Title: PSI 2019/2020 - Quantum Field Theory (Wohns/Xu) - Lecture 5

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Collection: PSI 2019/2020 - Quantum Field Theory (Wohns/Xu)

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Interaction Picture + Wick's Theorem

$$\sigma, \Gamma \longleftrightarrow \langle \Omega | T \varphi_1 \cdots \varphi_n | \Omega \rangle \xleftrightarrow{\text{int. pic}} \langle 0 | T \varphi_1 \cdots \varphi_m | 0 \rangle \xleftrightarrow{\text{Wick}} \langle 0 | T \varphi_1 \varphi_2 | 0 \rangle$$

$$H = H_0 + H_1$$

all agree at $t = t_0$

Heisenberg	Schrödinger	Interaction Picture
$\vert\psi\rangle = \text{constant}$	$\vert\psi(t)\rangle = e^{-iH(t-t_0)}\vert\psi\rangle$	$\vert\psi_I(t)\rangle = e^{iH_0(t-t_0)}\vert\psi(t)\rangle = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}\vert\psi\rangle$
$\mathcal{O}(t) = e^{iH(t-t_0)}\mathcal{O}\, e^{-iH(t-t_0)}$	$\mathcal{O} = \text{constant}$	$\mathcal{O}_I(t) = e^{iH_0(t-t_0)}\mathcal{O}\, e^{-iH_0(t-t_0)}$

$|0\rangle$

$$\varphi(\vec{x},t) = e^{iH(t-t_0)}\, e^{-iH_0(t-t_0)}\, \varphi_0(\vec{x},t)\, e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}$$

interaction picture field

$$= U^{\dagger}(t,t_0)\, \varphi_0(\vec{x},t)\, U(t,t_0)$$

$$U_{21} = U(t_2,t_1) = T\exp\left[-i\int_{t_1}^{t_2} dt'\, H_{1I}(t')\right]$$

time-ordering

int pic. int Hamiltonian

$U_{11} = \mathbb{1}$ time evolution for no time

$U_{21}^{-1} = U_{21}^{\dagger} = U_{12}$

$U_{32}U_{21} = U_{31}$

Interaction Picture + Wick's Theorem

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$\mathcal{O}(t) = e^{iH(t-t_0)}\mathcal{O}\, e^{-iH(t-t_0)}$	$\mathcal{O} = \text{constant}$	$\mathcal{O}_I(t) = e^{iH_0(t-t_0)}\mathcal{O}\, e^{-iH_0(t-t_0)} \rightarrow \mathcal{O} = e^{-iH_0(t-t_0)}\mathcal{O}_I(t)\, e^{iH_0(t-t_0)}$

$$|\Omega\rangle \overset{?}{\longleftrightarrow} |0\rangle$$

LSZ: assumed

interacting theory
$$\begin{cases} a_{\vec{p}}(t_0)\,|\Omega(t)\rangle = 0 \quad \text{for } t = \pm\infty \\ a_{\vec{p}}(t_0)\, e^{-iH(t-t_0)}|\Omega\rangle = 0 \end{cases}$$

Heisenberg

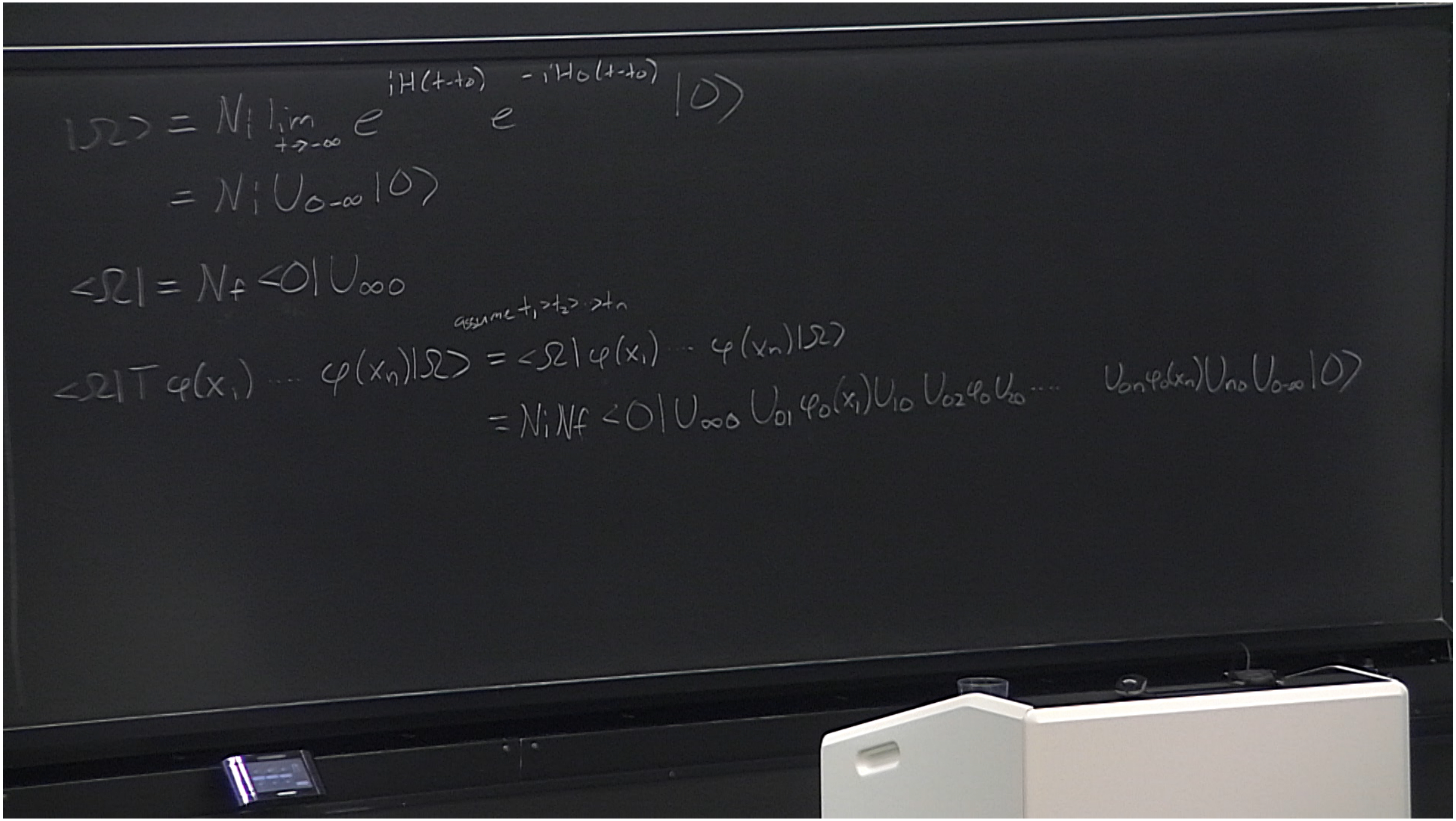
free theory
$$\begin{cases} a_{\vec{p}}(t_0)\,|0(t)\rangle = 0 \\ a_{\vec{p}}(t_0)\, e^{-iH_0(t-t_0)}|0\rangle = 0 \end{cases}$$

If we assume vacuum is unique $\rightarrow$
$$e^{iH(t-t_0)}|\Omega\rangle \propto e^{-iH_0(t-t_0)}|0\rangle$$

$|\Omega\rangle =$

$=$

$\langle\Omega| = N$



$|\Omega\rangle = N_i \lim_{t\to -\infty} e^{iH(t-t_0)} e^{-iH_0(t-t_0)} |0\rangle$
$= N_i U_{0-\infty} |0\rangle$
$\langle\Omega| = N_f \langle 0| U_{\infty 0}$
assume $t_1 > t_2 > \cdots > t_n$
$\langle\Omega| T\varphi(x_1) \cdots \varphi(x_n)|\Omega\rangle = \langle\Omega|\varphi(x_1) \cdots \varphi(x_n)|\Omega\rangle$
$= N_i N_f \langle 0| U_{\infty 0} U_{01} \varphi_0(x_1) U_{10} U_{02} \varphi_0 U_{20} \cdots U_{0n} \varphi_0(x_n) U_{n0} U_{0-\infty} |0\rangle$

$$\langle\Omega|T\,\varphi(x_1)\cdots\varphi(x_n)|\Omega\rangle = N_i N_f \langle 0|U_{\infty 1}\,\varphi_0(x_1)\,U_{12}\cdots U_{(n-1)n}\,\varphi_0(x_n)\,U_{n-\infty}|0\rangle$$

$$= N_i N_f \langle 0|T\,U_{\infty 1}\,\varphi_0(x_1)\,U_{12}\cdots\varphi_0(x_n)\,U_{n-\infty}|0\rangle$$

$$= N_i N_f \langle 0|T\,\varphi_0(x_1)\cdots\varphi_0(x_n)\,U_{\infty-\infty}|0\rangle$$

$$\langle\Omega|\Omega\rangle = 1$$

$$\langle\Omega|\Omega\rangle = N_i N_f \langle 0|U_{\infty-\infty}|0\rangle$$

$$N_i N_f = \frac{1}{\langle 0|U_{\infty-\infty}|0\rangle}$$

$$U_{32}U_{21} = U_{31}$$

$$\boxed{\langle\Omega|T\,\varphi(x_1)\cdots\varphi(x_n)|\Omega\rangle = \frac{\langle 0|T\,\varphi_0(x_1)\cdots\varphi_0(x_n)\exp\left[-i\int_{-\infty}^{\infty}dt'\,H_{1I}(t')\right]|0\rangle}{\langle 0|T\exp\left[-i\int_{-\infty}^{\infty}dt'\,H_{1I}(t')\right]|0\rangle}}$$

$$H_1(t) = \int d^3x\,\frac{\lambda}{4!}\varphi^4(\vec{x},t)$$

$$H_{1I}(t) = e^{iH_0(t-t_0)}\int d^3x\,\frac{\lambda}{4!}\varphi_0^4(\vec{x},t_0)\,e^{-iH_0(t-t_0)}$$

$$U_{32}U_{21} = U_{31}$$

$$\langle 0 | T \exp\left[-i\int_{-\infty}^{\infty} dt'\, H_{1I}(t')\right] |0\rangle$$

$$H_1(t) = \int d^3x\, \frac{\lambda}{4!}\varphi^4(\vec{x},t)$$

$$H_{1I}(t) = e^{iH_0(t-t_0)} \int d^3x\, \frac{\lambda}{4!}\varphi_0^4(\vec{x},t_0)\, e^{-iH_0(t-t_0)}$$

$$U_{\infty,-\infty} = T\exp\left[-i\int_{-\infty}^{\infty} dt\, H_{1I}(t)\right]$$

$$= T\exp\left[+i\int d^4x\, \mathcal{L}_{int}[\varphi_0]\right]$$

$$\boxed{\langle\Omega|T\varphi(x_1)\cdots\varphi(x_n)|\Omega\rangle = \frac{\langle 0|T\,\varphi_0(x_1)\cdots\varphi_0(x_n)\exp\left[i\int d^4x\,\mathcal{L}_{int}[\varphi_0]\right]|0\rangle}{\langle 0|T\exp\left[i\int d^4x\,\mathcal{L}_{int}[\varphi_0]\right]|0\rangle}}$$

$$\exp\left[i\int\mathcal{L}_i\right] = 1 + i\int\mathcal{L}_i + i\left(\int\mathcal{L}_i\right)^2 + \cdots$$

each term $\langle 0|T\varphi_0(x_1)\cdots\varphi_0(x_m)|0\rangle$

Wick's Theorem

include terms with $0, 1, 2, \ldots \left[\frac{m}{2}\right]$ contractions

$$T\,\varphi_0(x_1)\cdots\varphi_0(x_m) \equiv \,:\text{all possible contractions}:$$

normal ordering

$$\varphi_0(x_1)\,\varphi_0(x_2) = \varphi_0(x_1)\,\varphi_0(x_2) \equiv \langle 0|\,T\varphi_0(x_1)\,\varphi_0(x_2)\,|0\rangle = \Delta_F(x_1 - x_2)$$

$::$ eliminates all partially contracted terms in $\langle 0|\quad|0\rangle$

$$T\,\underbrace{\varphi_0(x_1)}_{\varphi_1}\,\underbrace{\varphi_0(x_2)}_{\varphi_2}\,\underbrace{\varphi_0(x_3)}_{\varphi_3} = \,:\varphi_1\varphi_2\varphi_3: + :\varphi_1\varphi_2\varphi_3: + :\varphi_1\varphi_2\varphi_3: + :\varphi_1\varphi_2\varphi_3:$$

Sketch of Proof: Define
$$\varphi_+(x) = \int \frac{d^3p}{(2\pi)^3\, 2E_{\vec p}}\, a^\dagger_{\vec p}\, e^{ip\cdot x}\Big|_{p^0=E_{\vec p}}$$
$$\varphi_-(x) = \int \frac{d^3p}{(2\pi)^3\, 2E_{\vec p}}\, a_{\vec p}\, e^{-ip\cdot x}\Big|_{p^0=E_{\vec p}}$$

Claim: $\Delta_F(x-y) = \Theta(x^0-y^0)\,[\varphi_-(x), \varphi_+(y)] + \Theta(y^0-x^0)\,[\varphi_-(y), \varphi_+(x)]$

$$T\,\varphi_0(x_1)\,\varphi_0(x_2) \overset{t_1>t_2}{=} \varphi_+(x_1)\,\varphi_+(x_2) + \varphi_+(x_1)\,\varphi_-(x_2) + \underbrace{\varphi_-(x_1)\,\varphi_+(x_2)}_{\text{not normal ordered}} + \varphi_-(x_1)\,\varphi_-(x_2)$$
$$= \,:\varphi_0(x_1)\,\varphi_0(x_2):\, + [\varphi_-(x_1), \varphi_+(x_2)]$$

$$T\,\varphi(x_1)\,\varphi(x_2) = \,:\varphi_0(x_1)\,\varphi_0(x_2):\, + \Delta_F(x_1-x_2)$$

Induction: Assume for n-1

$t_1 > t_2 \cdots > t_n$

$$T\varphi_0(x_1)\cdots\varphi_0(x_n) = \varphi_0(x_1) : \text{contractions involving } \varphi_0(x_2), \dots \varphi_0(x_n) :$$

$$= \big(\varphi_+(x_1) + \varphi_-(x_1)\big) : \text{contractions not involving } \varphi_1 :$$

can put inside : :

creates contractions

e.g. $\big(\varphi_+(x_1) + \varphi_-(x_1)\big)\big(:\varphi_2\,\varphi_3: + \Delta_F(x_2 - x_3)\big)$

Feynman Propagator

if $x^0 > y^0$ $\quad \langle 0|\varphi(x)\varphi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p}$

$$\Delta_F(x-y) = \langle 0|T\varphi_0(x)\varphi_0(y)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_{\vec p}} e^{-ip\cdot(x-y)}\,\Theta(x^0-y^0) + \underbrace{\frac{1}{2E_{\vec p}} e^{-ip\cdot(y-x)}\,\Theta(y^0-x^0)}_{\left.\frac{1}{2E_{\vec p}} e^{-ip(x-y)}\,\Theta(y^0-x^0)\right|_{p^0=-E_{\vec p}}}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \int_{C_F} \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-ip\cdot(x-y)}$$

Feynman Propagator

if $x^0 > y^0$ $\langle 0|\varphi(x)\varphi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_{\vec p}} \int \frac{d^3k}{(2\pi)^3 2E_{\vec k}} \langle 0$

$$\Delta_F(x-y) = \langle 0|T\varphi_0(x)\varphi_0(y)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_{\vec p}} e^{-ip\cdot(x-y)}\,\Theta(x^0-y^0) + \underbrace{\frac{1}{2E_{\vec p}} e^{-ip\cdot(y-x)}\,\Theta(y^0-x^0)}_{\frac{1}{2E_{\vec p}} e^{-ip(x-y)}\,\Theta(y^0-x^0)\big|_{p^0=-E_{\vec p}}}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \int_{C_F} \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-ip\cdot(x-y)} = \boxed{\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)} = \Delta_F(x-y)}$$

$O(t) = e^{iH(t-t_0)}\, O\, e^{-iH(t-t_0)}$

$O = \text{constant}$

$O_I(t) = e^{iH_0(t-t_0)}\, O\, e^{-iH_0(t-t_0)} \rightarrow O = e^{-iH_0(t-t_0)}\, O_I(t)\, e^{iH_0(t-t_0)}$

$$(\partial^2+m^2)\Delta_F(x-y) = \int_{C_F} \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2}(-p^2+m^2)\, e^{-ip\cdot(x-y)}$$

$$= \int_{C_F} \frac{d^4p}{(2\pi)^4}(-i)\, e^{-ip\cdot(x-y)}$$

$$= -i\,\delta^4(x-y)$$

$0)[\varphi_-(y),$

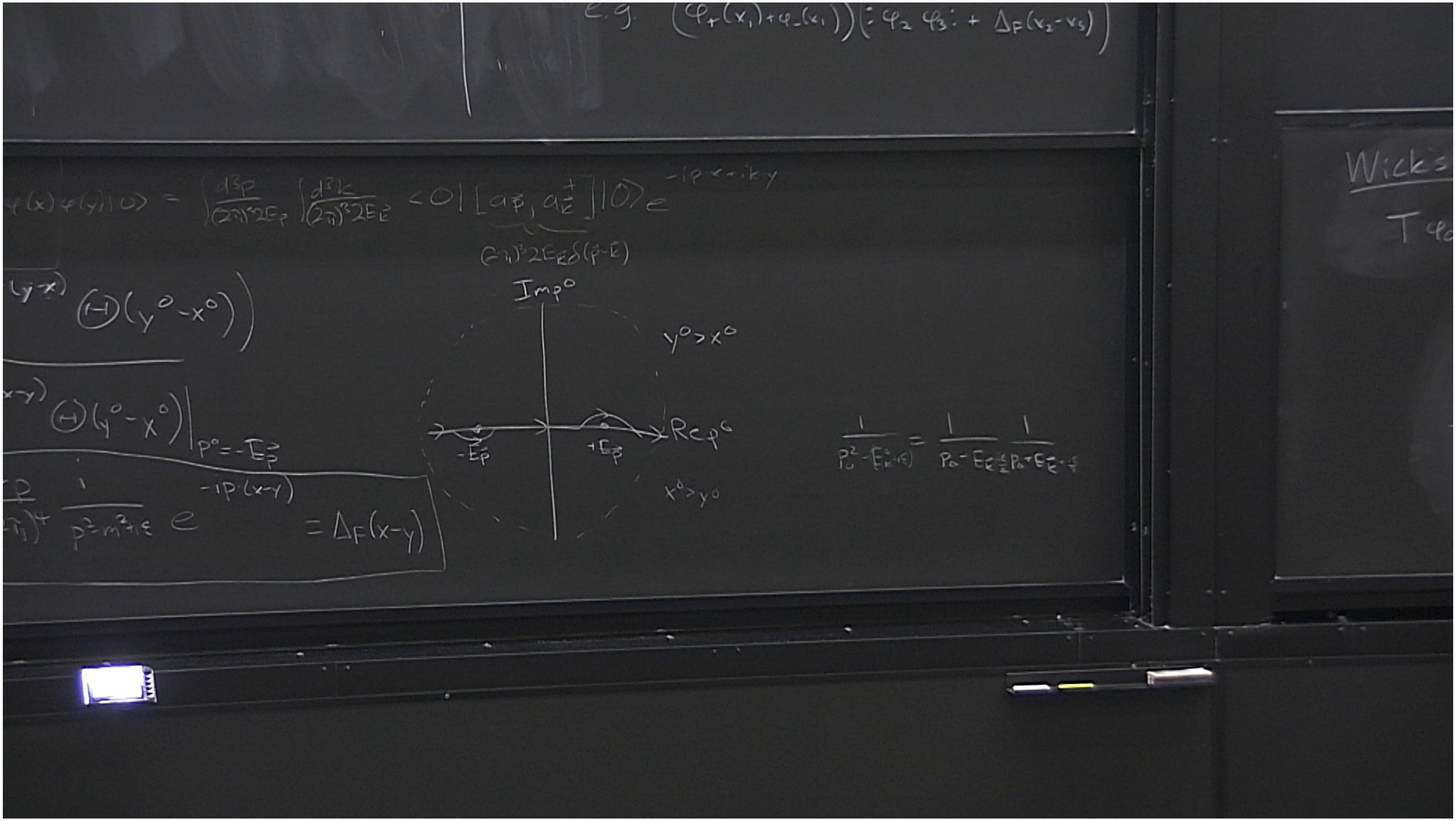
$(x_1)\,\varphi_+(x_2)$

normal-ordered

Feynman Propagator

if $x^0 > y^0$ $\langle 0|\varphi(x)\varphi(y)|0\rangle = \int \frac{d^3}{(2\pi)}$

$= \langle 0|T\varphi_0(x)\varphi_0(y)|0\rangle$


e.g. (φ+(x1)+φ-(x1))(:φ2 φ3: + ΔF(x2-x3))
Wick's
Im p0
y0 > x0
-Ep
+Ep
Re p0
x0 > y0
p0 = -Ep
= ΔF(x-y)