Title: PSI 2019/2020 - Quantum Field Theory (Wohns/Xu) - Lecture 8

Speakers: Dan Wohns, Gang Xu

Collection: PSI 2019/2020 - Quantum Field Theory (Wohns/Xu)

Date: October 21, 2019 - 9:00 AM

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Quantum Field Theory (QFT) I part 2 spin ≠ 0

Motivation spin = $\frac{1}{2}$ → spinor

real world

Ultimate Goal: compute $|\mathcal{M}|^2$ → G_F in

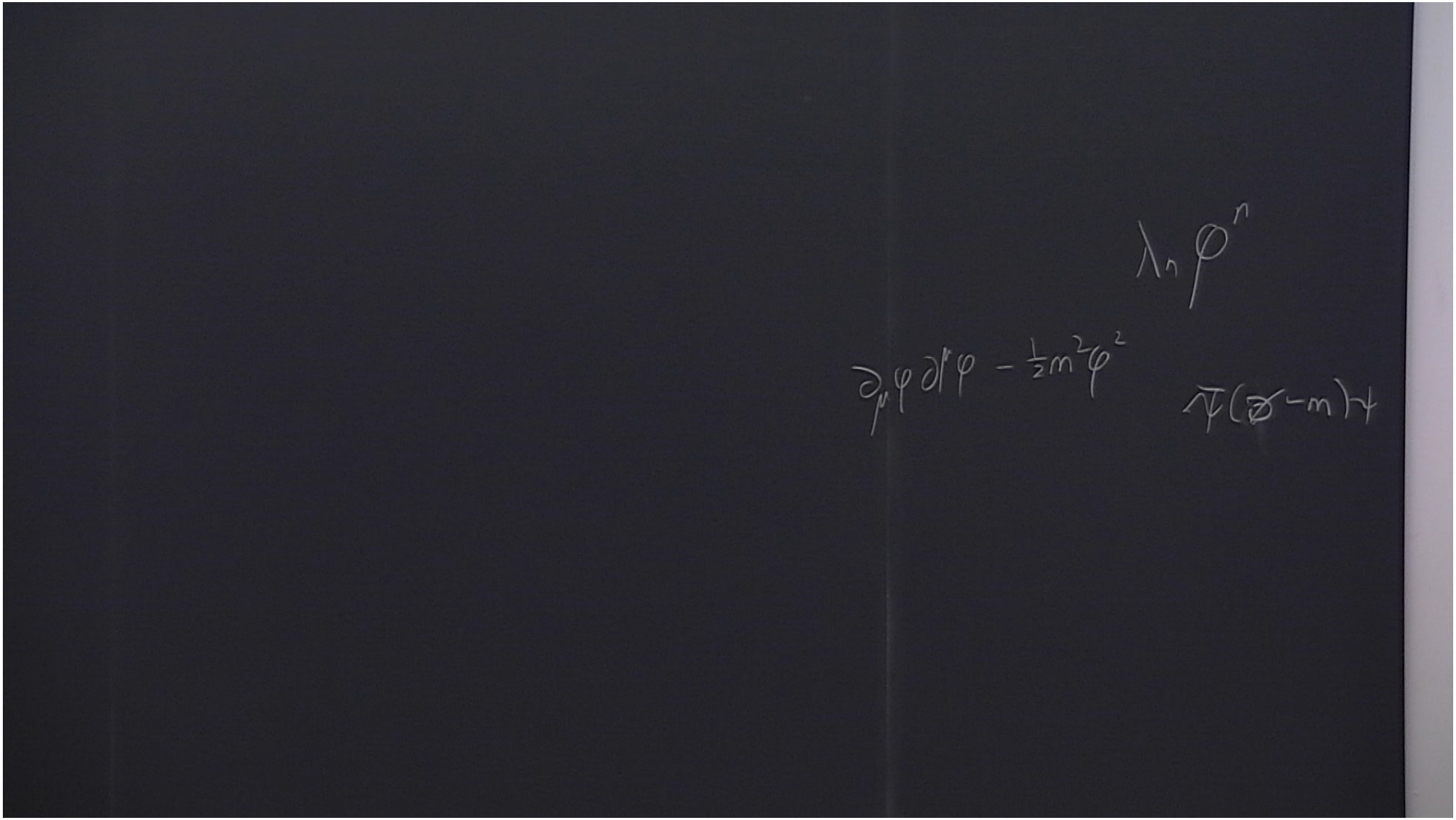
Quantum electrodynamics (QED)

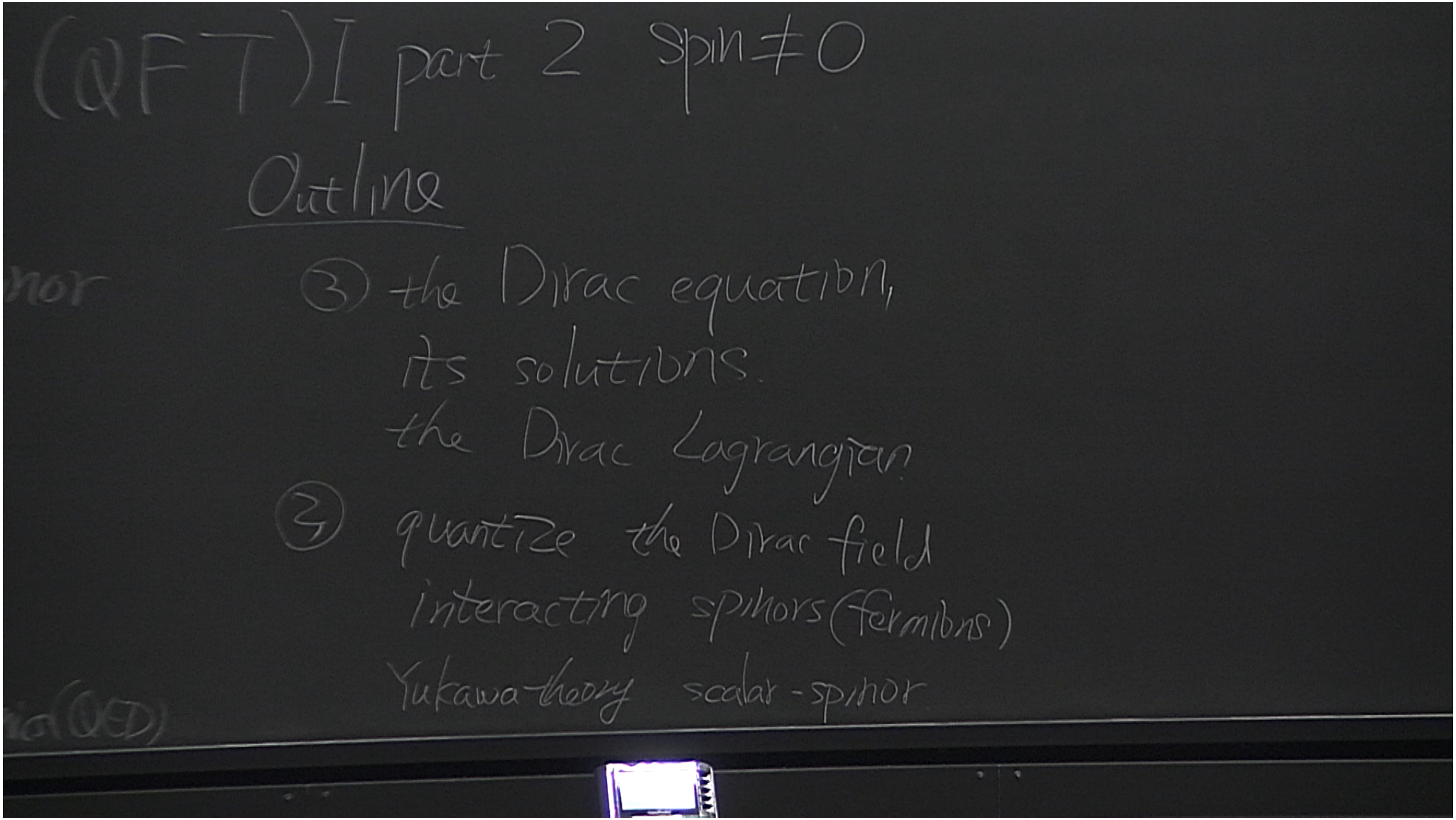
Quantum Field Theory (QFT

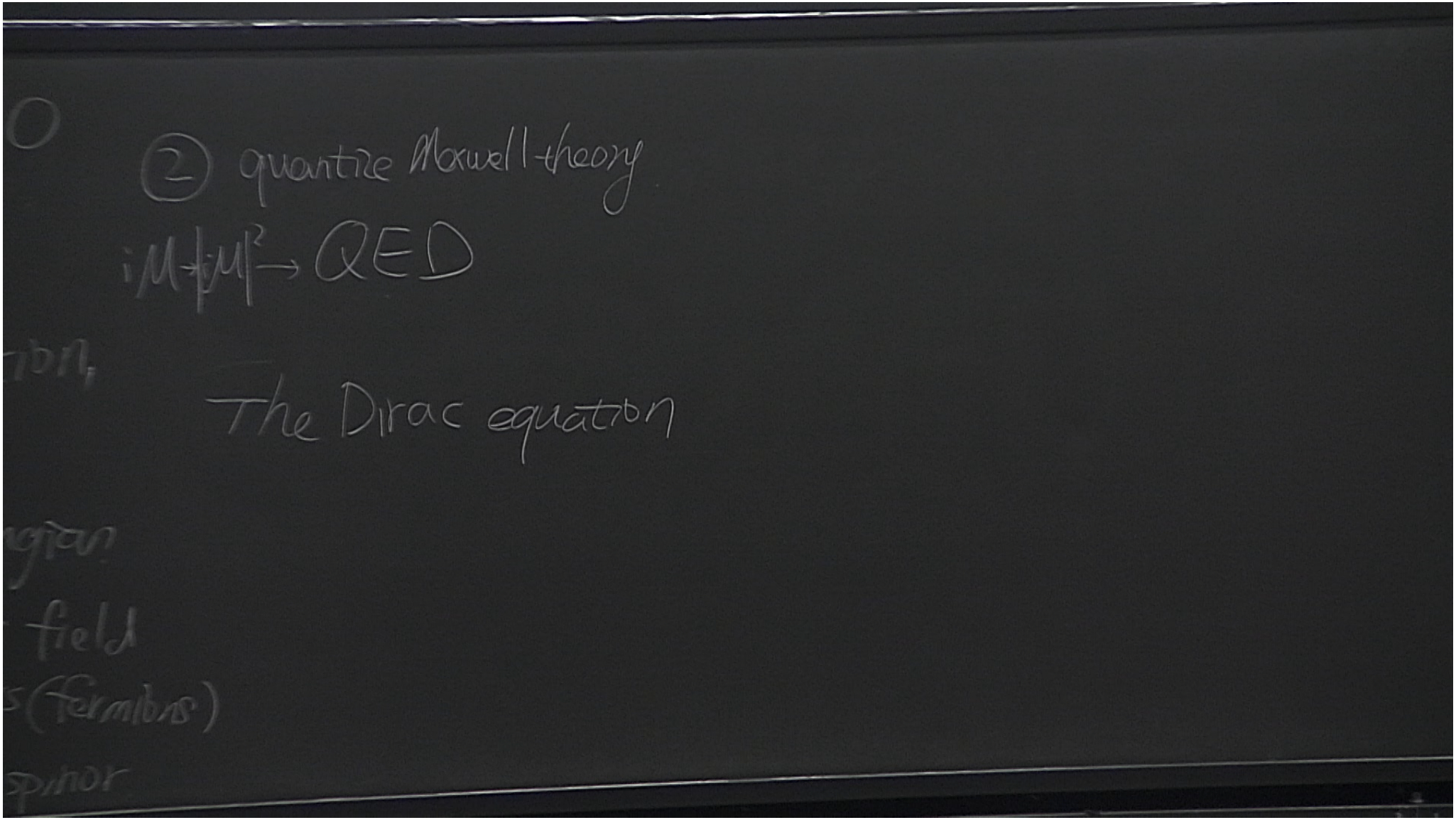
Motivation spin $= \frac{1}{2}$ → spinor

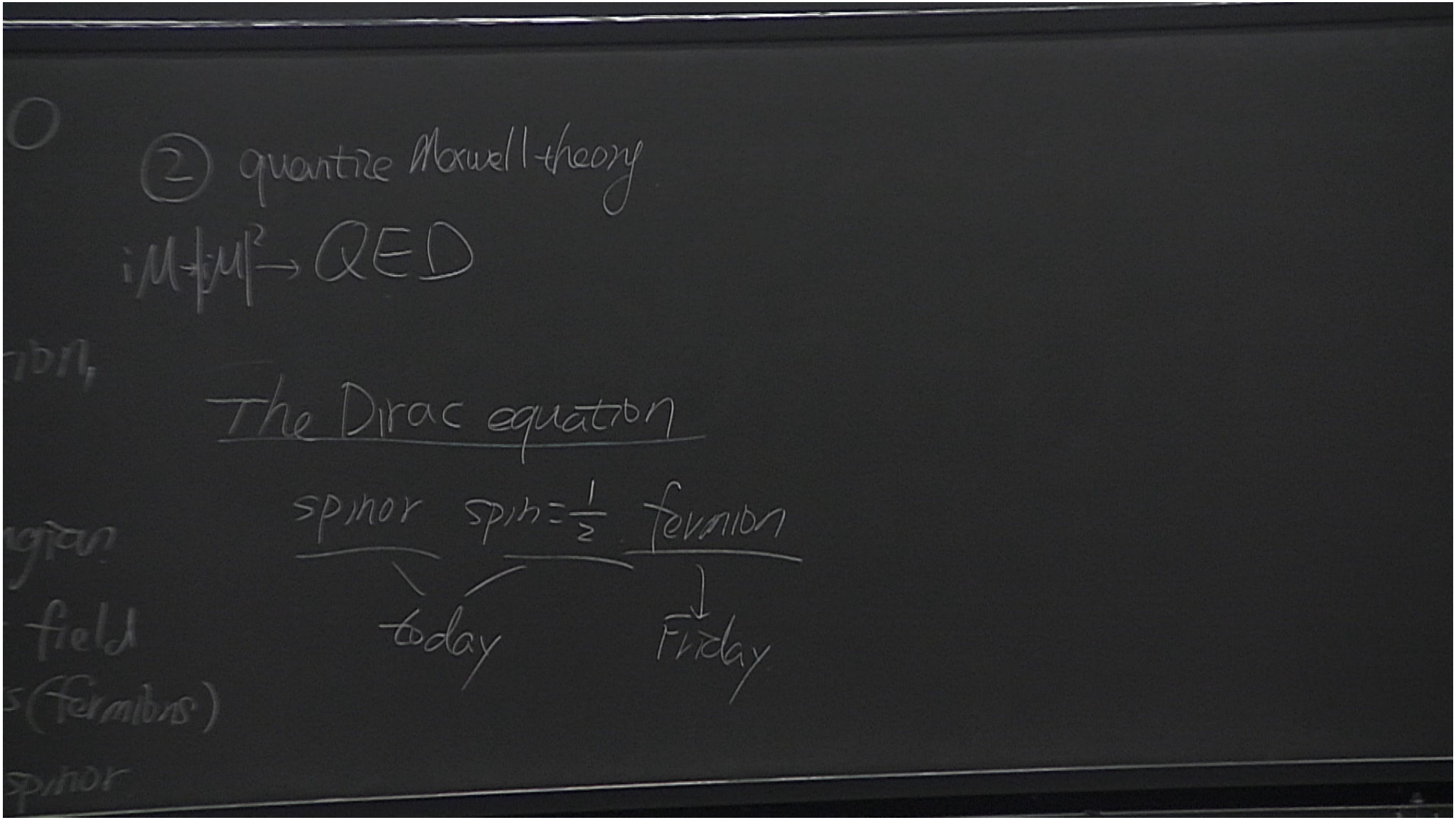
real world

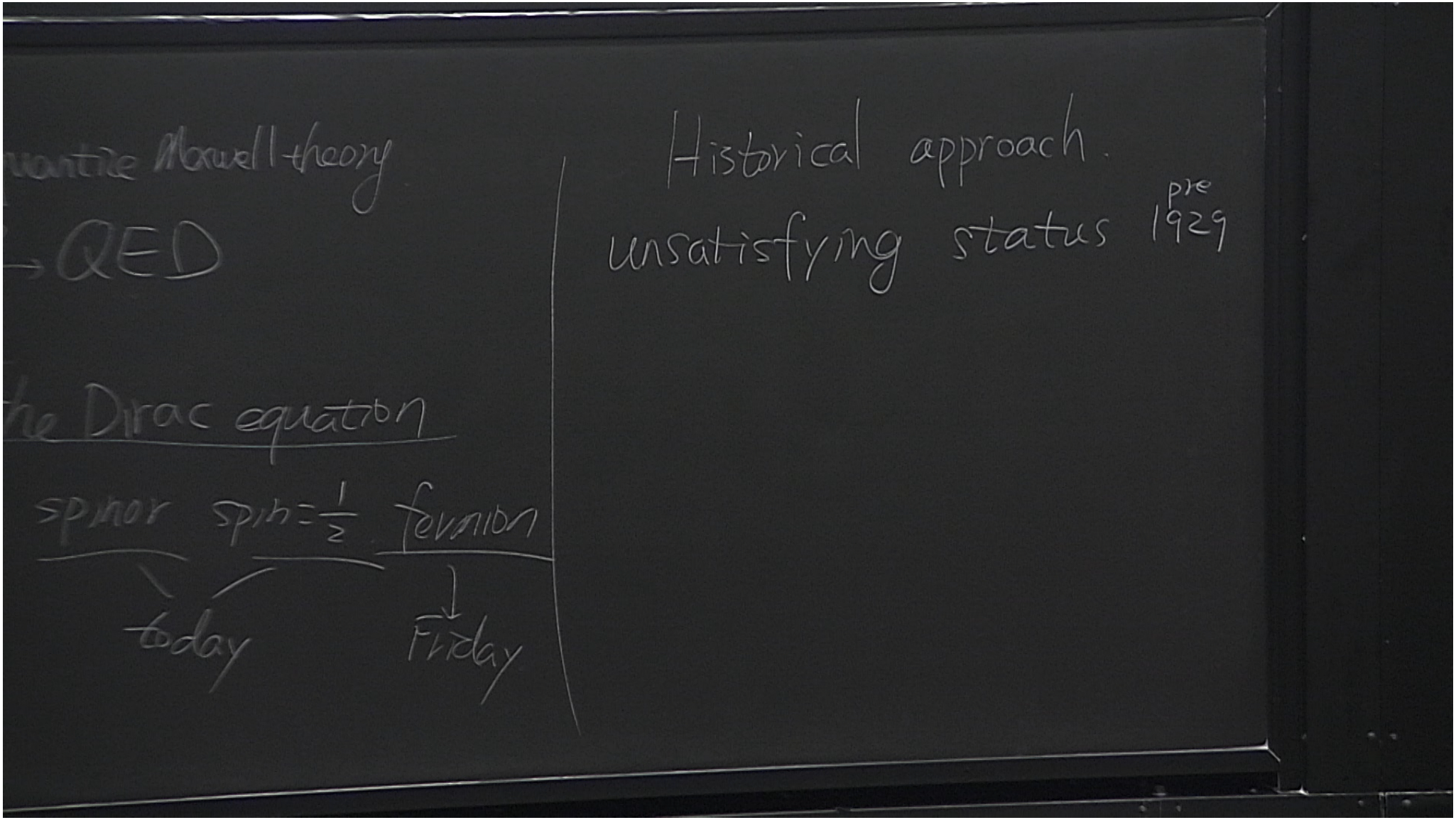
Ultimate Goal: compute $|\mathcal{M}|^2$ → Γ, σ, in Quantum electrodynamics (QED)

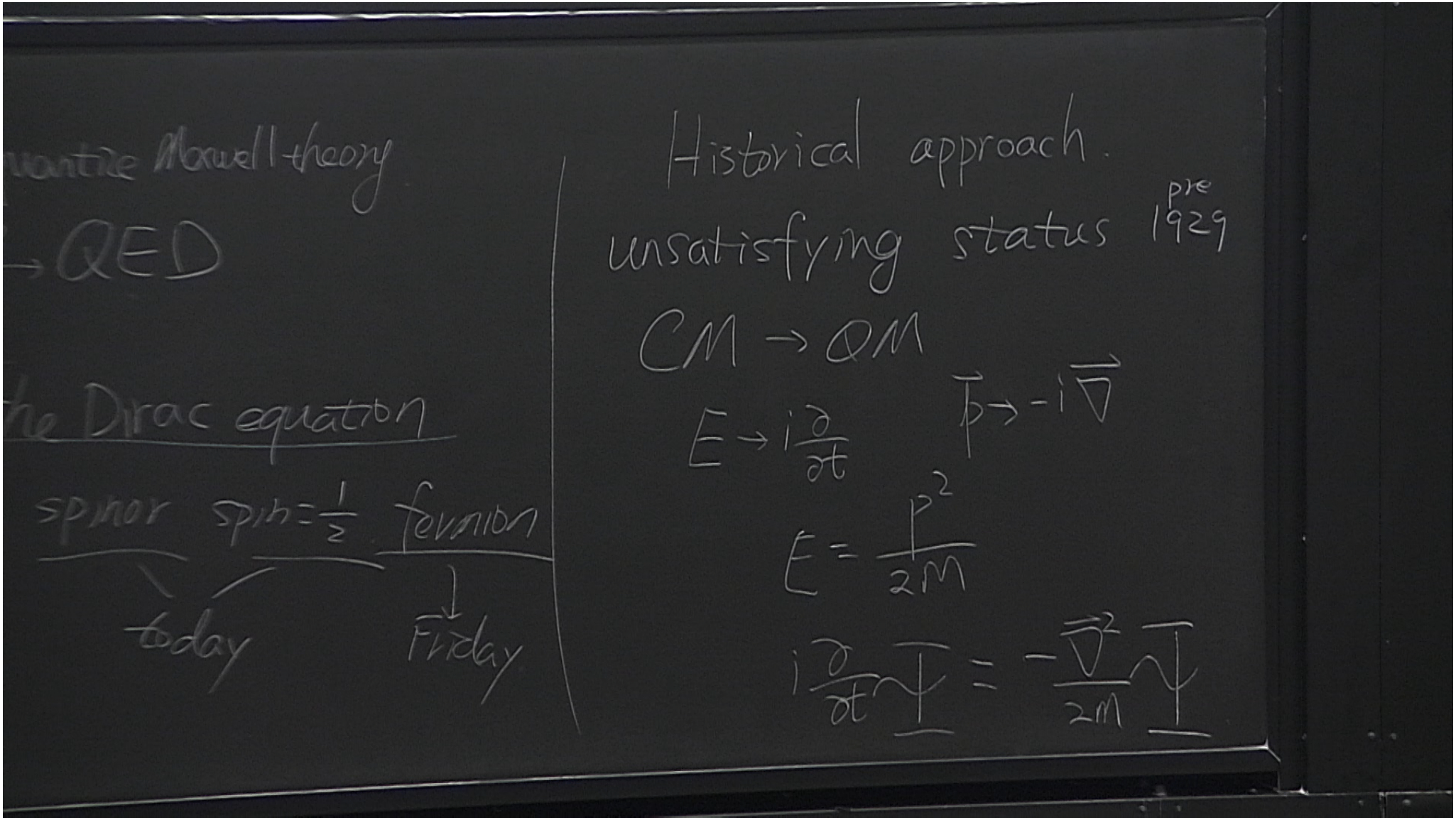

$\lambda_n \varphi^n$
$\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$
$\bar{\psi}(\not{\partial} - m)\psi$

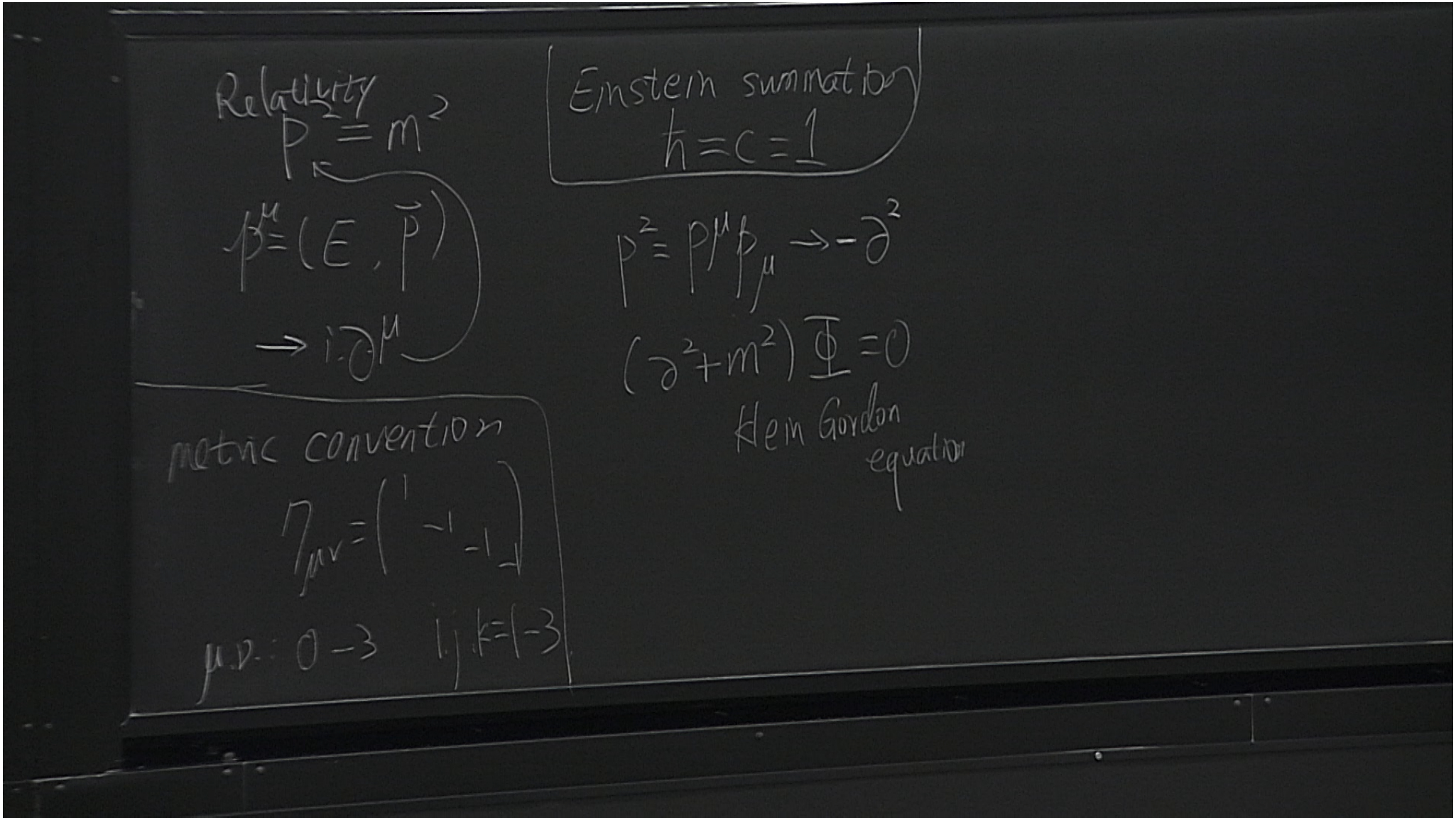

(QFT) I part 2 spin ≠ 0
Outline
③ the Dirac equation,
its solutions.
the Dirac Lagrangian
④ quantize the Dirac field
interacting spinors (fermions)
Yukawa theory scalar - spinor


② quantize Maxwell-theory
$i\mathcal{M} + i\mathcal{M}^2 \rightarrow$ QED
The Dirac equation
field
(fermions)
spinor


② quantize Maxwell-theory
$i\mathcal{M}+|\mathcal{M}|^2 \rightarrow$ QED
The Dirac equation
spinor spin $=\frac{1}{2}$ fermion
today
Friday


uantic Maxwell theory
→ QED
the Dirac equation
spinor spin = 1/2 fermion
today
Friday
Historical approach.
unsatisfying status pre 1929


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Historical approach.
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CM → QM
E → i ∂/∂t
p → -i ∇
E = p²/2M
i ∂/∂t ψ = -∇²/2M ψ


Relativity
$P^2 = m^2$
$p^\mu = (E, \vec{p})$
$\rightarrow i\partial^\mu$
Einstein summation
$\hbar = c = 1$
$p^2 = p^\mu p_\mu \rightarrow -\partial^2$
$(\partial^2 + m^2)\Phi = 0$
Klein Gordon equation
metric convention
$\eta_{\mu\nu} = (1, -1, -1, -1)$
$\mu, \nu : 0 - 3$
$i, j, k = 1 - 3$

② quantize Maxwell theory

$|M|^2 \rightarrow$ QED

The Dirac equation

spinor spin $= \frac{1}{2}$ fermion

today Friday

Historical approach.

unsatisfying status pre 1929

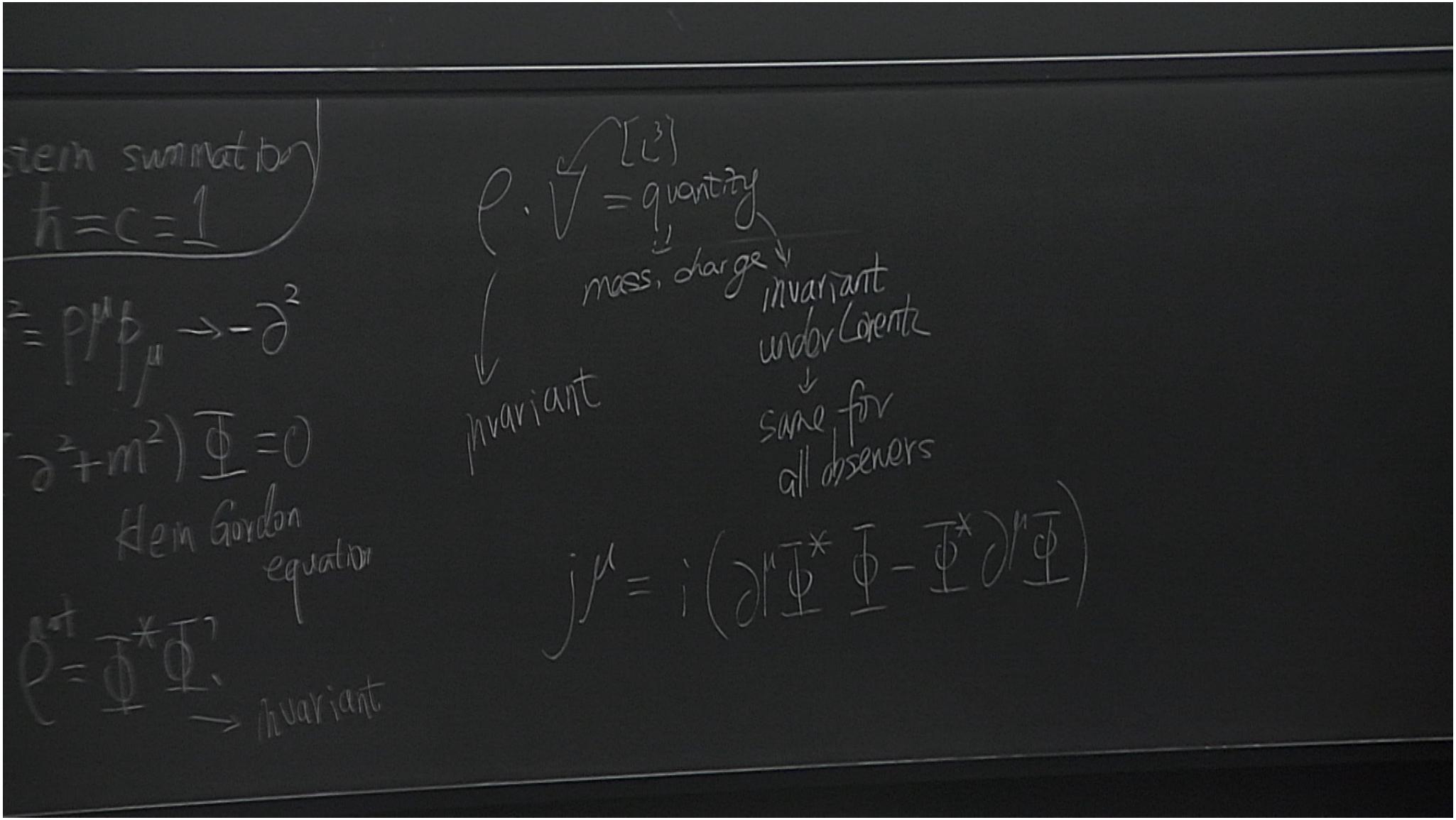
CM $\rightarrow$ QM

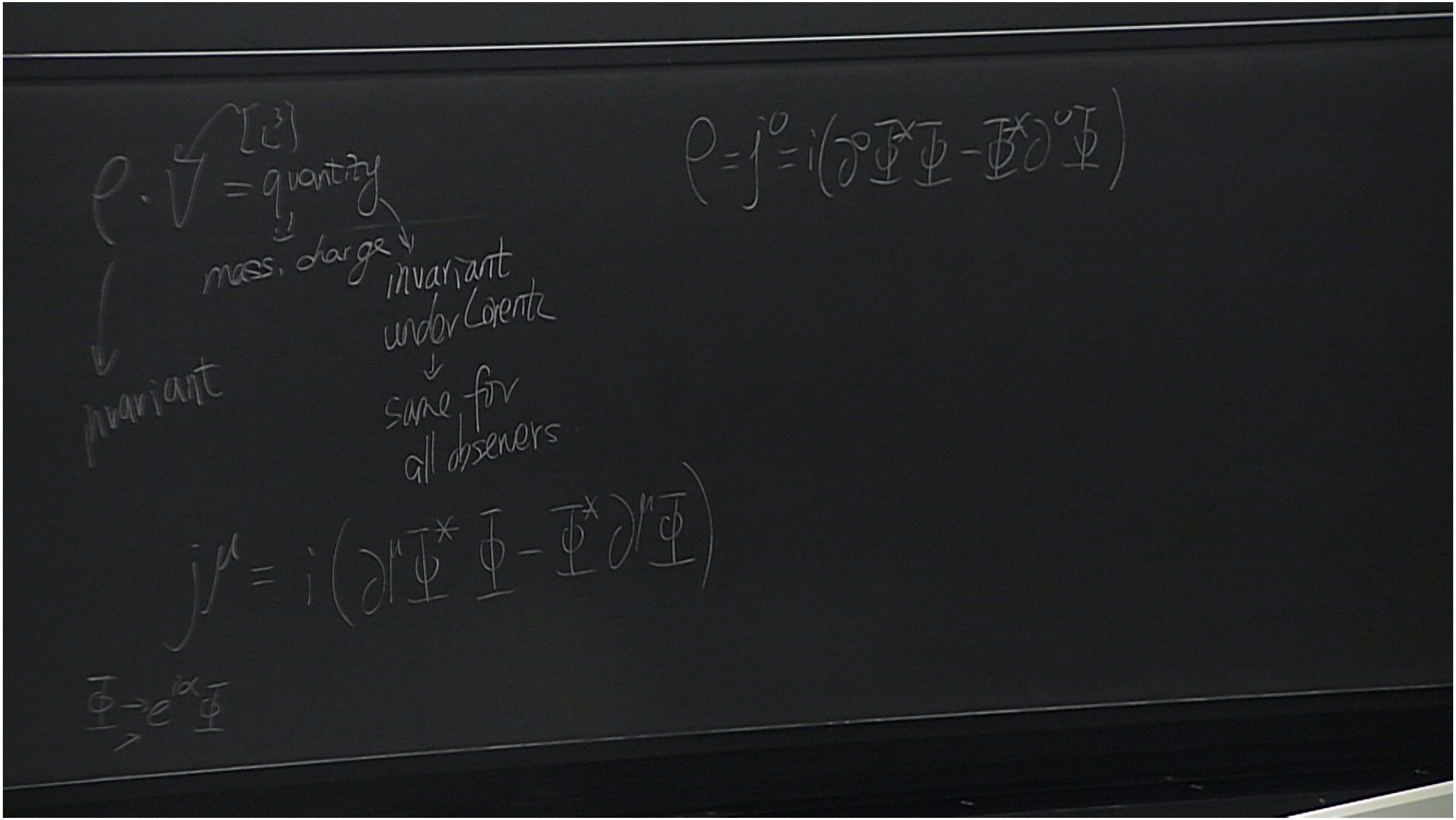
$E \rightarrow i\frac{\partial}{\partial t}$ $\quad \vec{p} \rightarrow -i\vec{\nabla}$

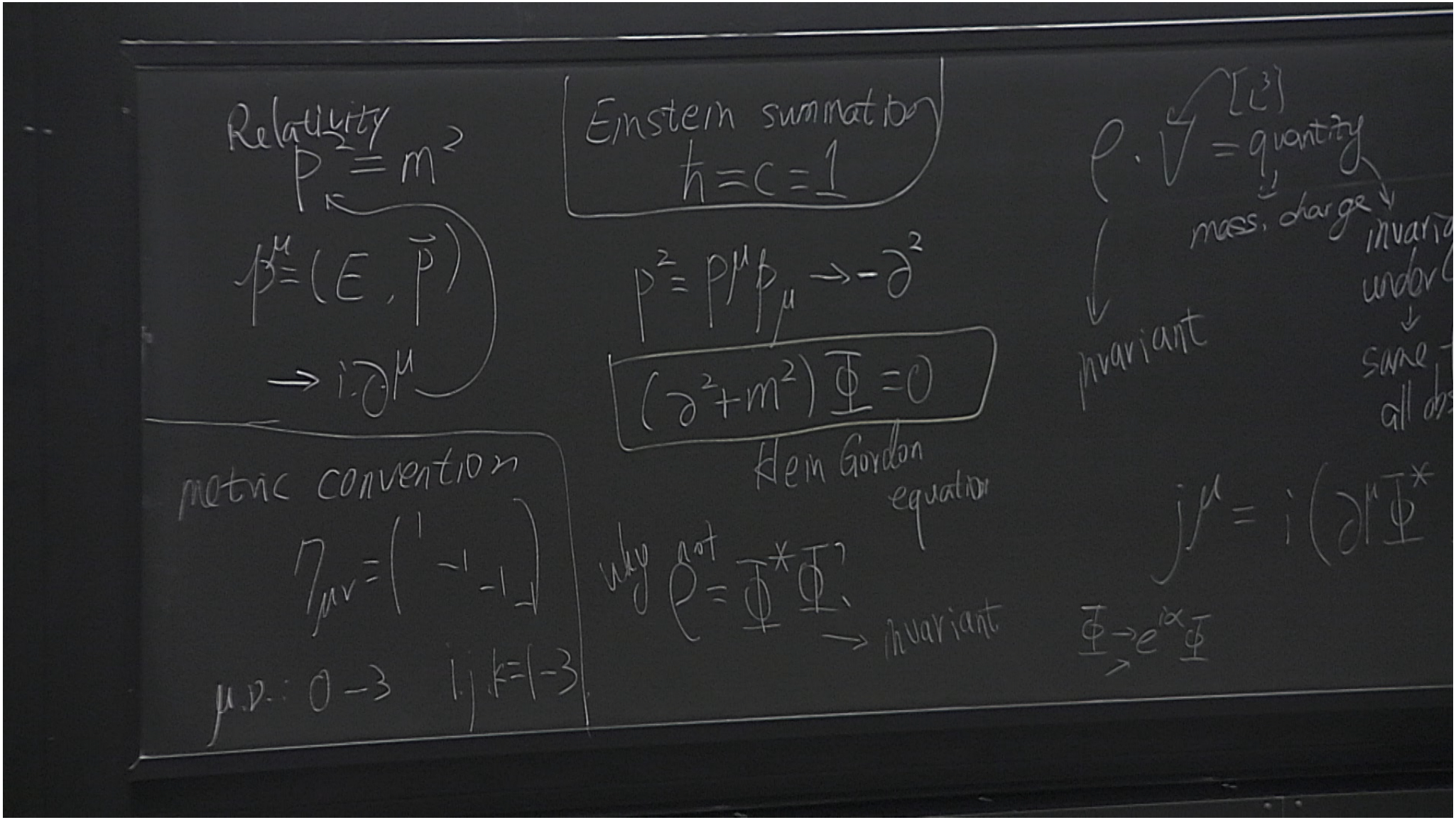
$E = \frac{p^2}{2M}$

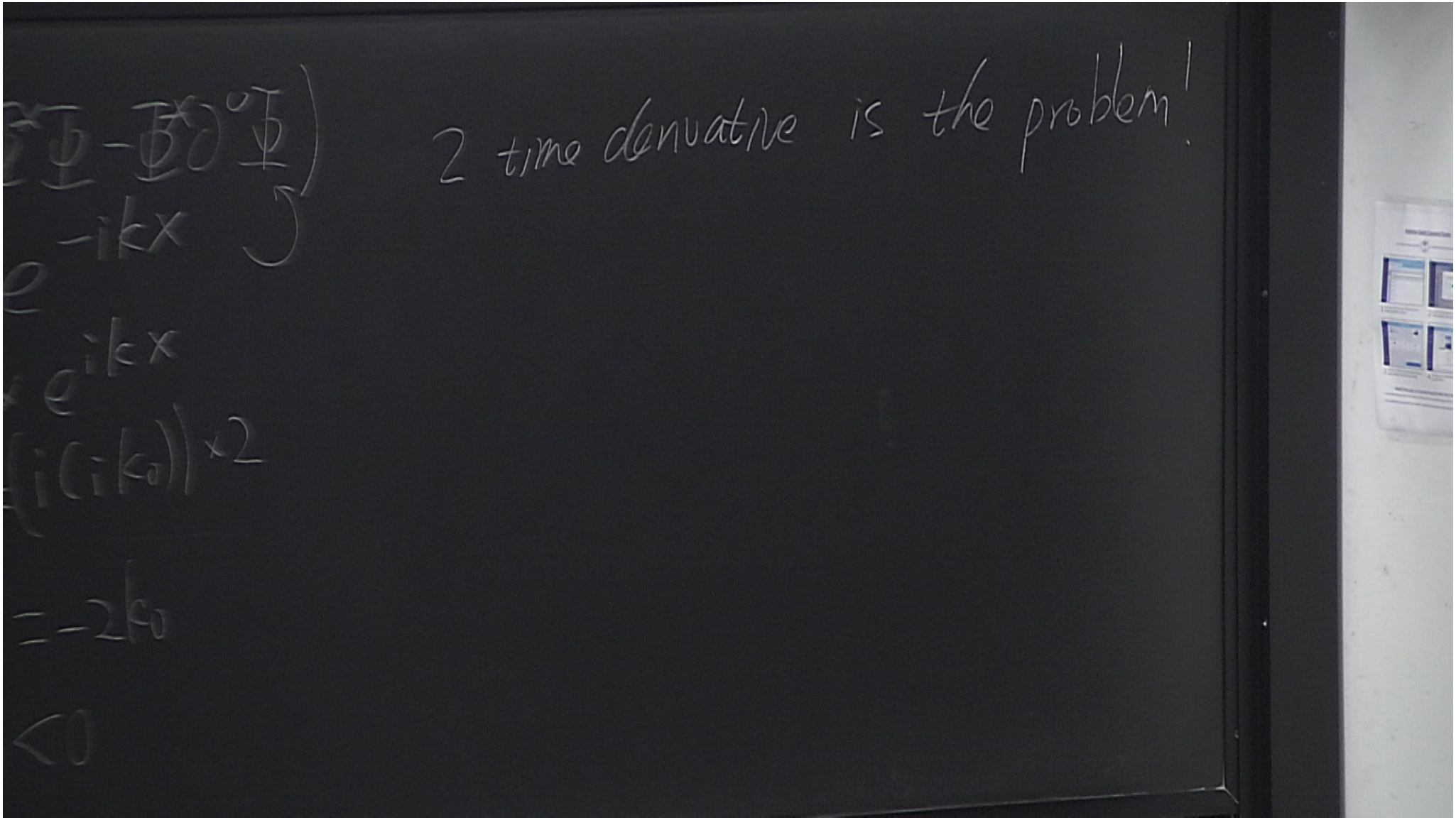
probability density

$\rho = \psi^* \psi$ $\quad i\frac{\partial}{\partial t}\psi = -\frac{\vec{\nabla}^2}{2M}\psi$


stein summation
$\hbar = c = 1$
$= p^\mu p_\mu \rightarrow -\partial^2$
$\partial^2 + m^2)\,\Phi = 0$
Klein Gordon equation
$\rho = \Phi^* \Phi$
invariant
$\rho \cdot V$ = quantity
$[L^3]$
mass, charge
invariant
invariant under Lorentz
same for all observers
$j^\mu = i(\partial^\mu \Phi^* \Phi - \Phi^* \partial^\mu \Phi)$


$\rho \cdot V$ = quantity
[?]
mass, charge
invariant under Lorentz
same for all observers
invariant
$\rho = j^0 = i(\partial^0 \bar{\Phi}^* \bar{\Phi} - \bar{\Phi}^* \partial^0 \bar{\Phi})$
$j^\mu = i(\partial^\mu \bar{\Phi}^* \bar{\Phi} - \bar{\Phi}^* \partial^\mu \bar{\Phi})$
$\bar{\Phi} \to e^{i\alpha} \bar{\Phi}$


Relativity
$p^2 = m^2$
$p^\mu = (E, \vec{p})$
$\rightarrow i\partial^\mu$
Einstein summation
$\hbar = c = 1$
$p^2 = p^\mu p_\mu \rightarrow -\partial^2$
$(\partial^2 + m^2)\Phi = 0$
Klein Gordon equation
metric convention
$\eta_{\mu\nu} = (1, -1, -1, -1)$
$\mu, \nu: 0 \rightarrow 3$
$i, j, k = 1 \rightarrow 3$
why not $\rho = \Phi^* \Phi$?
$\rightarrow$ invariant
$\rho \cdot V = q$
$[L^3]$
quantity
mass, charge
invariant
$j^\mu = i(\partial^\mu \Phi^*$
$\Phi \rightarrow e^{i\alpha} \Phi$


2 time derivative is the problem!

What Dirac wants:

① one time derivative

② one space derivative

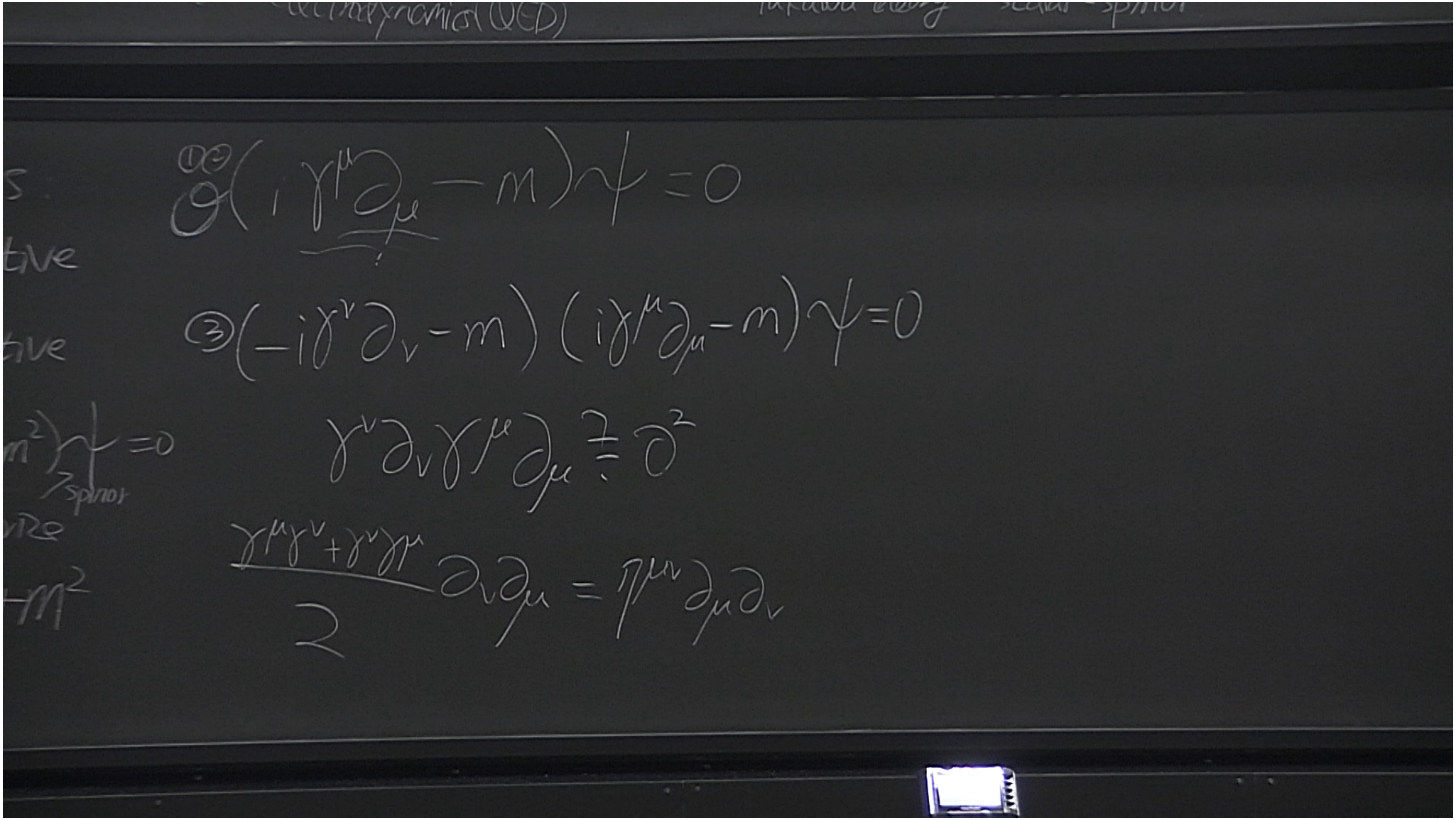
③ $p^2 = m^2$ $\quad (\partial^2 + m^2)\psi = 0$ ← spinor

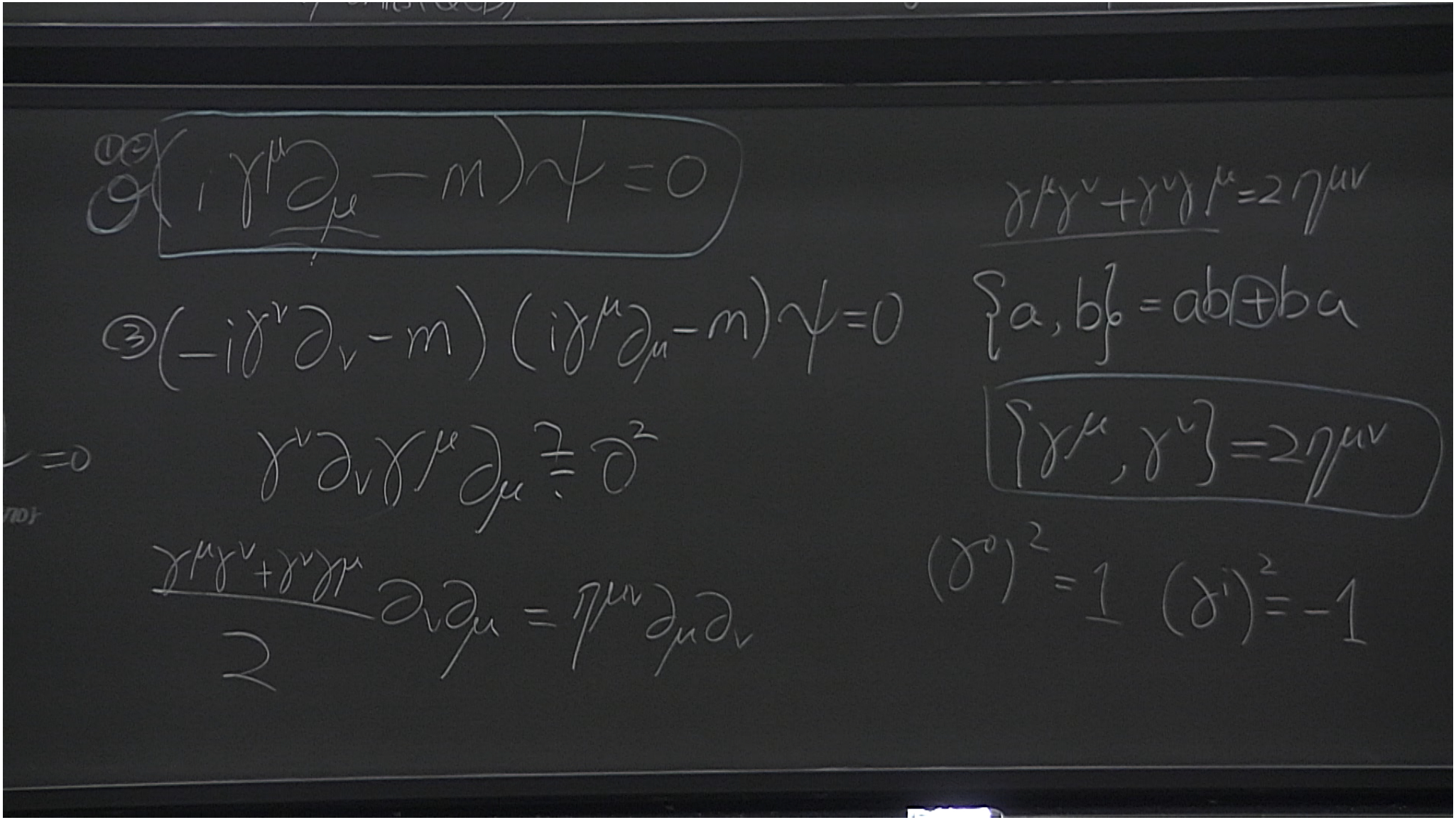
↓ factorize

$(-ia - m)(ia - m) = a^2 + m^2$

①② $(i\gamma^\mu \partial_\mu - m)\psi = 0$

③ $(-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m)$


$(i\gamma^\mu\partial_\mu - m)\psi = 0$
$(-i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu - m)\psi = 0$
$\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu = \partial^2$
$\frac{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu}{2}\partial_\nu\partial_\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu$


$(i\gamma^\mu\partial_\mu - m)\psi = 0$
$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$
$(-i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu - m)\psi = 0$
$\{a, b\} = ab + ba$
$\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu = \partial^2$
$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$
$\frac{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu}{2}\partial_\nu\partial_\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu$
$(\gamma^0)^2 = 1 \quad (\gamma^i)^2 = -1$

$=2\eta^{\mu\nu}$

$\{\gamma^a, \gamma^b\} = 2\eta^{\mu\nu}$

γ^μ has to be matrix

$(\gamma^\mu)_{ab}$

a.b.c.d spinor index

$$i(\gamma^\mu)_{ab}\partial_\mu - m\mathbb{1}_{ab}\psi_b = 0$$

$)^2 = -1$

$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$

γ^μ has to be matrix

$(\gamma^\mu)_{ab}$

a b c d spinor index

$\psi = 0$ $\{a, b\} = ab + ba$

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}$

matrix matrix

$(\gamma^0)^2 = 1$ $(\gamma^1)^2 = -1$

$i(\gamma^\mu)_{ab} \partial_\mu - m \mathbb{1}_{ab} \psi_b = 0$

def: $\gamma_\mu \equiv \eta_{\mu\nu} \gamma^\nu$ bunch of numbers

$$i\frac{\partial}{\partial t}\psi = H\psi$$

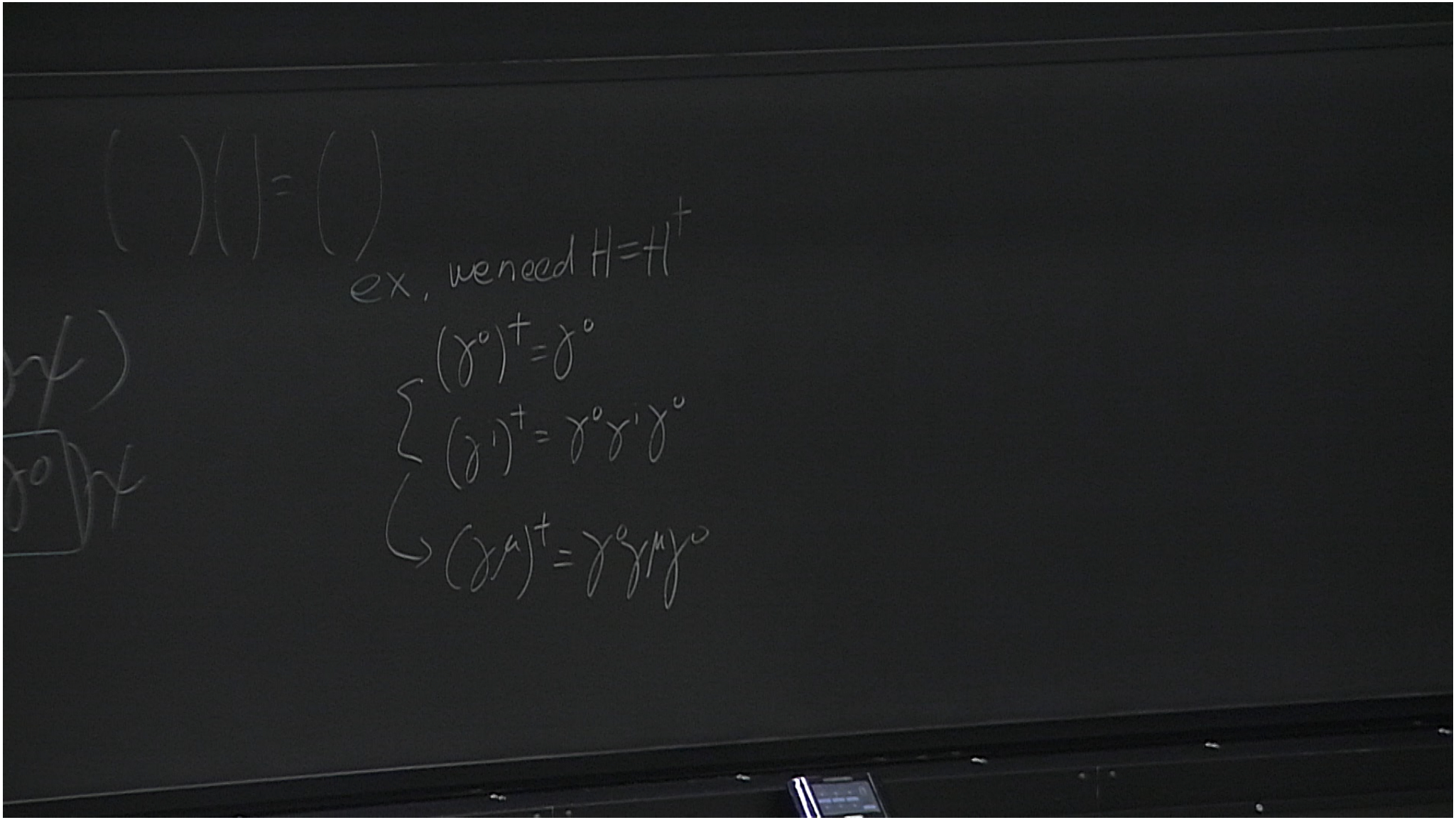
$$(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi = 0$$

$$\gamma^0 \times \left(i\gamma^0\partial_0\psi = (-i\gamma^i\partial_i + m)\psi\right)$$

$$i\partial_0\psi = \boxed{-i\gamma^0\gamma^i\partial_i + m\gamma^0}\,\psi$$

$$\| \\ H$$

$$(\,)(\,) = (\,)$$


ex. we need $H = H^\dagger$
$(\gamma^0)^\dagger = \gamma^0$
$(\gamma^i)^\dagger = \gamma^0 \gamma^i \gamma^0$
$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$

$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$

γ^μ has to be matrix

$(\gamma^\mu)_{ab}$

$\{a, b\}_\oplus = ab \oplus ba$

a.b.c.d spinor index.

Dirac Algebra

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}$

$i(\gamma^\mu)_{ab} \partial_\mu - m \mathbb{1}_{ab} \psi_b = 0$

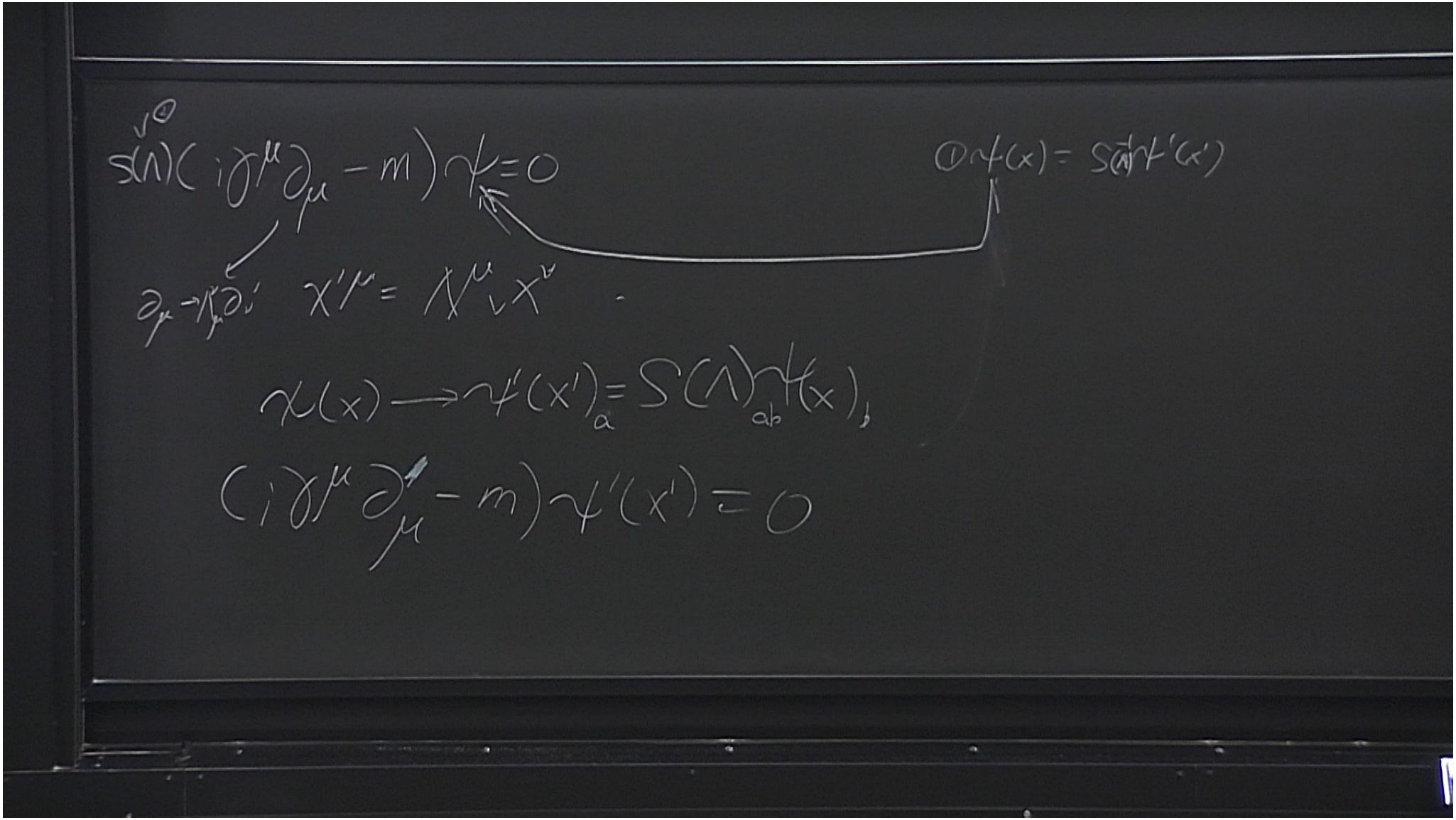
$\mathbb{1}^2 = \mathbb{1}$ $(\gamma^i)^2 = -1$

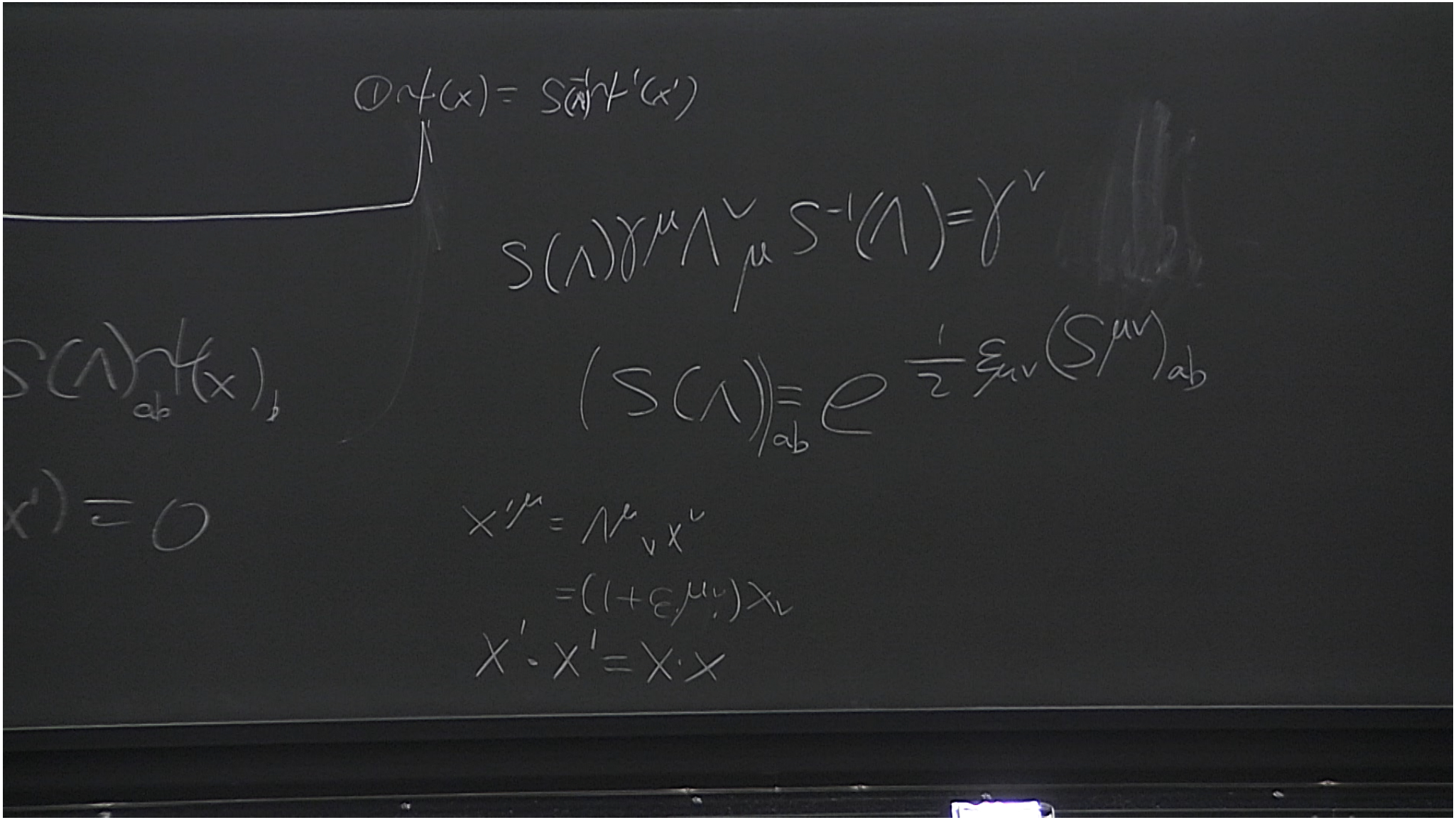
def: $\gamma_\mu \equiv \eta_{\mu\nu} \gamma^\nu$ bunch of numbers

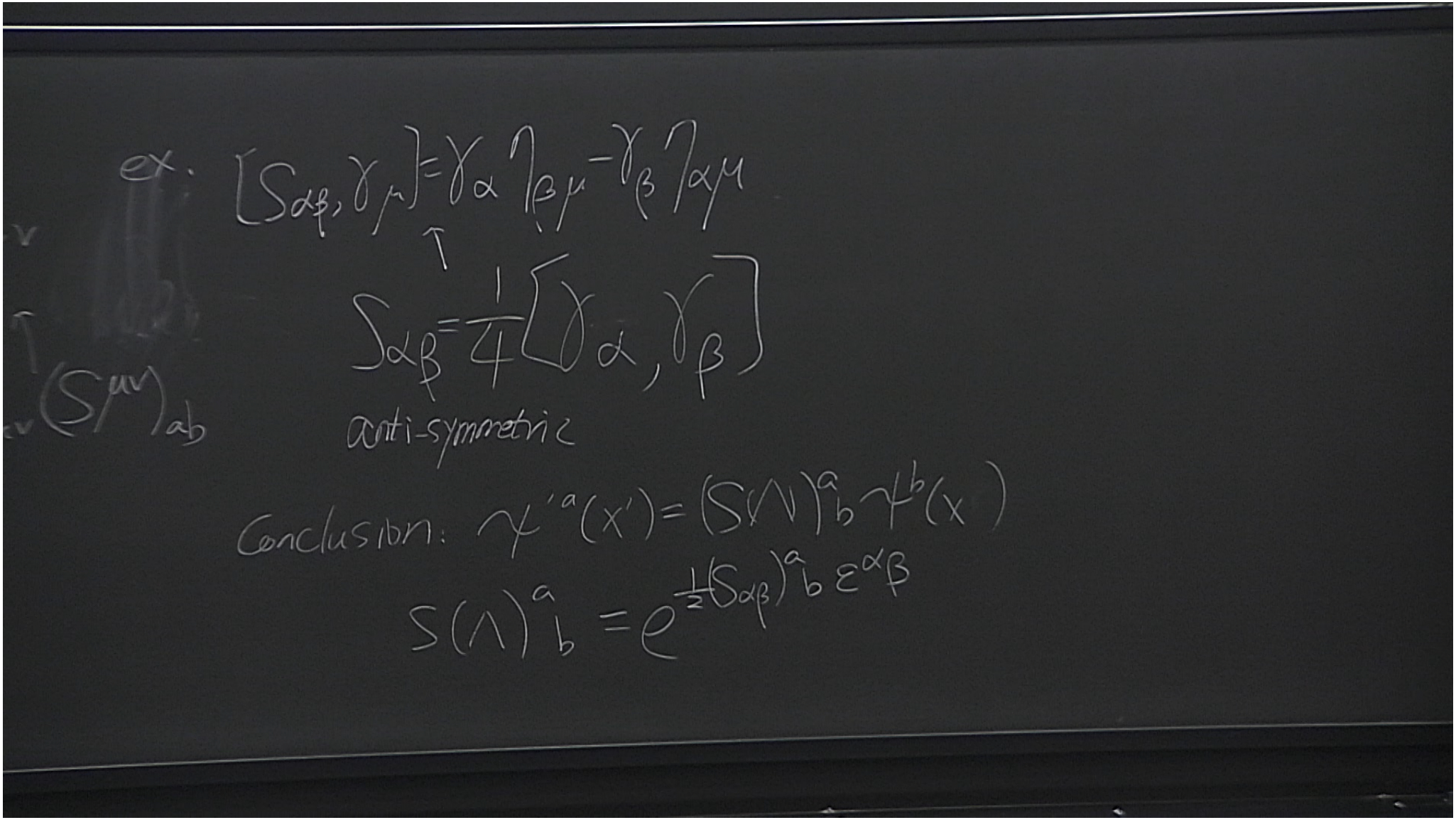
$$N-1-\frac{N-1}{3}-1=3k$$

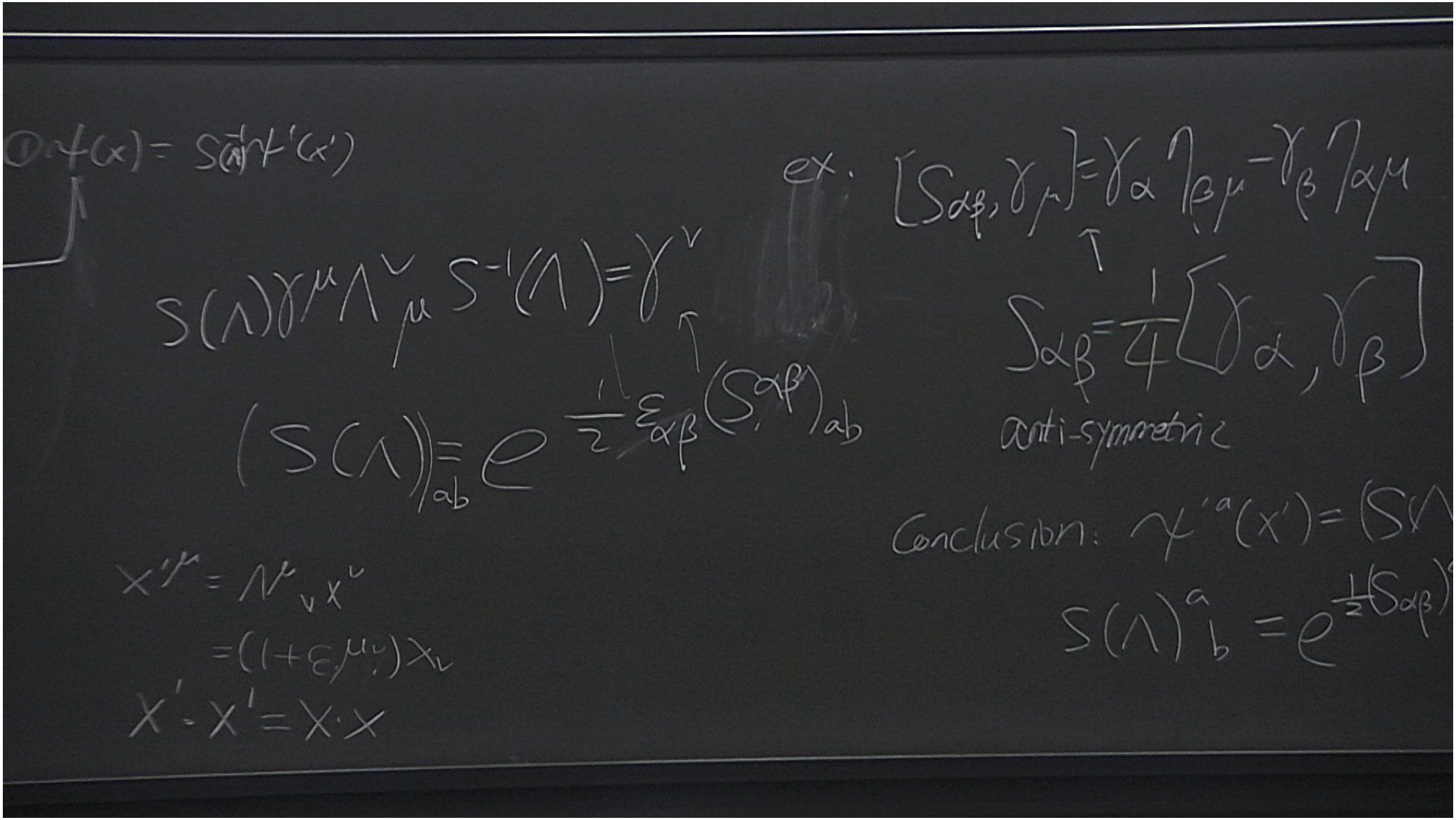
$$k=\frac{N-1}{3}$$

$N=-2$ ← minus fish! Dirac Sea


$S(\Lambda)(i\gamma^\mu\partial_\mu - m)\psi = 0$
$\partial_\mu \to \Lambda^\nu_\mu \partial'_\nu \quad x'^\mu = \Lambda^\mu_\nu x^\nu$
$\psi(x) \to \psi'(x')_a = S(\Lambda)_{ab}\psi(x)_b$
$(i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0$
$\psi(x) = S(\Lambda)\psi'(x')$


① ψ(x) = S(Λ)ψ'(x')
S(Λ)γ^μ Λ^ν_μ S^{-1}(Λ) = γ^ν
S(Λ)_ab ψ(x)_b
(S(Λ))_ab = e^{(1/2) ε_μν (S^μν)_ab}
x') = 0
x'^μ = Λ^μ_ν x^ν
= ((1 + ε^μ_ν) x_ν
x' · x' = x · x


ex: $[S_{\alpha\beta}, \gamma_\mu] = \gamma_\alpha \eta_{\beta\mu} - \gamma_\beta \eta_{\alpha\mu}$
$S_{\alpha\beta} = \frac{1}{4}[\gamma_\alpha, \gamma_\beta]$
anti-symmetric
$(S^{\mu\nu})_{ab}$
Conclusion: $\psi'^a(x') = (S(\Lambda))^a_b \psi^b(x)$
$S(\Lambda)^a_b = e^{\frac{1}{2}(S_{\alpha\beta})^a_b \varepsilon^{\alpha\beta}}$


① ψ(x) = S(Λ) ψ'(x')
S(Λ) γ^μ Λ^ν_μ S^{-1}(Λ) = γ^ν
(S(Λ))_ab = e^{-1/2 ε_αβ (S^αβ)_ab}
ex: [S_αβ, γ_μ] = γ_α η_βμ − γ_β η_αμ
S_αβ = 1/4 [γ_α, γ_β]
anti-symmetric
Conclusion: ψ'^a(x') = (S(Λ))^a_b ψ(x)
S(Λ)^a_b = e^{-1/2 (S_αβ)
x'^μ = Λ^μ_ν x^ν
= (1 + ε^μ_ν) x_ν
X' − X' = X − X

$\begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$ small rotation in xy-plane

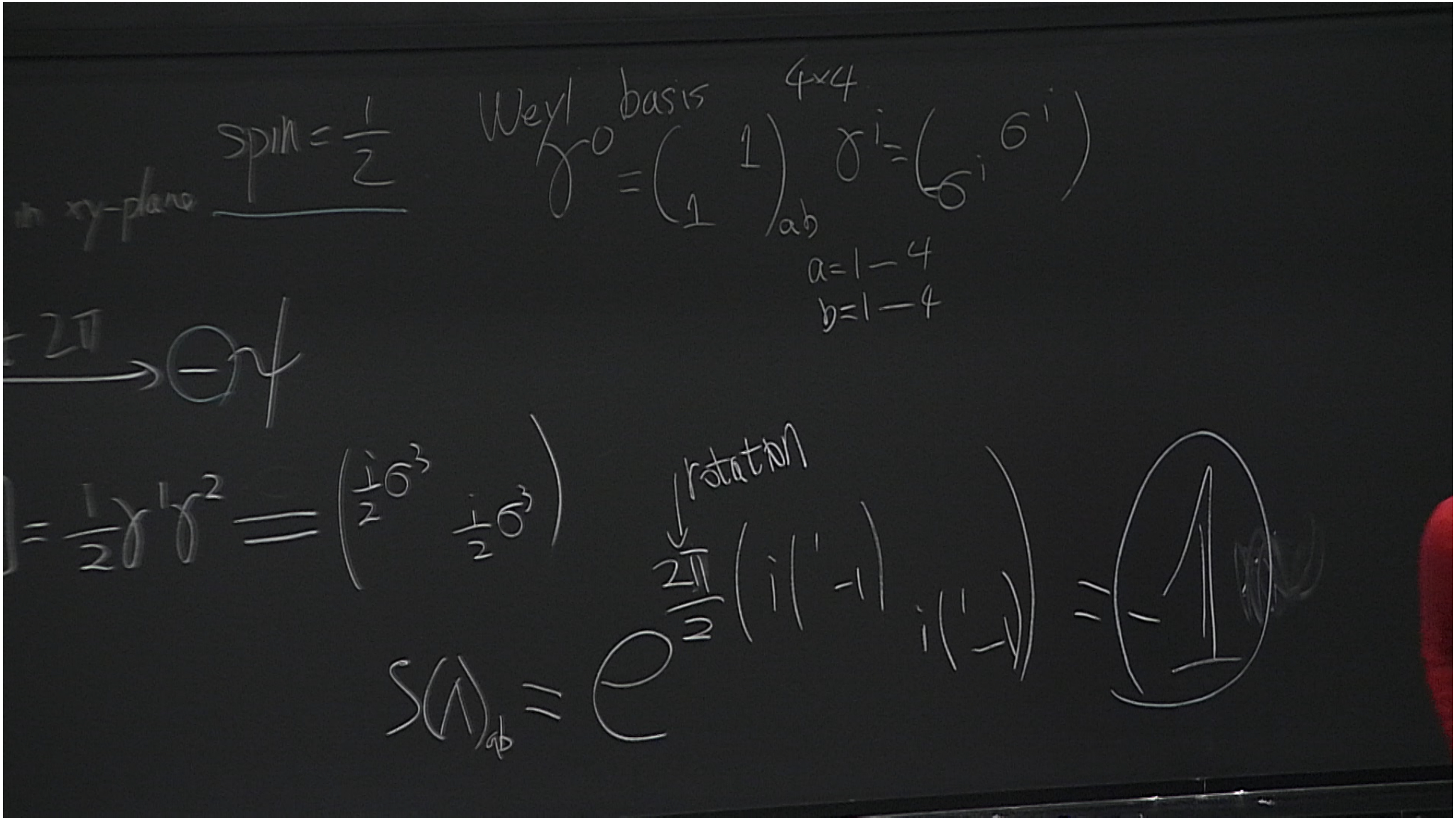
spin $= \frac{1}{2}$

Weyl basis

$\gamma^0 = \begin{pmatrix} & 1 \end{pmatrix}$

$\chi \xrightarrow{\text{rot } 2\pi} -\chi$

$S^{12} = \frac{1}{4}[\gamma^1, \gamma^2] = \frac{1}{2}\gamma^1\gamma^2 = \begin{pmatrix} \frac{1}{2}\sigma^3 & \\ & \frac{1}{2}\sigma^3 \end{pmatrix}$


spin = 1/2
Weyl basis
4×4
$\gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}_{ab}$
$\gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}$
a = 1 − 4
b = 1 − 4
xy-plane
$= 2\pi \rightarrow -\psi$
$= \frac{1}{2}\gamma^1\gamma^2 = \begin{pmatrix} \frac{1}{2}\sigma^3 & \\ & \frac{1}{2}\sigma^3 \end{pmatrix}$
rotation
$S(\Lambda)_{ab} = e^{\frac{2\pi}{2}\begin{pmatrix} i(1-1) & \\ & i(1-1) \end{pmatrix}} = -1$