = 0, \lambda_+ = 2\cosh h \rightarrow Z = (e^{h} + e^{-h})^{L}$
$Z = (\cosh J)^{L} [1 + v^{L}]$
(L decoupled spins, $E = hS$)

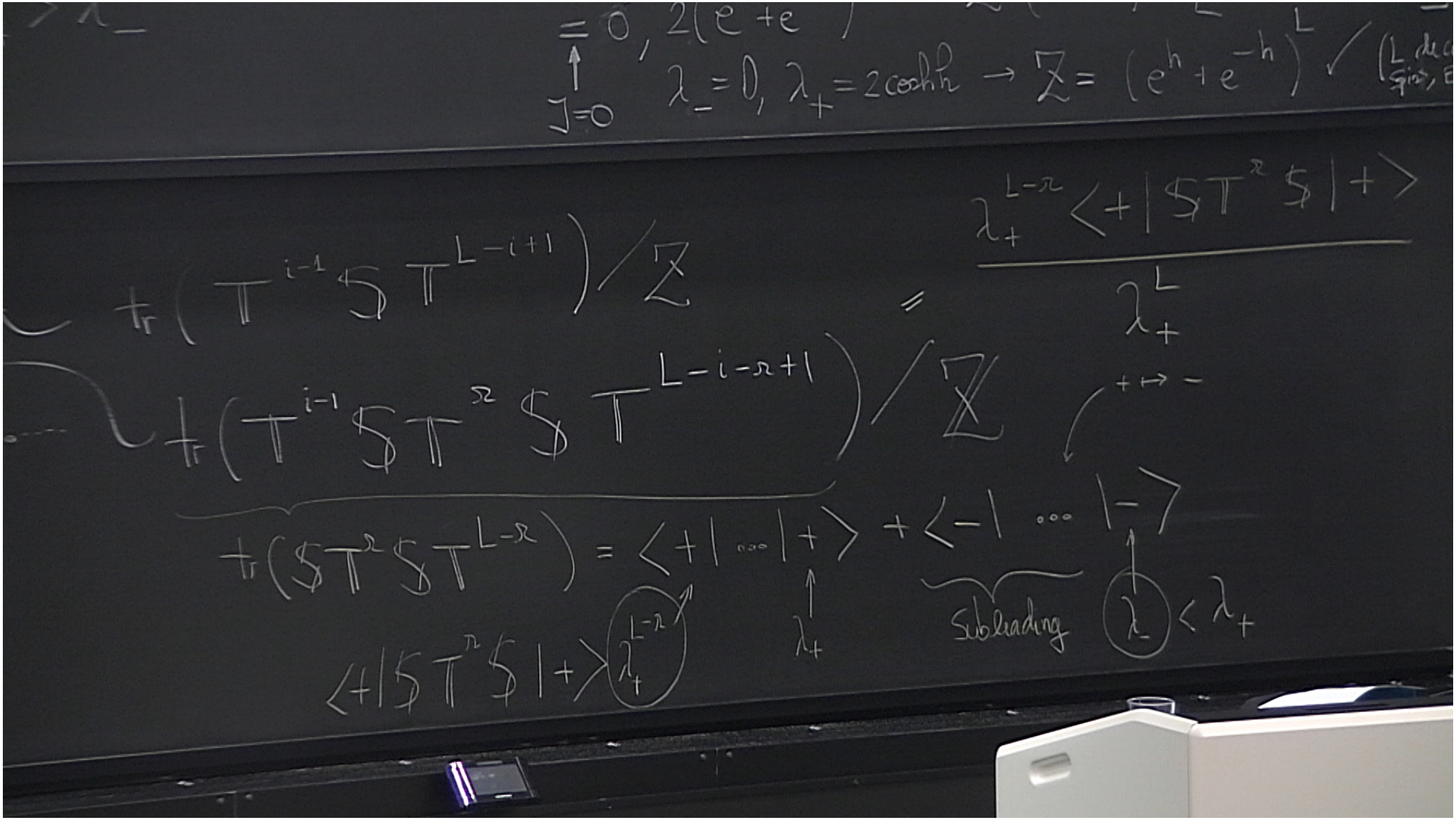
$$(e^{J+h} - \lambda)(e^{J-h} - \lambda) - e^{-2J} = 0$$

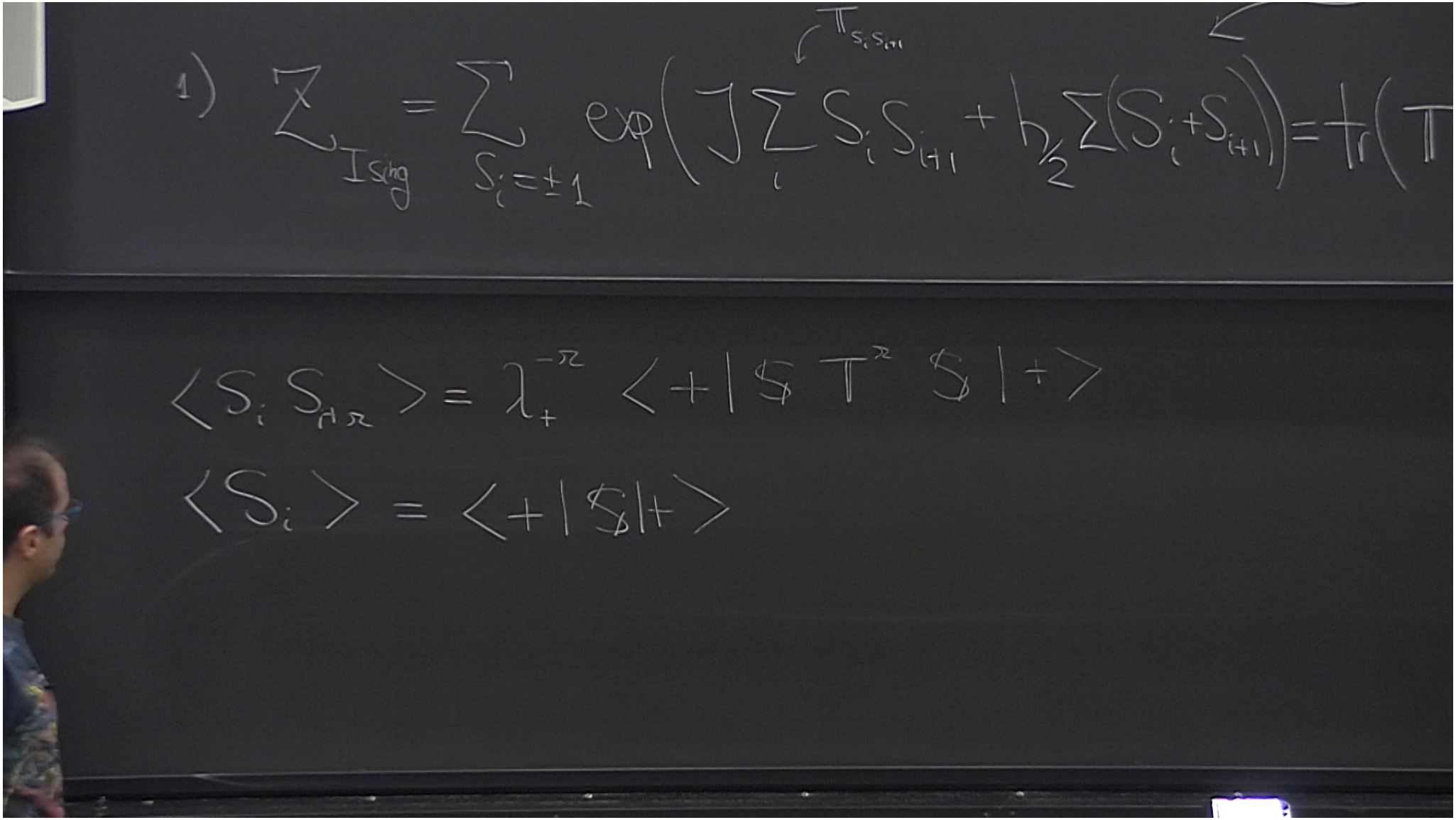
$$2\lambda_{\pm} = e^{J+h} + e^{J-h} \pm \sqrt{\left(e^{J+h} + e^{J-h}\right)^2 - 4\left(e^{2J} - e^{-2J}\right)}$$

$$Z(J,h) \simeq \lambda_+^L \quad \text{since } \lambda_+ > \lambda_-$$


$\langle S_i \; S_{i+R} \rangle$
or
$\langle S_i \rangle$
$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$T \cdot S \cdot T \cdot T \cdots$
$T_{S_{i+1} S_i}$
$S_i S_i' \; S_i' S_{i+1}$
$T_{S_{i+1} S_{i+2}} \cdots$


$\langle S_i \rangle \sim \mathrm{tr}\left(T^{i-1} S T^{L-i+1}\right)/Z$
$\sim \mathrm{tr}\left(T^{i-1} S T^{2} S T^{L-i-2+1}\right)/Z$
$T \cdot S \cdot T \cdot T \cdots$


λ_-
$= 0, 2(e^{+} + e^{-})$
$J=0$
$\lambda_- = 0, \lambda_+ = 2\cosh h \to Z = (e^h + e^{-h})^L$
$\frac{\lambda_+^{L-r} \langle + | S T^r S | + \rangle}{\lambda_+^L}$
$+ \leftrightarrow -$
$\mathrm{Tr}(T^{i-1} S T^{L-i+1}) / Z$
$\mathrm{Tr}(T^{i-1} S T^r S T^{L-i-r+1}) / Z$
$\mathrm{Tr}(S T^r S T^{L-r}) = \langle + | \cdots | + \rangle + \langle - | \cdots | - \rangle$
$\langle + | S T^r S | + \rangle \lambda_+^{L-r}$
λ_+
Subleading
$\lambda_- < \lambda_+$


1) $Z_{Ising} = \sum_{S_i = \pm 1} \exp\left(J \sum_i S_i S_{i+1} + h/2 \sum (S_i + S_{i+1}) \right) = \mathrm{Tr}(T$
$T_{S_i S_{i+1}}$
$\langle S_i S_{i+r} \rangle = \lambda_+^{-r} \langle + | \$ T^r \$ | + \rangle$
$\langle S_i \rangle = \langle + | \$ | + \rangle$

2) Some spoilers / motivation for random walks from Ising in $D > 1$

1) $$Z_{\text{Ising}} = \sum_{S_i = \pm 1} \exp\left(J \sum_i S_i S_{i+1} + h \sum_i \left(S_i + S_{i+1} \right) \right) = \text{Tr}\left(T \cdot T \cdot T \cdots \right) = \text{Tr}\left(T^L \right)$$

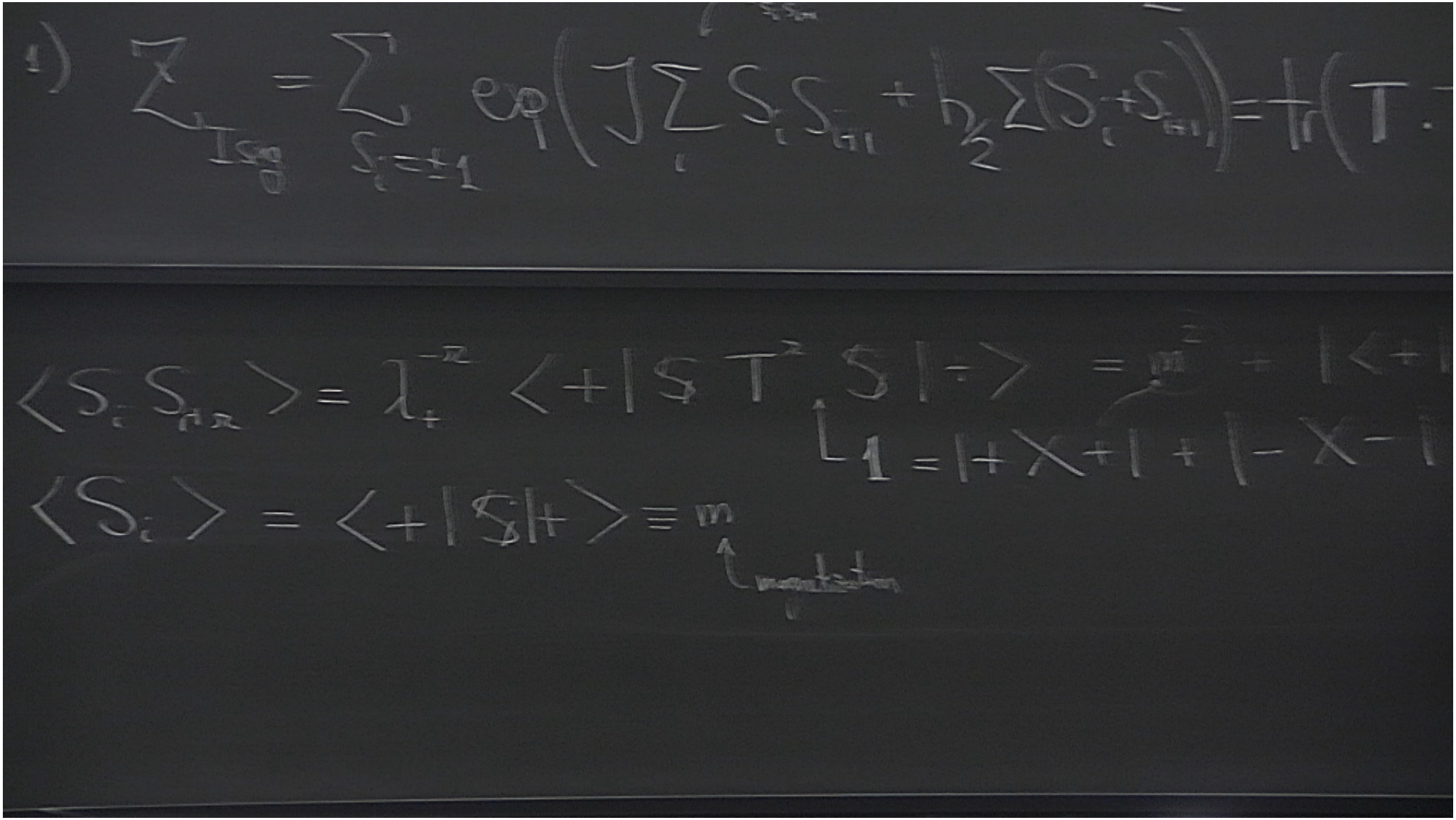
$T_{S_i S_{i+1}}$

$$\langle S_i S_{i+r} \rangle = \lambda_+^{-r} \langle + | S T^r S | + \rangle = m^2 + | \langle + | S | - \rangle |^2 \left(\frac{\lambda_-}{\lambda_+} \right)^r$$

$$\mathbb{1} = | + \rangle\langle + | + | - \rangle\langle - |$$

$$\langle S_i \rangle = \langle + | S | + \rangle \equiv m$$

magnetization


1) $Z_N = \sum_{\{S_i\}} \exp\left(J\sum_i S_i S_{i+1} + h_2 \sum (S_i + S_{i+1})\right) = \mathrm{Tr}(T^N)$
$\langle S_i S_{i+2} \rangle = \lambda_+^{-2} \langle + | S T^2 S | + \rangle = m^2 + |\langle + |$
$1 = |+\rangle\langle +| + |-\rangle\langle -|$
$\langle S_i \rangle = \langle + | S_i | + \rangle \equiv m$

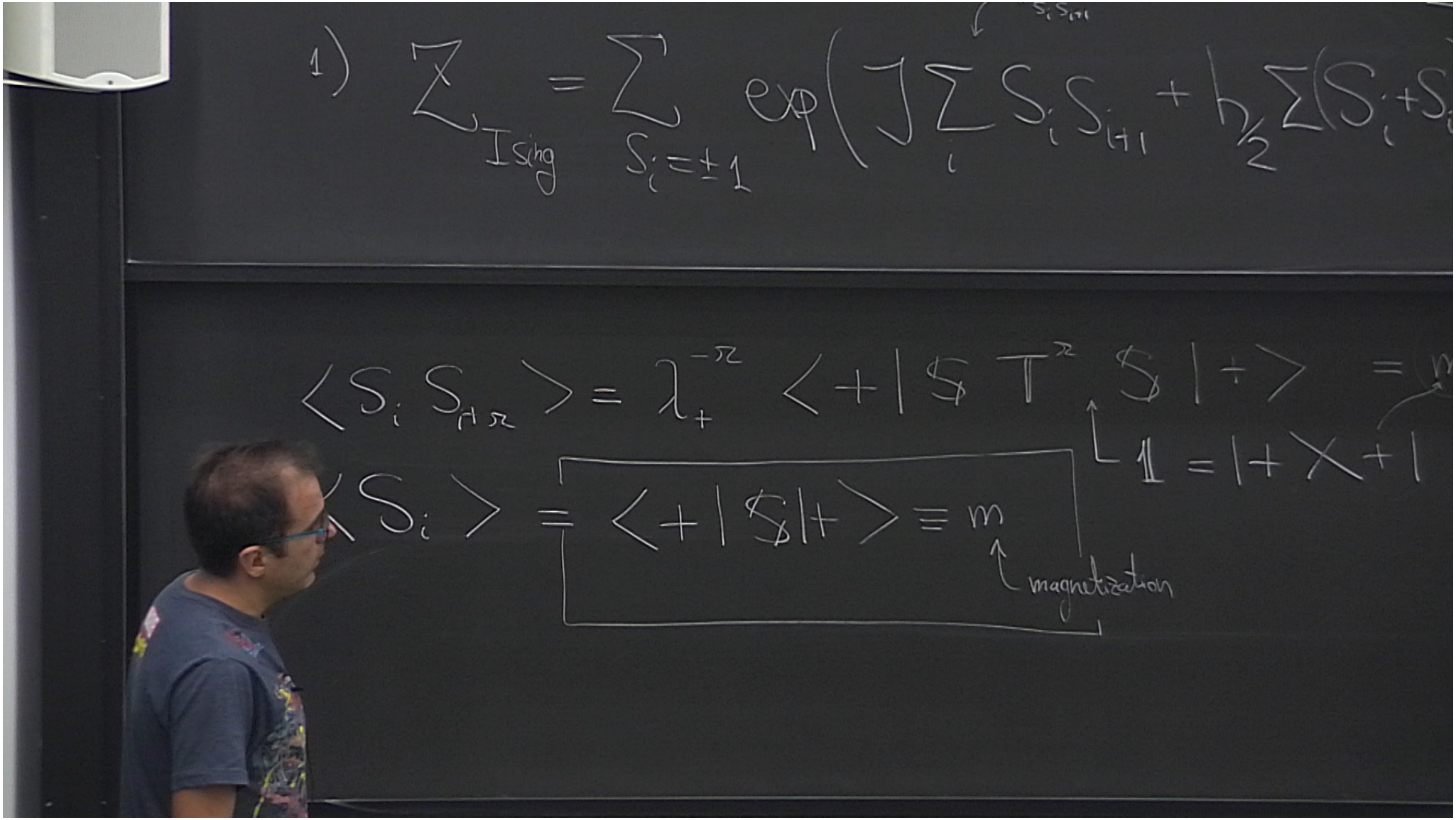
$$\left(S_i^+ S_{i+1}^-\right)\Big) = \mathrm{Tr}\left(T \cdot T \cdot T \cdots\right) \approx \mathrm{Tr}\left(T^L\right) = \lambda_+^L + \lambda_-^L$$

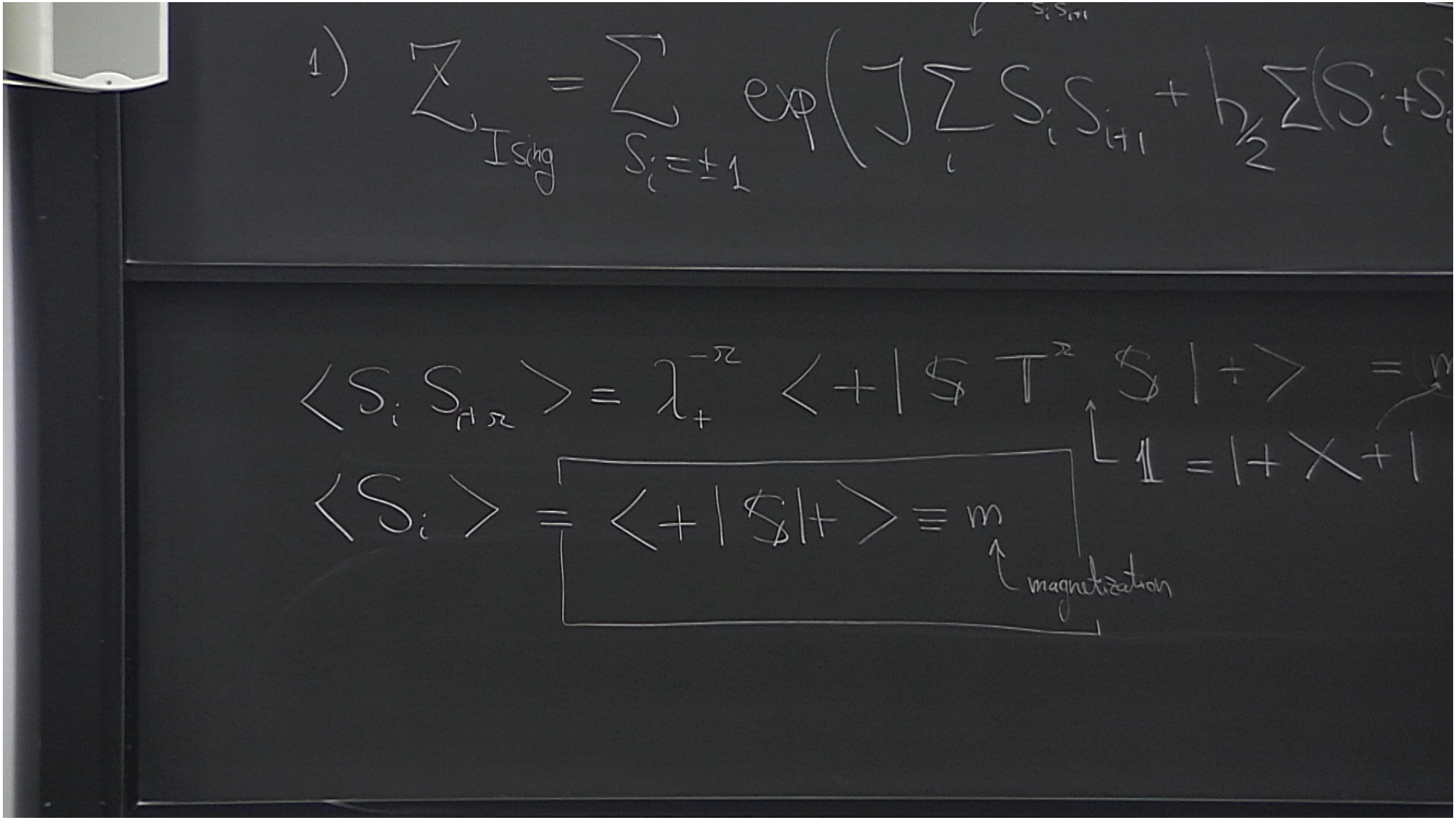
$$= m^2 + |\langle +|S|-\rangle|^2 \left(\frac{\lambda_-}{\lambda_+}\right)^x$$

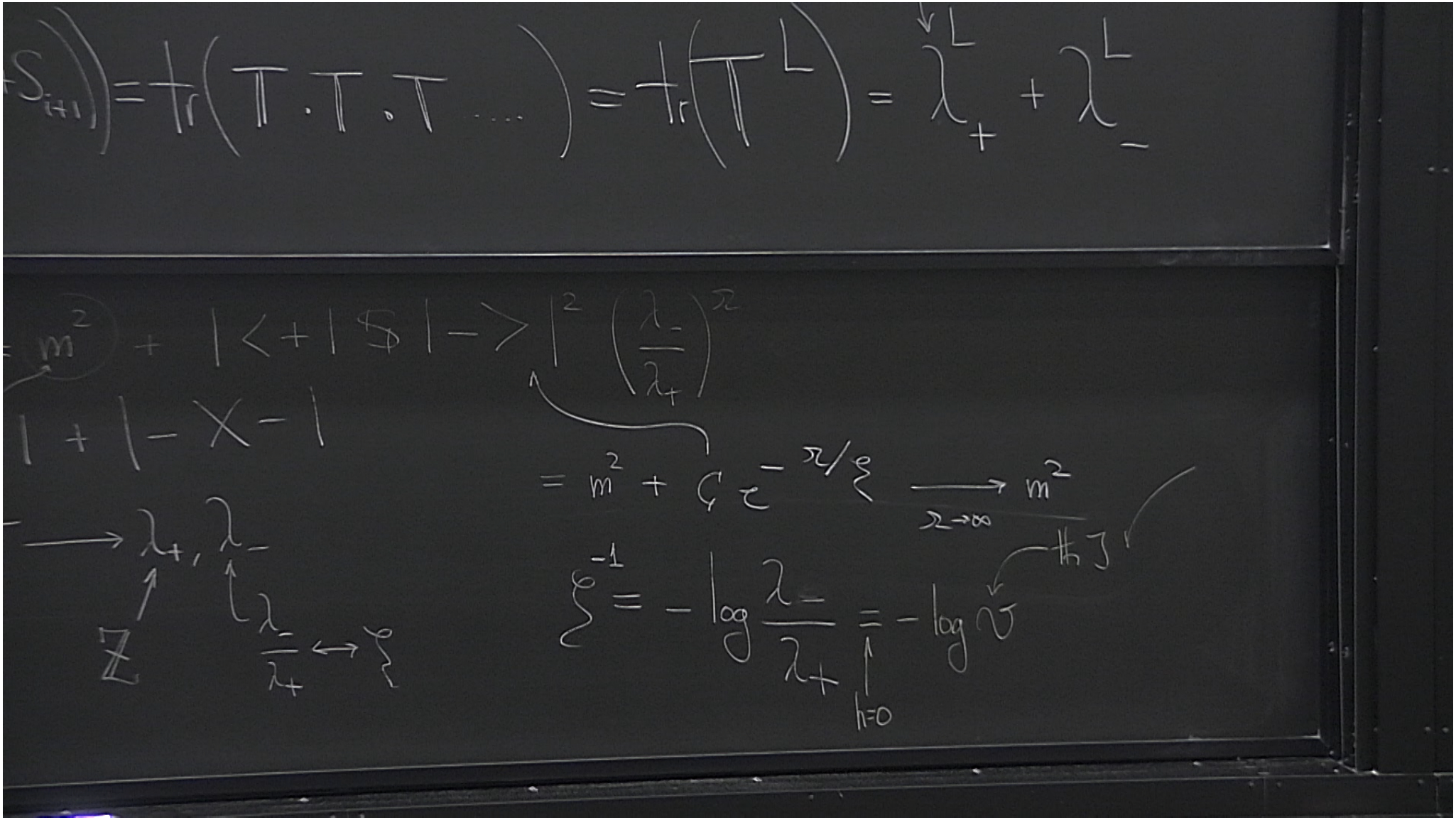
$$X+|+|-X-|$$

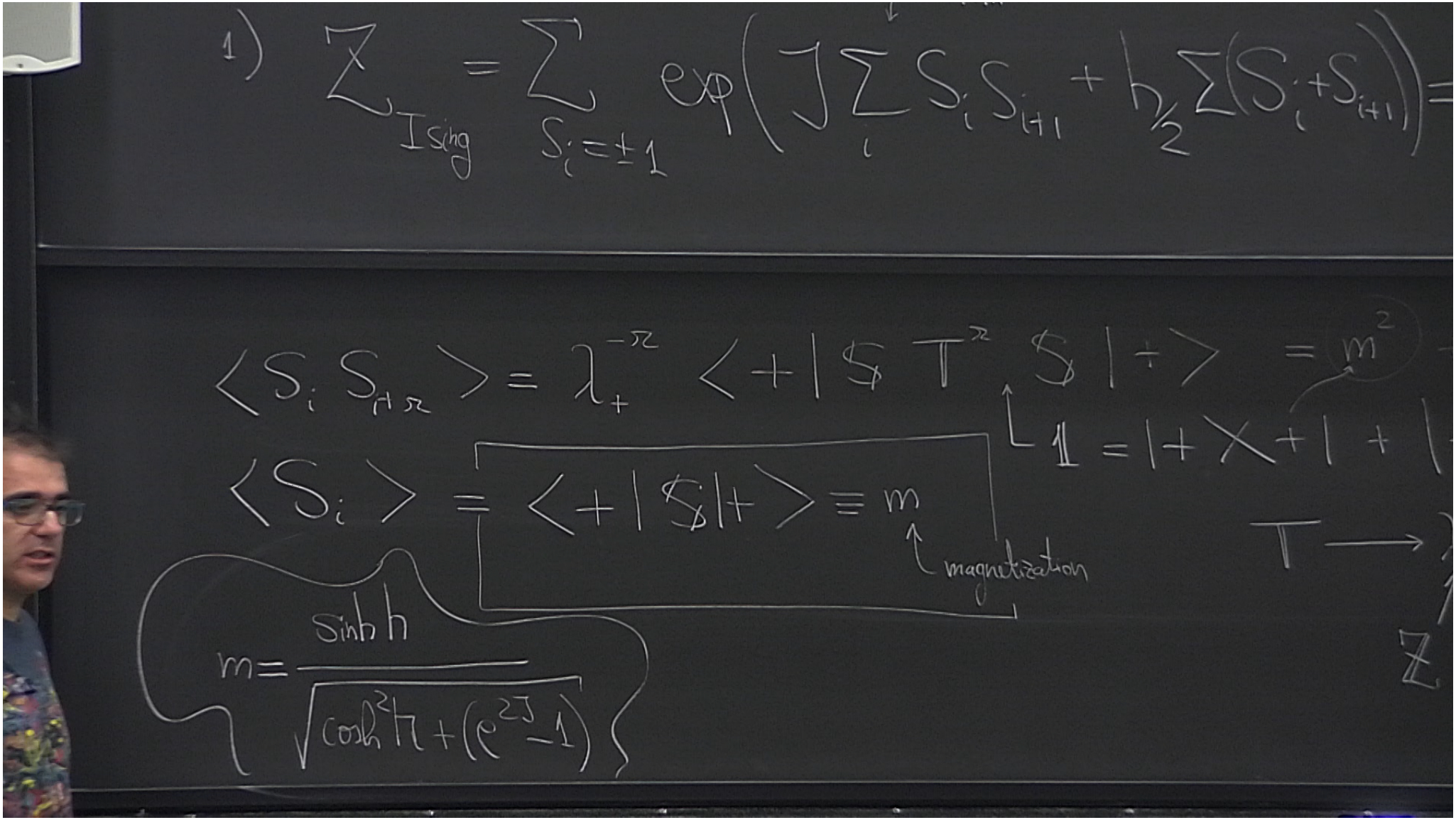
$$= m^2 + C e^{-x/\xi} \xrightarrow[x\to\infty]{} m^2$$

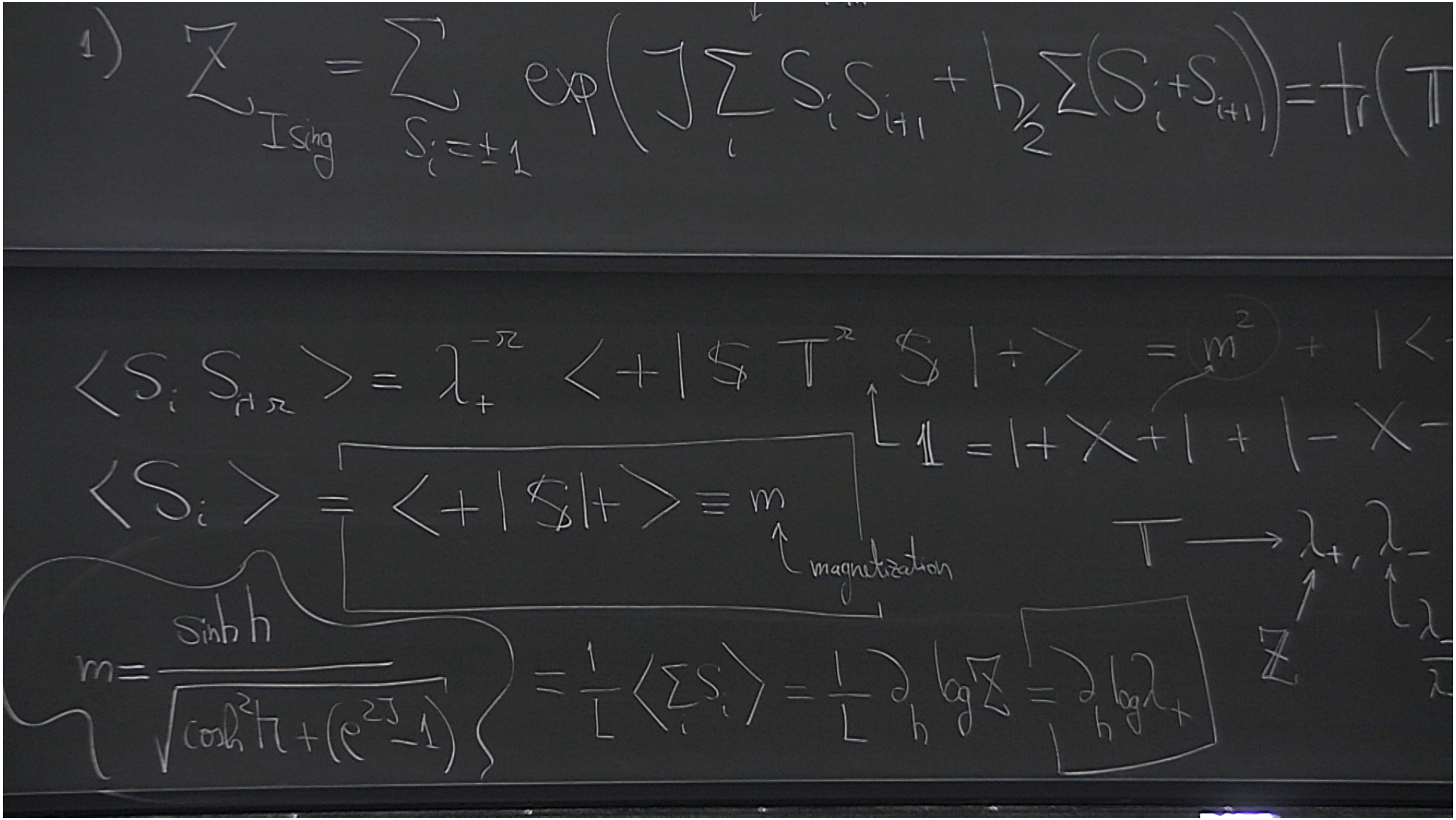
$$\xi^{-1} = -\log\frac{\lambda_-}{\lambda_+}$$

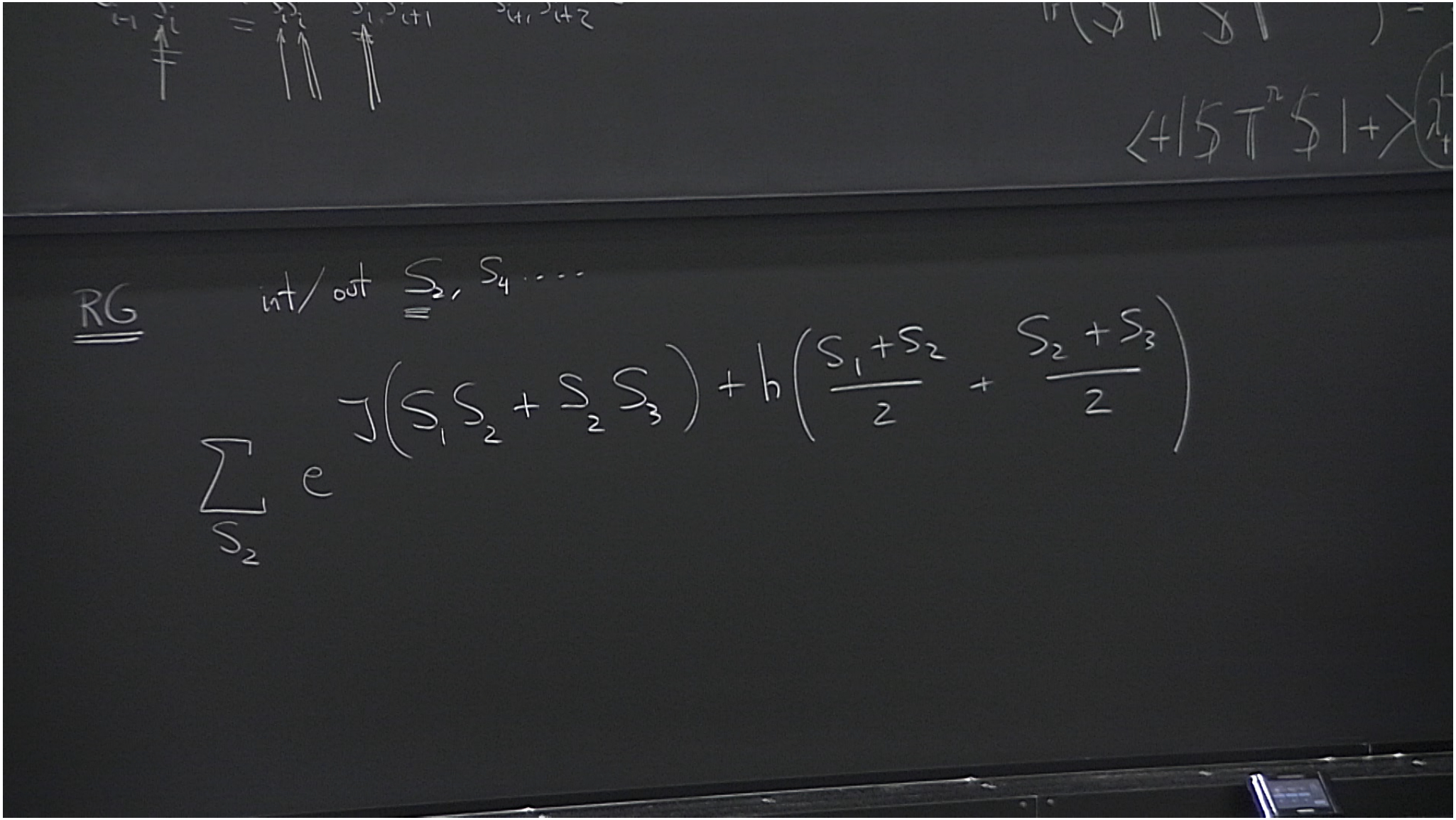

1) $Z_{Ising} = \sum_{S_i = \pm 1} \exp\left(J \sum_i S_i S_{i+1} + h/2 \Sigma (S_i + S_{i+1}\right.$
$S_i S_{i+1}$
$\langle S_i S_{i+r} \rangle = \lambda_+^{-r} \langle + | \$ T^r \$ | + \rangle$
$\mathbb{1} = |+\rangle\langle+| + |-\rangle\langle-|$
$\langle S_i \rangle = \langle + | \$ | + \rangle \equiv m$
magnetization

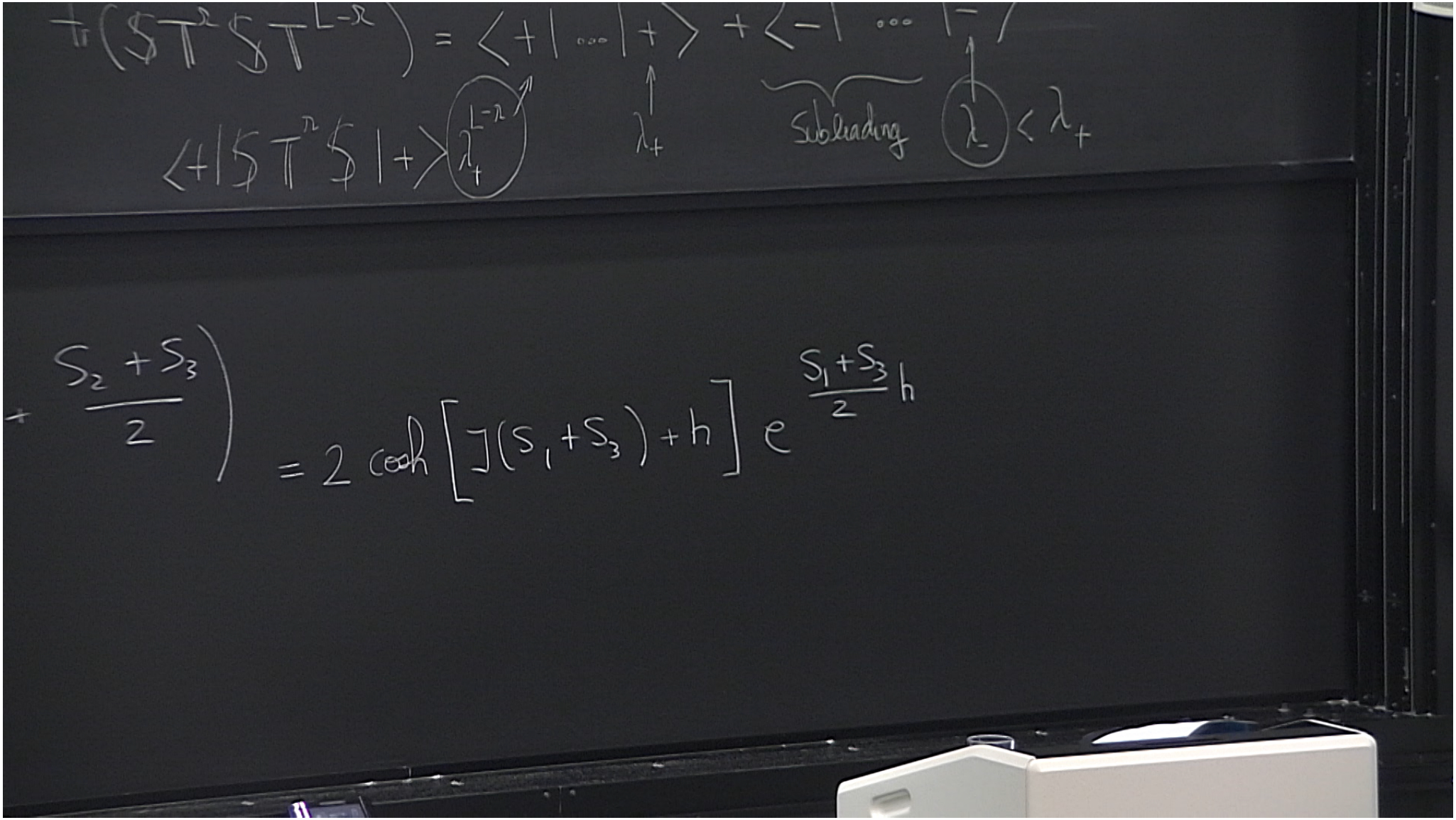

1) $Z_{Ising} = \sum_{S_i = \pm 1} \exp\left(J \sum_i S_i S_{i+1} + h/2 \sum (S_i + S_i$
$S_i S_{i+1}$
$\langle S_i S_{i+r} \rangle = \lambda_+^{-r} \langle + | \$ T^r \$ | + \rangle = m$
$\mathbb{1} = |+\rangle\langle+|$
$\langle S_i \rangle = \langle + | \$_i | + \rangle \equiv m$
magnetization

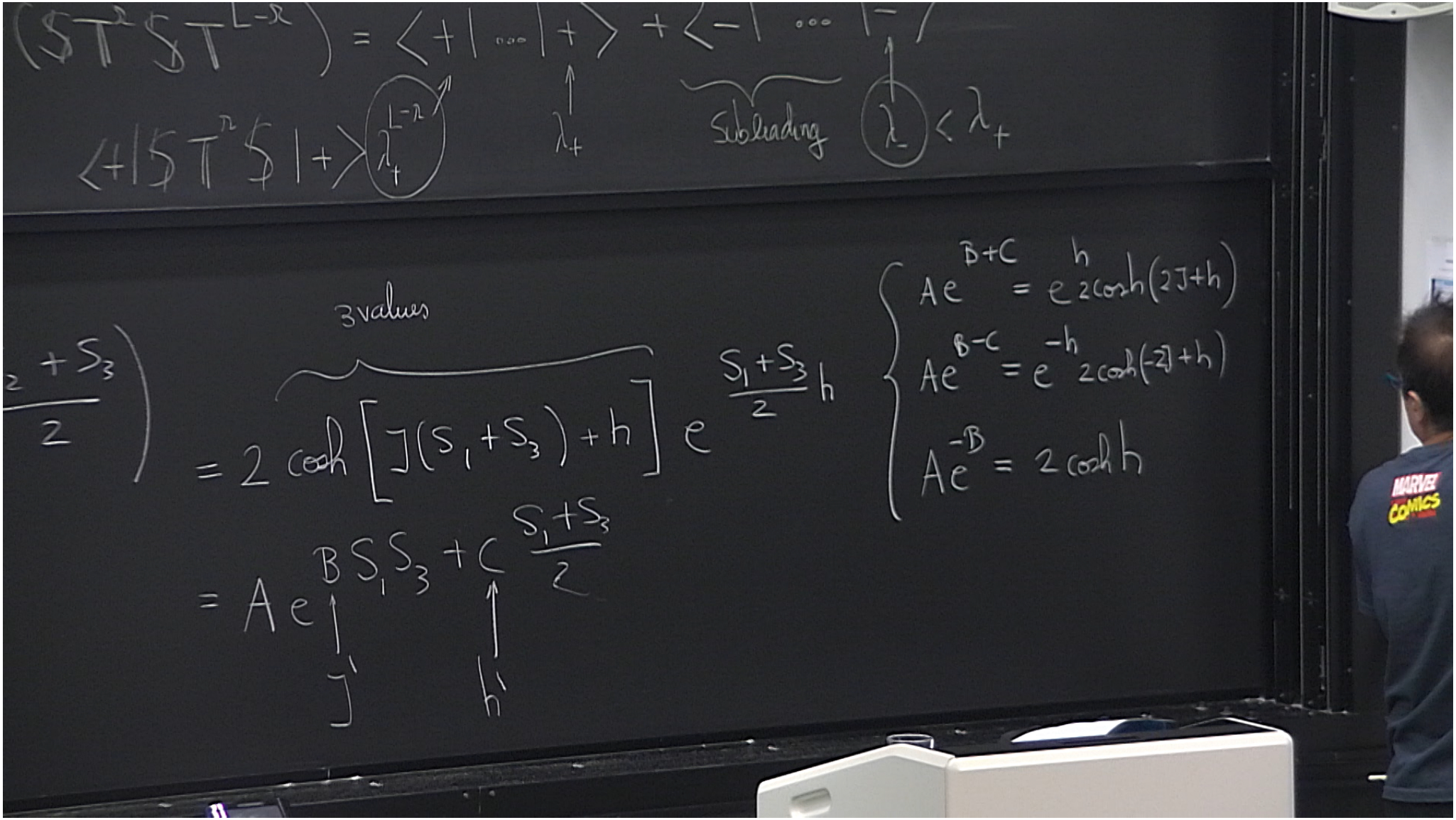

$S_{i+1}))) = \mathrm{Tr}(T \cdot T \cdot T \cdots) = \mathrm{Tr}(T^L) = \lambda_+^L + \lambda_-^L$
$= m^2 + |\langle + | \$ | - \rangle|^2 \left(\frac{\lambda_-}{\lambda_+}\right)^x$
$1 + |- X - 1$
$= m^2 + C e^{-x/\xi} \longrightarrow m^2$
$x \to \infty$
$\longrightarrow \lambda_+, \lambda_-$
$\xi^{-1} = -\log \frac{\lambda_-}{\lambda_+} = -\log v$
$\tanh J$
$h=0$
Z
$\frac{\lambda_-}{\lambda_+} \leftrightarrow \xi$


1) $Z_{Ising} = \sum_{S_i=\pm 1} \exp\left(J\sum_i S_i S_{i+1} + h/2 \sum (S_i + S_{i+1})\right) =$
$\langle S_i S_{i+r} \rangle = \lambda_+^{-r} \langle + | \$ T^r \$ | + \rangle = m^2$
$\mathbb{1} = |+\rangle\langle +| + |$
$\langle S_i \rangle = \langle + | \$ | + \rangle \equiv m$
magnetization
$T \longrightarrow$
Z
$m = \frac{\sinh h}{\sqrt{\cosh^2 h + (e^{2J} - 1)}}$

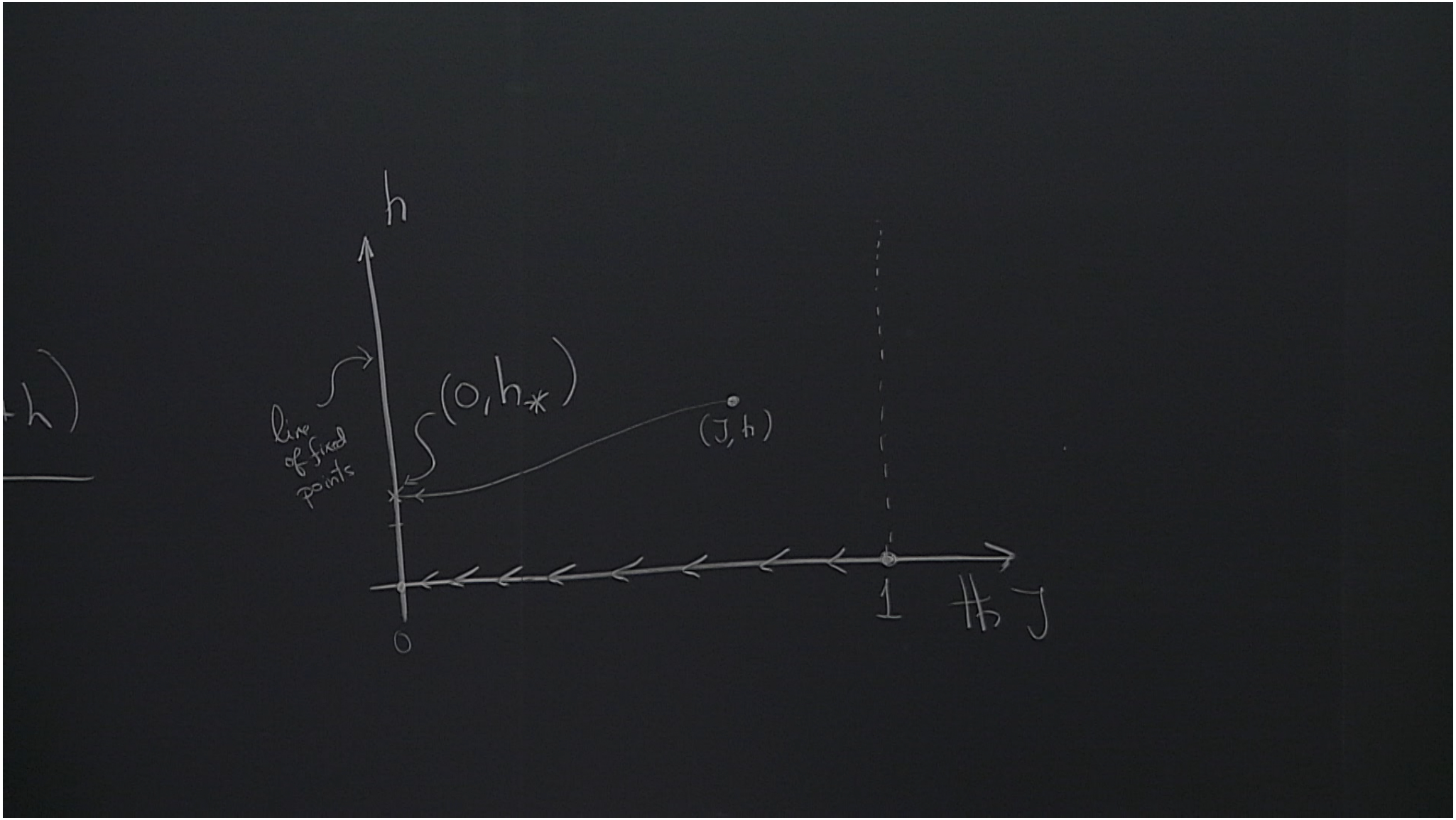

1) $Z_{Ising} = \sum_{S_i=\pm 1} \exp\left(J\sum_i S_i S_{i+1} + h/2 \sum (S_i + S_{i+1})\right) = \mathrm{Tr}(T$
$\langle S_i S_{i+r} \rangle = \lambda_+^{-r} \langle + | \$ T^r \$ | + \rangle = m^2 + |<$
$\mathbb{1} = |+\rangle\langle+| + |-\rangle\langle-|$
$\langle S_i \rangle = \langle + | \$_i | + \rangle \equiv m$
magnetization
$T \longrightarrow \lambda_+, \lambda_-$
Z
$m = \frac{\sinh h}{\sqrt{\cosh^2 h + (e^{2J} - 1)}}$
$= \frac{1}{L}\langle \sum_i S_i \rangle = \frac{1}{L} \partial_h \log Z = \partial_h \log \lambda_+$

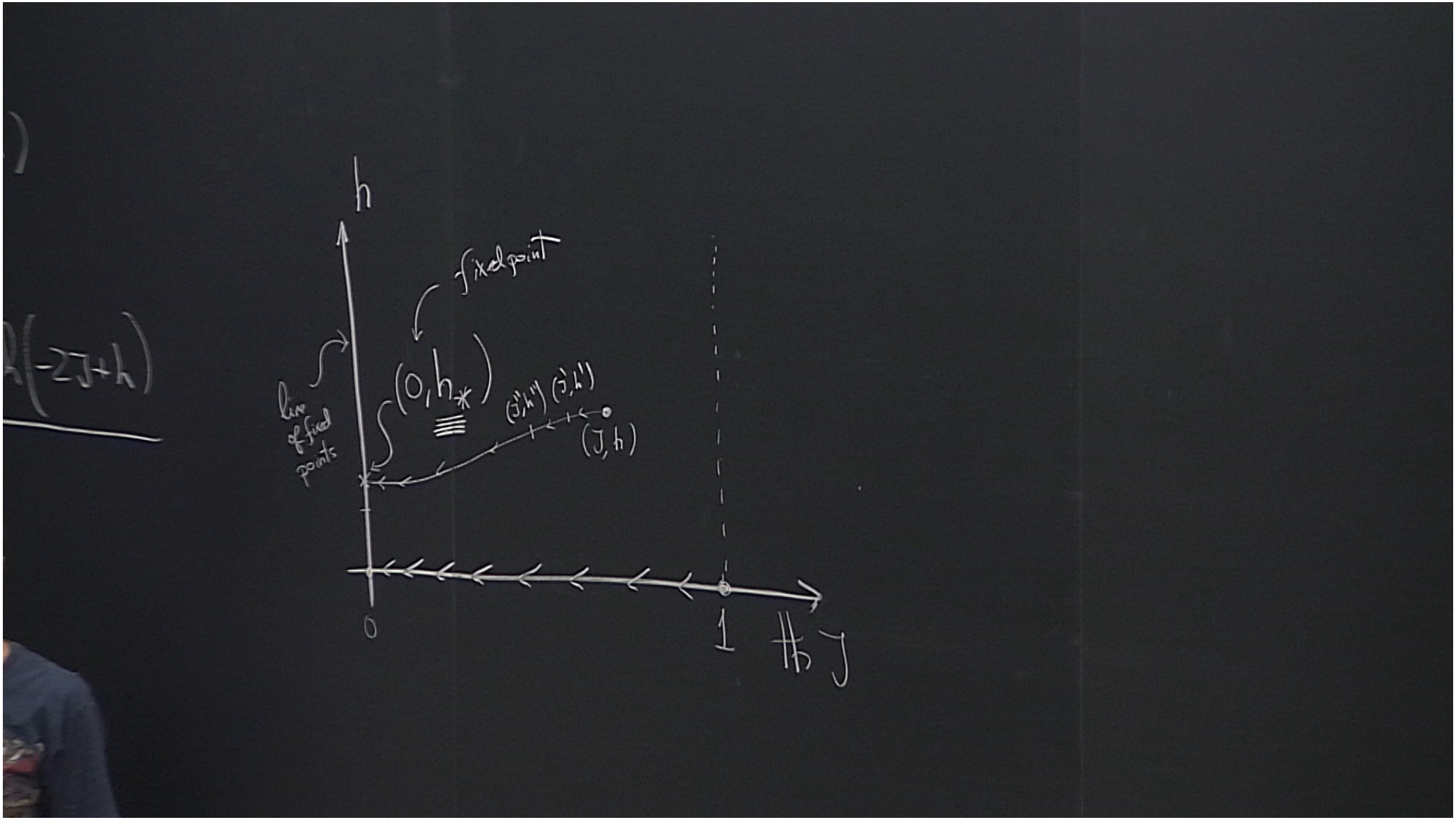

RG
int/out $S_2, S_4 \cdots$
$\sum_{S_2} e^{J(S_1 S_2 + S_2 S_3) + h\left(\frac{S_1+S_2}{2} + \frac{S_2+S_3}{2}\right)}$

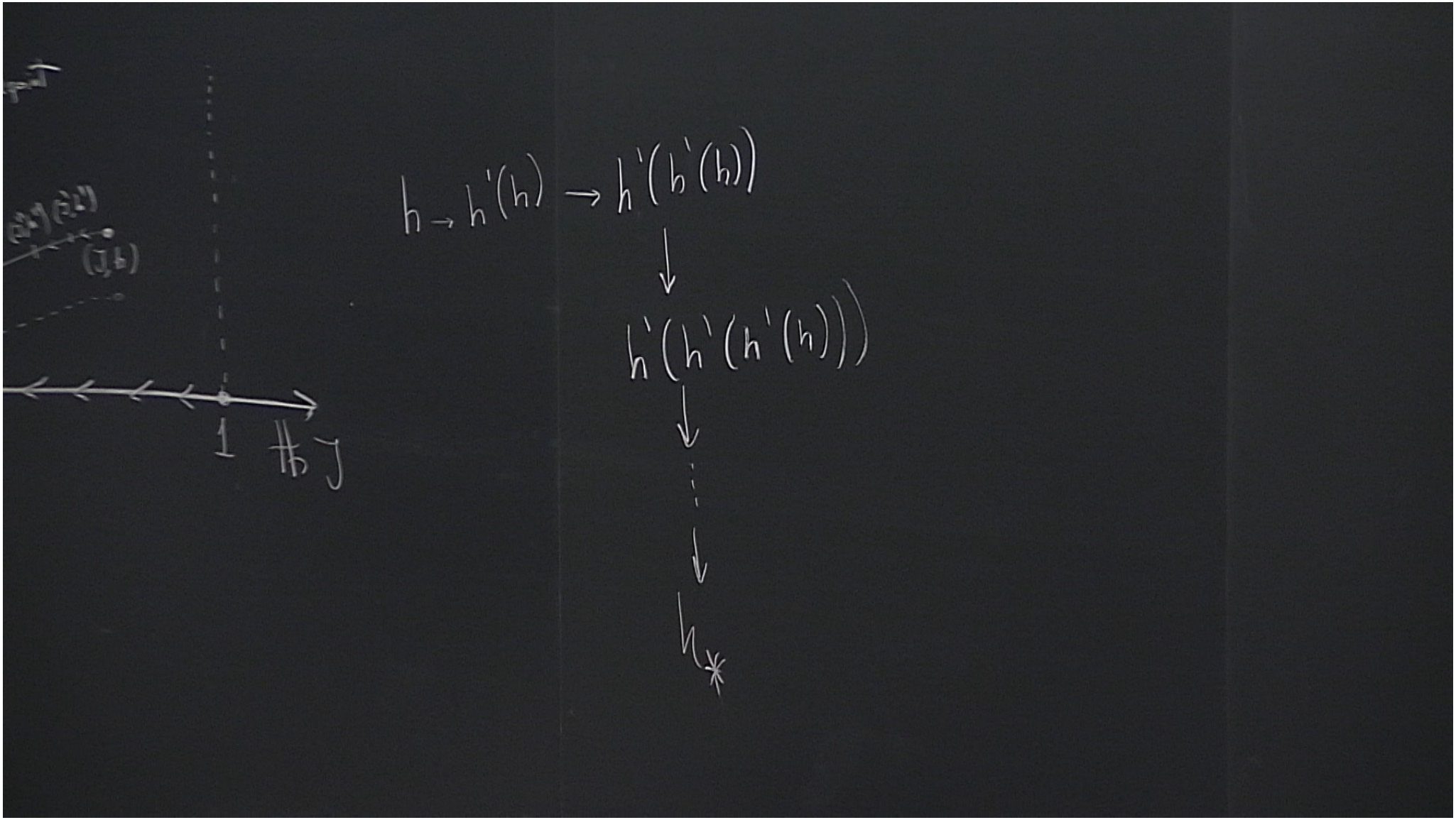

Subleading
λ_+
λ_-
$\lambda_- < \lambda_+$
$= 2\cosh\left[J(S_1+S_3)+h\right] e^{\frac{S_1+S_3}{2}h}$


Subleading
3values
$\lambda_+ > \lambda_-$
$= 2\cosh\left[J(S_1+S_3)+h\right] e^{\frac{S_1+S_3}{2}h}$
$= A e^{B S_1 S_3 + C \frac{S_1+S_3}{2}}$
J'
h'
$Ae^{B+C} = e^{h} 2\cosh(2J+h)$
$Ae^{B-C} = e^{-h} 2\cosh(-2J+h)$
$Ae^{-B} = 2\cosh h$

$$\begin{cases} h' = h + \frac{1}{2}\log\frac{\cosh(2J+h)}{\cosh(2J-h)} \\ J' = \frac{1}{4}\log\frac{\cosh(2J+h)\cosh(-2J+h)}{\cosh^2 h} \end{cases}$$

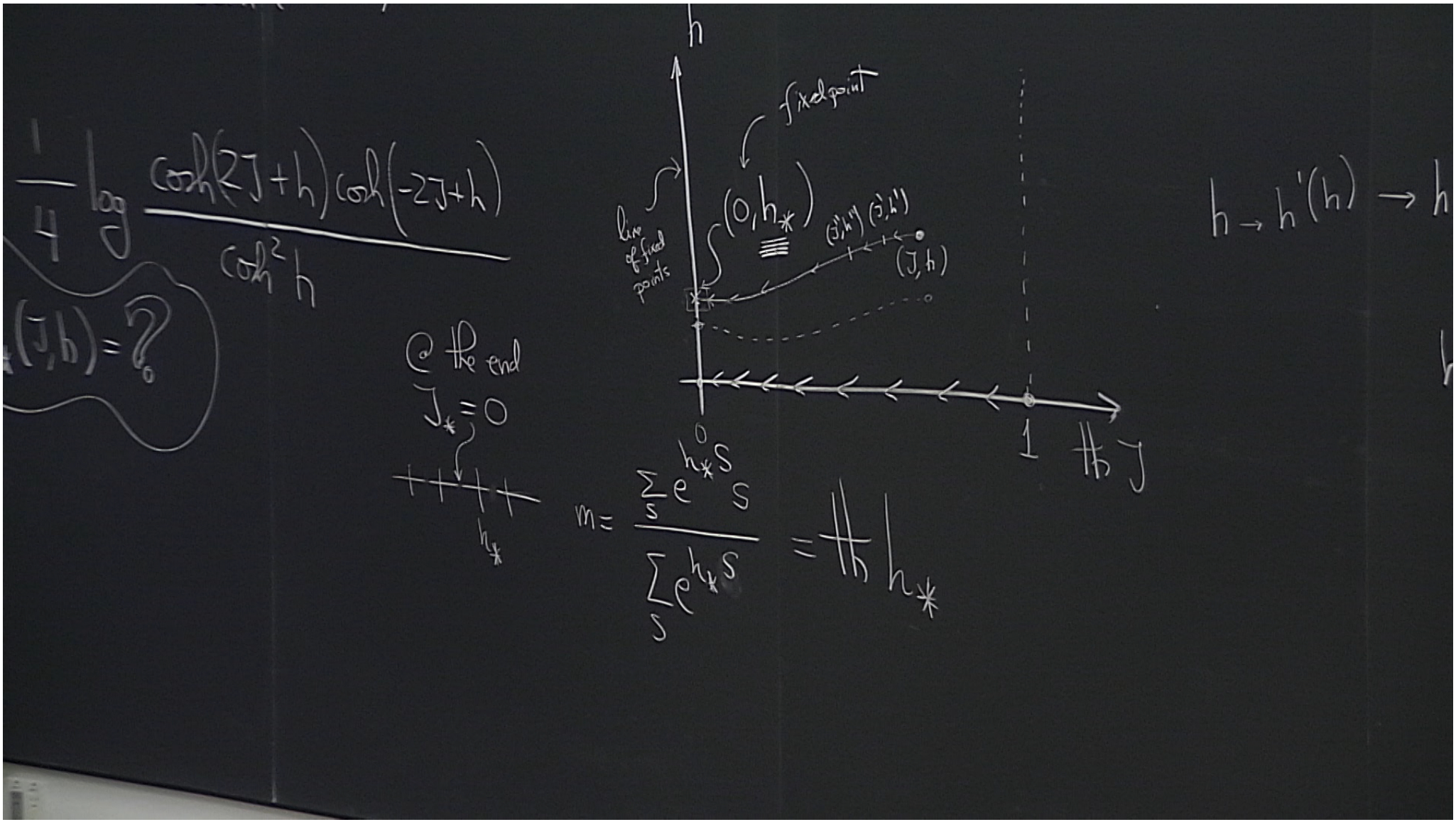

h
line of fixed points
(0, h*)
(J, h)
1
J

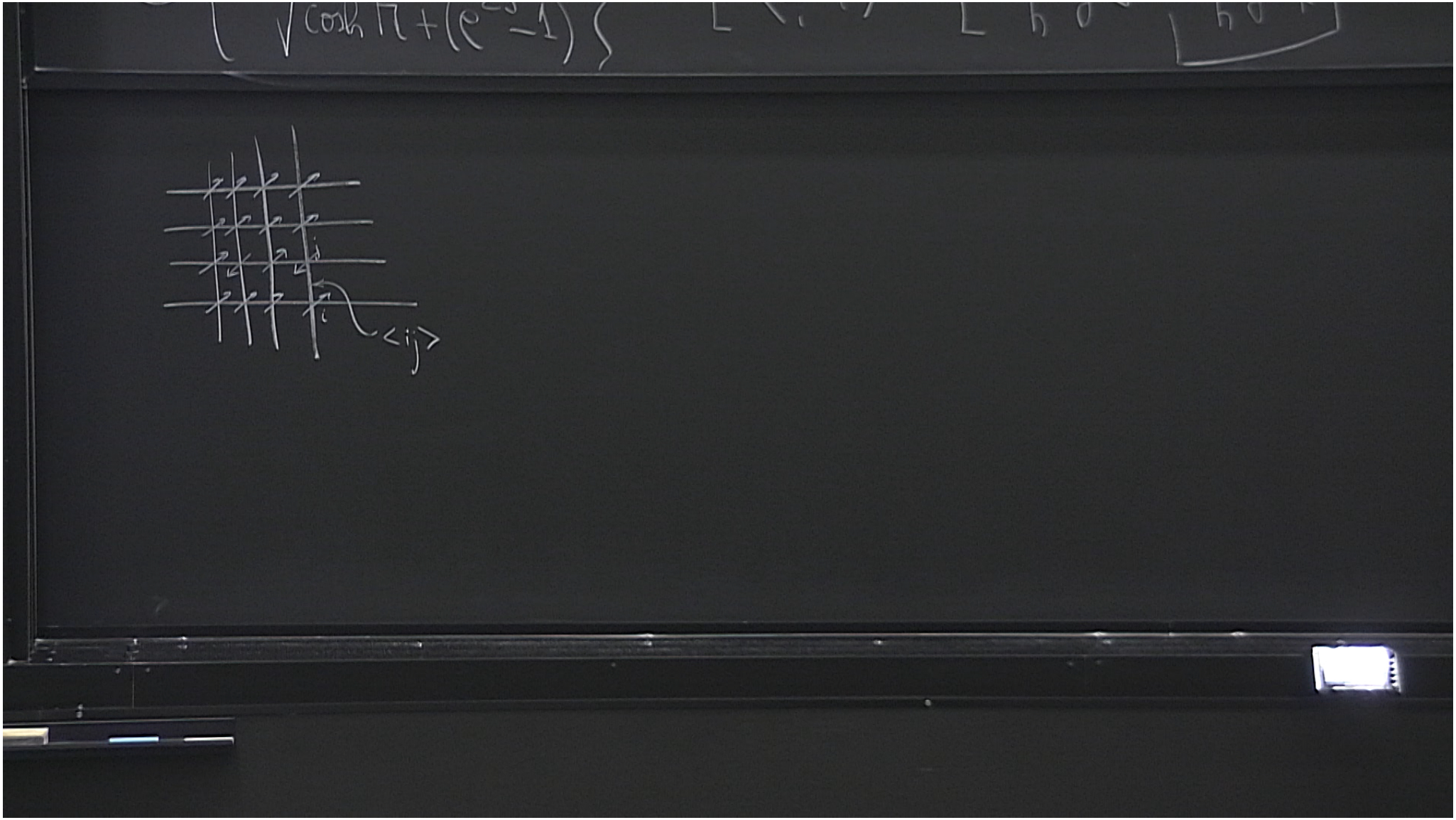

h
fixed point
line of fixed points
(0, h*)
(J', h'')
(J', h')
(J, h)
0
1
J

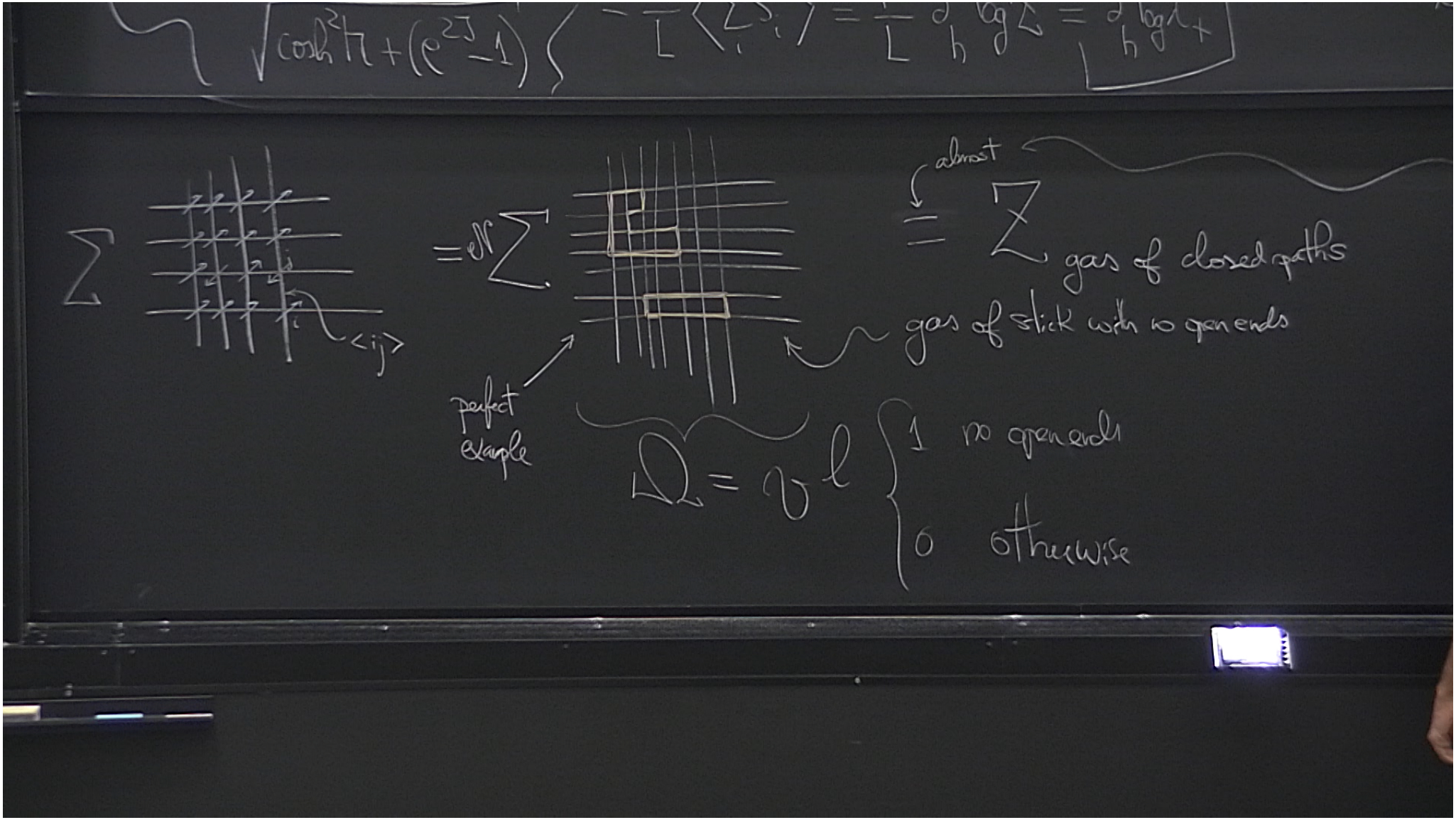

(j,k)
1
$h \to h'(h) \to h'(h'(h))$
$h'(h'(h'(h)))$
h_*

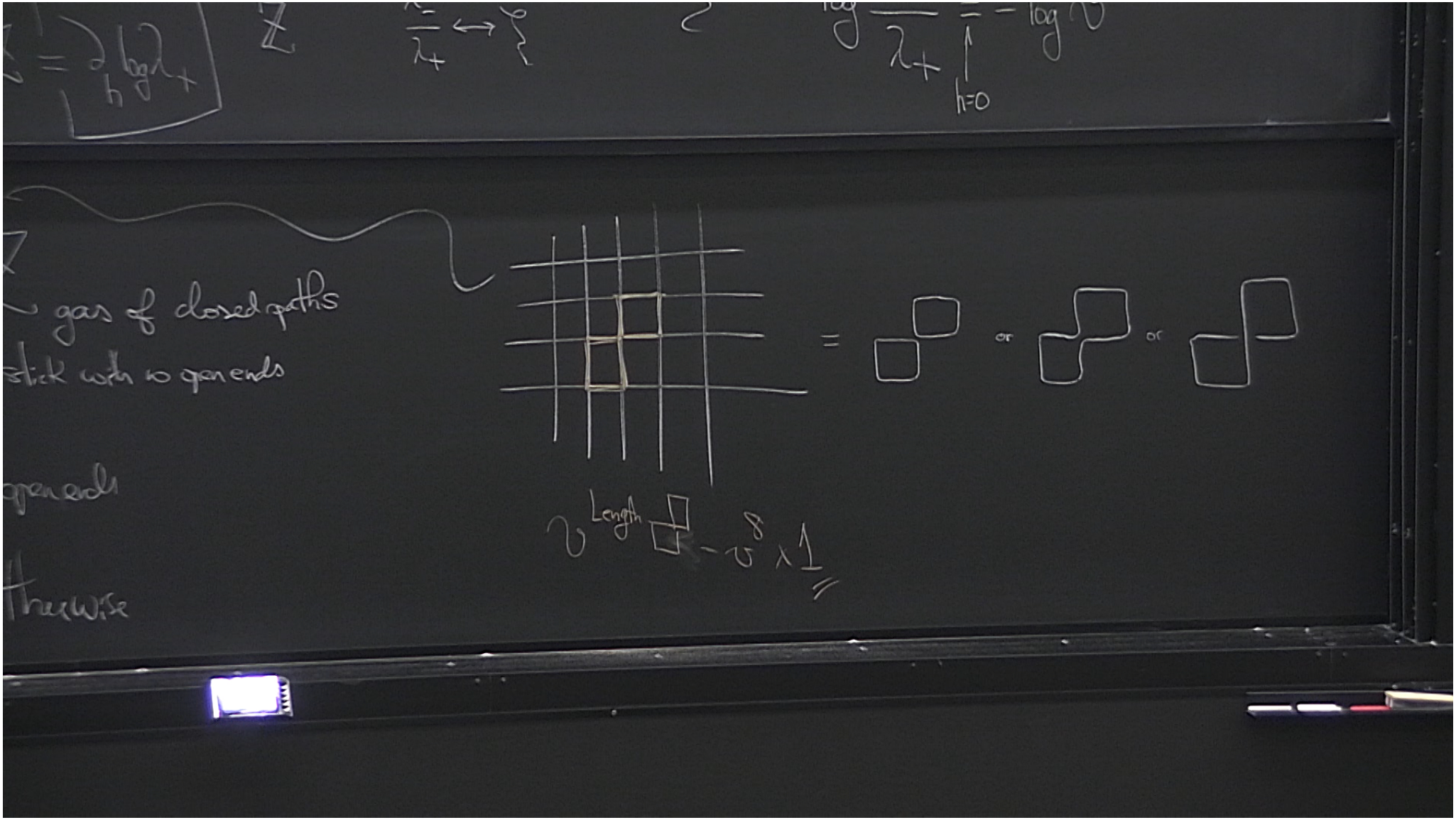
$$\frac{1}{4}\log\frac{\cosh(2J+h)\cosh(-2J+h)}{\cosh^2 h}$$

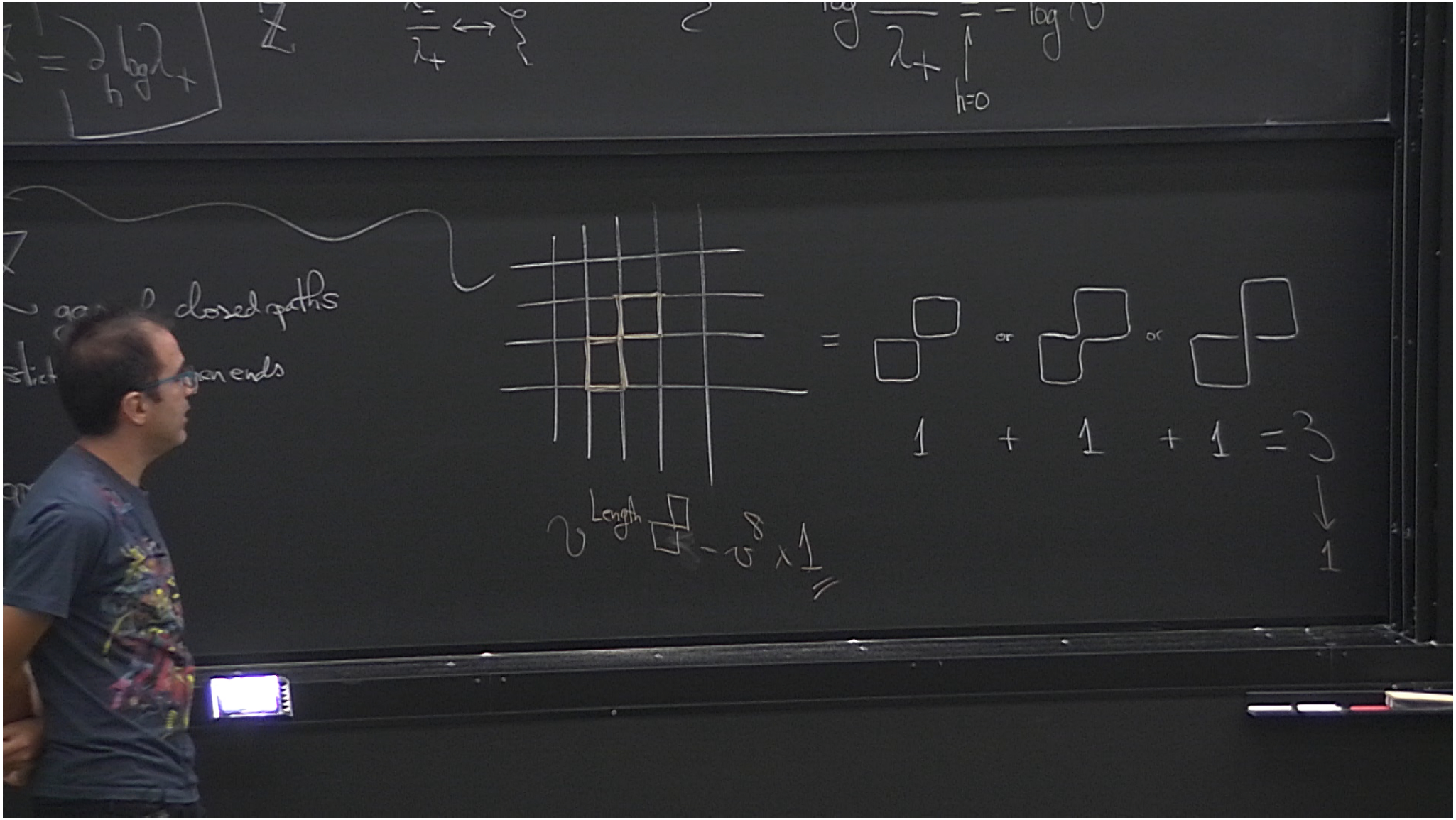
$h_*(J,h) = ?$

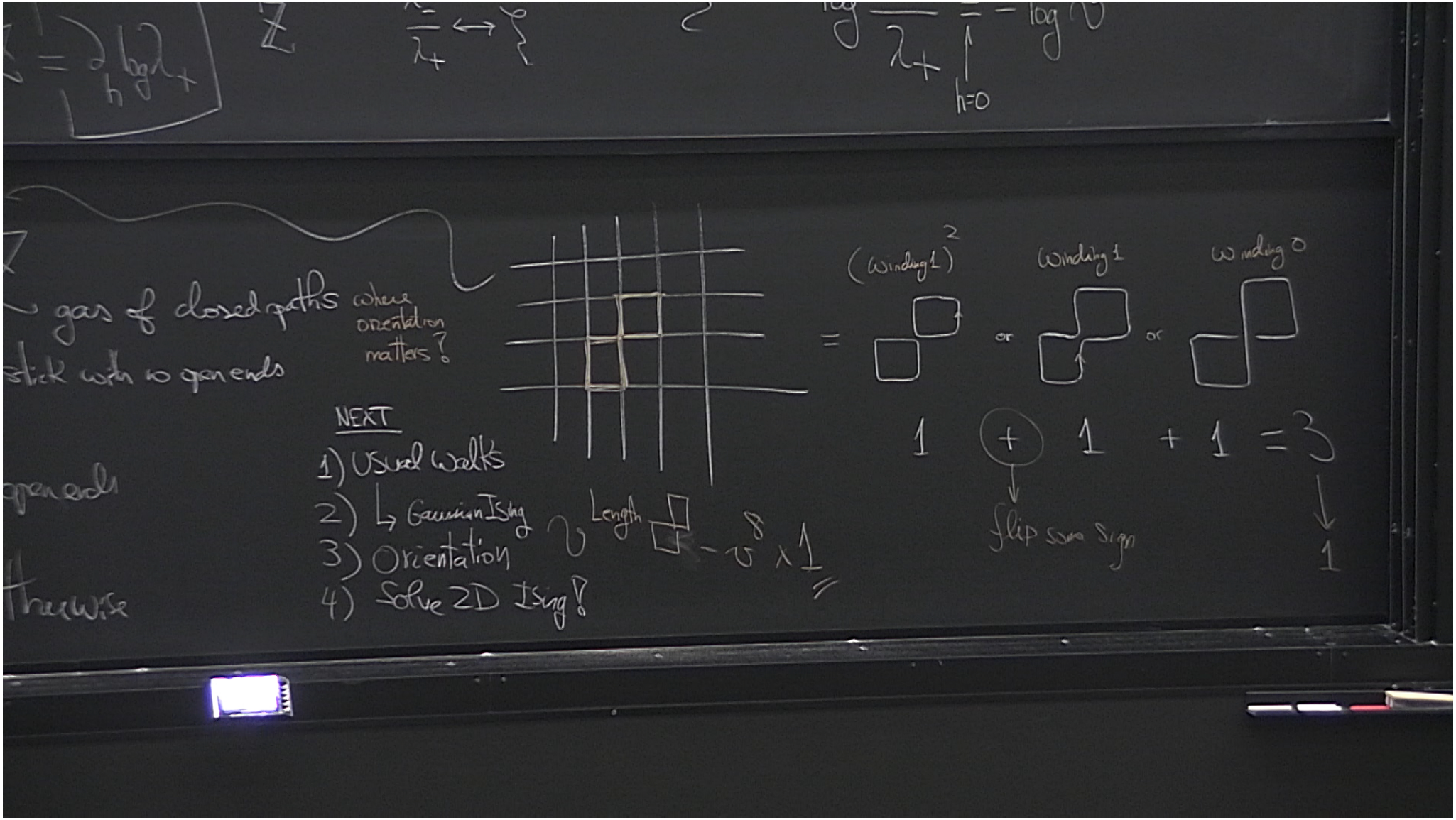



cosh π + (e^{-1})
i
j
<ij>


$\sqrt{\cosh^2 h + (e^{2J} - 1)}$
$\sum$
$\langle ij \rangle$
$= e^{\prime\prime} \sum$
perfect example
almost
$= Z$, gas of closed paths
gas of stick with no open ends
$\Omega = v \cdot l$
1 no open ends
0 otherwise


gas of closed paths
stick with no gon ends
Length
or
or


closed paths
or
or
1 + 1 + 1 = 3
Length
1


gas of closed paths
where orientation matters?
stick with no open ends
open ends
otherwise
NEXT
1) Usual walks
2) ↳ Gaussian Ising
3) Orientation
4) Solve 2D Ising!
Length
(Winding 1)²
Winding 1
Winding 0
or
or
1 + 1 + 1 = 3
flip some sign
1