Title: General Relativity for Cosmology - Lecture 14

Speakers: Achim Kempf

Collection: General Relativity for Cosmology (Kempf)

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GR for Cosmology, Achim Kempf Lecture 12

Note Title

Plan: I The dynamics of matter & radiation in curved spacetime

II Energy-momentum tensor

III The dynamics of spacetime itself.

1. **Recall:** On a (pseudo)-Riemannian mfld, equations are well-defined only if defined independently of any chart.

⇒ Any eqn, including the eqns of motions

II Energy-momentum tensor

III The dynamics of spacetime itself.

1. **Recall:** On a (pseudo)-Riemannian mfld, equations are well-defined only if defined independently of any chart.

⇒ Any eqn, including the eqns of motions for matter fields must be eqns among tensors and their covariant derivatives.

⇒ Need a tensor field, Ψ, for each species of particle:

e^-, q, gluon, $\pi^{\substack{+\\-\\0}}$, photon, $W^\pm$, etc...

Notation:

$$\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$$

contravariant

covariant

species label

Note: any spinor equation can also be expressed as a (complicated) tensor equation (see e.g. Hawking & Ellis, p 59)

Question:

Notation:

$$\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$$

contravariant

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Note: any spinor equation can also be expressed as a (complicated) tensor equation (see e.g. Hawking & Ellis, p 59)

Question:

Could we have also an additional connection field $\tilde{\Gamma}^k_{ij}$?

Yes, we could: But, the difference field $Q^{k}{}_{ij} := \Gamma^{k}{}_{ij} - \tilde{\Gamma}^{k}{}_{ij}$ is actually a tensor field!

$$\Gamma^{r}{}_{ab} \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial^2 x^{j}}{\partial \bar{x}^{a} \partial \bar{x}^{b}} + \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \Gamma^{k}{}_{ij}$$

$$\tilde{\Gamma}^{r}{}_{ab} \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial^2 x^{j}}{\partial \bar{x}^{a} \partial \bar{x}^{b}} + \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \tilde{\Gamma}^{k}{}_{ij}$$

$$\Rightarrow \left(\Gamma^{r}{}_{ab} - \tilde{\Gamma}^{r}{}_{ab}\right) \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \left(\Gamma^{k}{}_{ij} - \tilde{\Gamma}^{k}{}_{ij}\right)$$

$$\Rightarrow \boxed{Q^{r}{}_{ab} \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \; Q^{r}{}_{ab}}$$

i.e. $Q^{r}{}_{ab}$ is a tensor due to having the correct transformation property according to the physicist's definition of a tensor.

$\Rightarrow$ Introducing an additional connection $\tilde{\Gamma}$ is same as introducing simply a new tensor field Q.

$$\Rightarrow \quad \left(\Gamma^{r}{}_{ab}-\tilde{\Gamma}^{r}{}_{ab}\right) \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{k}}\frac{\partial x^{i}}{\partial \bar{x}^{a}}\frac{\partial x^{j}}{\partial \bar{x}^{b}}\left(\Gamma^{k}{}_{ij}-\tilde{\Gamma}^{k}{}_{ij}\right)$$

$$\Rightarrow \quad \boxed{Q^{r}{}_{ab} \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{k}}\frac{\partial x^{i}}{\partial \bar{x}^{a}}\frac{\partial x^{j}}{\partial \bar{x}^{b}}\; Q^{r}{}_{ab}}$$

i.e. $Q^{r}{}_{ab}$ is a tensor due to having the correct transformation property according to the physicist's definition of a tensor.

⇒ Introducing an additional connection $\tilde{\Gamma}$ is same as introducing simply a new tensor field Q.

Remark: ⇒ "variations" $\delta\Gamma^{r}{}_{ab}$ will behave tensorially!

Eqns of motion of matter fields?

Action principle: (As in special relativity)

Any theory of matter fields can be defined by specifying the so-called Lagrangian function, L, namely a scalar function of the matter fields $\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}$ and their first covariant derivatives, and now also of the metric g:

we'll sometimes omit the indices

☐ Define the action functional:

$$S[\psi] := \int_B \underbrace{\overbrace{L(\psi)}^{\text{scalar}}\ \overbrace{\sqrt{g}\, d^4x}^{\Omega = \text{volume form}}}_{n\text{-form}} \quad \in \mathbb{R}$$

← some bounded
and closed 4-dim
region in M.

Thus, each physical field $\psi(x,t)$ (as a function of both space <u>and time</u>) is mapped into a number $S[\psi]$.

☐ Action principle (or postulate) of classical physics:

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Thus, each physical field $\psi(x,t)$ (as a function of both space <u>and time</u>) is mapped into a number $S[\psi]$.

☐ **Action principle (or postulate) of classical physics:**

In nature, physical fields ψ are such that $S[\psi]$ is extremal in the space of all fields ψ.

- **Thus:** The matter fields ψ obey:

$$\boxed{\frac{\delta S[\psi]}{\delta \psi} = 0} \qquad (*)$$

These will be the eqns of motion for the fields ψ.

- **Definition of (*) ?**

$p \in B \subset \mathcal{M}$

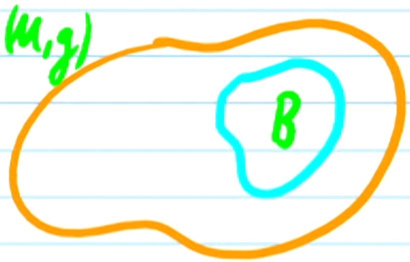
Def: A "variation $\delta \psi$" of the fields $\psi_{(i)}(p)$ in a region B is a one-parameter

$$\boxed{\frac{\delta S[\Psi]}{\delta \Psi} = 0} \qquad (*)$$

These will be the eqns of motion for the fields Ψ.

- **Definition of (*)?**

Def: A "variation $\delta\Psi$" of the fields $\Psi_{(i)}(\overset{\overset{p \in B \subset M}{\downarrow}}{p})$ in a region B is a one-parameter deformation, $\Psi_{(i)}(\lambda, p)$, with $\underset{\uparrow\ \text{deformation parameter}}{\lambda} \in \overbrace{(-\varepsilon, \varepsilon)}^{\text{some finite interval}}$,



so that

i.e. $\lambda=0$ is non-deformation

1.) $\psi_{(i)}(0,p) = \psi_{(i)}(p) \quad \forall p \in M$

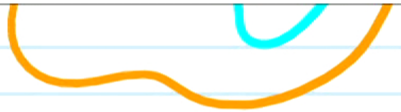
2.) $\psi_{(i)}(\lambda,p) = \psi_{(i)}(p) \quad \forall \lambda,$ if $p \in M-B$

i.e. no deformation at all outside region B.

Def: Then, we define:

$$\delta\psi_{(i)}(p) := \left.\frac{\partial \psi_{(i)}(\lambda,p)}{\partial \lambda}\right|_{\lambda=0}$$

Def: The action principle now reads:



2.) $\Psi_{(i)}(\lambda, p) = \Psi_{(i)}(p) \quad \forall \lambda,$ if $p \in M - B$

i.e. no deformation at all outside region B.

Def: Then, we define:

$$\delta \Psi_{(i)}(p) := \left. \frac{\partial \Psi_{(i)}(\lambda, p)}{\partial \lambda} \right|_{\lambda=0}$$

Def: The action principle now reads:

$$\boxed{0 = \left. \frac{\partial S[\Psi]}{\partial \lambda} \right|_{\lambda=0} \qquad \text{for all variations } \delta \Psi_{(i)} .}$$

Evaluate:

$$0 = \frac{\partial S}{\partial \lambda}\bigg|_{\lambda=0} = \sum_i \int_B \Bigg[\overbrace{\frac{\partial \mathcal{L}}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} \underbrace{\delta \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}_{\text{recall: } = \frac{d\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}{d\lambda}\big|_{\lambda=0}}}^{\text{Term I}}$$

$$+ \overbrace{\frac{\partial \mathcal{L}}{\underbrace{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}_{\substack{\text{by assumption,} \\ \mathcal{L} \text{ depends also on} \\ \text{the 1st cov. derivatives.}}}} \delta\left(\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)}^{\text{Term II}} \Bigg] \sqrt{g}\, d^4x$$

Term II:

We notice:

Recall: At origin of geodesic coordinate system, $\Gamma^k{}_{ij}=0$, i.e. $\psi_{;e}=\psi_{,e}$. But then $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial \lambda}$ commute. True in any coordinate system.

$$\delta\left(\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)=\left(\delta\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e}$$

$$\Rightarrow \text{Term II} = \sum_i \int_B \frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\left(\delta\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e}\sqrt{g}\,d^4x$$

$$= \sum_i \int_B \Bigg[\Bigg(\overbrace{\frac{\partial L}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\,\delta\psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{=:K^e}\Bigg)_{;e}$$

(use Leibniz rule to verify)

II We notice:

i.e. $\psi_{;e} = \psi_{,e}$. But then $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial \lambda}$ commute. True in any coordinate system.

$$\delta\left(\psi_{(i)}{}^{a\dots b}{}_{c\dots d;e}\right) = \left(\delta \psi_{(i)}{}^{a\dots b}{}_{c\dots d}\right)_{;e}$$

$$\Rightarrow \text{Term II} = \sum_i \int_B \frac{\partial L}{\partial \psi_{(i)}{}^{a\dots b}{}_{c\dots d;e}} \left(\delta \psi_{(i)}{}^{a\dots b}{}_{c\dots d}\right)_{;e} \sqrt{g}\, d^4x$$

$$= \sum_i \int_B \Bigg[\Bigg(\overbrace{\frac{\partial L}{\partial \psi_{(i)}{}^{a\dots b}{}_{c\dots d;e}}\, \delta \psi_{(i)}{}^{a\dots b}{}_{c\dots d}}^{=:K^e} \Bigg)_{;e}$$

(use Leibniz rule to verify)

$$- \left(\frac{\partial L}{\partial \psi_{(i)}{}^{a\dots b}{}_{c\dots d;e}}\right)_{;e} \delta \psi_{(i)}{}^{a\dots b}{}_{c\dots d} \Bigg] \sqrt{g}\, d^4x$$

One term is a "boundary term":

$$\sum_i \int_B K^e{}_{;e} \sqrt{g}\, d^4x$$

⇐

Exercise:

show that for all ξ^μ:

$$\xi^\mu{}_{;\mu}\, \Omega = \mathrm{div}_\Omega \xi$$

if $\Omega = \sqrt{g}\, dx^1 \wedge \ldots \wedge dx^n$

$$= \sum_i \int_B \mathrm{div}_\Omega K$$

Gauß' theorem ⇒

$$= \sum_i \int_{\partial B} \overset{\swarrow \text{ inner derivation}}{i_K} \Omega$$

$$\left(\begin{aligned} \text{Recall: } \mathrm{div}_\Omega K &= L_K \Omega \\ &= (i_K \circ d + d \circ i_K)\, \Omega \\ &= d \circ i_K \Omega \end{aligned} \right)$$

but: $K \propto \delta \psi$ and $\delta \psi(p) = 0$ if $p \in \partial B$

by property 2) of variations.

$$\sum_i \int_B$$

$$= \sum_i \int_B \mathrm{div}_\Omega K$$

show that for all ξ^μ:

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Gauß' theorem ⇒

$$= \sum_i \int_{\partial B} \underset{}{i_K}\,\Omega$$

inner derivation

(Recall: $\mathrm{div}_\Omega K = L_K \Omega$

$$= (i_K \circ d + d \circ i_K)\,\Omega$$

$$= d \circ i_K\, \Omega \;)$$

but: $K \propto \delta\psi$ and $\delta\psi(p) = 0$ if $p \in \partial B$ by property 2) of variations.

$$\Rightarrow \qquad = 0 \qquad !$$

Thus, term II simplifies and we obtain:

$$0 = \frac{\partial S}{\partial \lambda}\bigg|_{\lambda=0} = \sum_i \int_B \Bigg[\overbrace{\frac{\partial \mathcal{L}}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} \,\delta \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{\text{Term I}} - \overbrace{\left(\frac{\partial \mathcal{L}}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \right)_{;e} \delta \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{\text{Term II}} \Bigg] \sqrt{g}\, d^4x$$

Since must hold for all variations $\delta \psi$

$$\Rightarrow \quad \boxed{\begin{gathered} \frac{\partial \mathcal{L}}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} - \left(\frac{\partial \mathcal{L}}{\partial \psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \right)_{;e} = 0 \\ \text{"Euler-Lagrange equations"} \end{gathered}}$$

Example: A real-valued scalar field ψ ↖ real-valued

- Such ψ describe e.g.:
  - π^0 meson (quark+antiquark)
  - inflaton
- Lagrangian?
  - Choose geodesic cds at arb. point and appeal to equiv. principle.
  - Obtain from spec. relat. Lagrangian:

$$L = -\frac{1}{2}\left(\psi_{;a}\psi_{;b}g^{ab} + \frac{m^2}{\hbar^2}\psi^2\right)$$

- Euler-Lagrange equation: Klein-Gordon equation

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☐ Euler-Lagrange equation: Klein-Gordon equation

(Exercise: verify)

$$\boxed{\psi_{;ab}\,g^{ab} - \frac{m^2}{\hbar^2}\psi = 0}$$

Example: The electromagnetic fields

- Assume there are no charges (i.e. there are only EM waves)
- Define the "EM 4-potential" as a real-number-valued one-form A.
- Consider the field strength tensor F:

$$F := dA$$

- Recall that the E and B fields are

- Assume there are no charges (i.e. there are only EM waves)

- Define the "EM 4-potential" as a real-number-valued one-form A.

- Consider the field strength tensor F:

$$F := dA$$

- Recall that the E and B fields are components of the 2-form F. (up to a factor of 2)

- The Lagrangian (from equiv. principle):

$$L = -\frac{1}{16\pi} F_{ab} F_{cd} g^{ac} g^{bd}$$

(Exercise: write in terms of forms)

- Varying w. resp. to A, the E.L. equations read:

$$\boxed{F_{ab;c}\, g^{bc} = 0}$$

recall: this is $\delta F = 0$

"Maxwell eqns".

- It is also true that

$$F_{ab;c} + F_{ca;b} + F_{bc;a} = 0$$

but this is not an Euler Lagrange eqn. It

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- It is also true that

$$F_{ab;c} + F_{ca;b} + F_{bc;a} = 0$$

but this is not an Euler Lagrange eqn. It is simply: $\boxed{dF = 0}$ (which holds because $F = dA$ and $d^2 = 0$)

Example: A charged scalar field ψ, ← complex-valued

(such ψ describe, e.g., $\pi^{\pm}$ mesons)

together with electromagnetism.

Equiv. principle yields from spec. relativity:

Why ψ complex?

Mixed term is Lorentz force

If ψ was real, it would be absent:

$-ieA_a\psi^*\psi_{;b}g^{ab}$

$+ieA_b\psi^*_{,a}\psi g^{ab}$

$-ieA_a{}^{ab}(\psi^*_{,a}\psi-\psi_{,a}\psi^*)$

electric charge constant

$$L=-\frac{1}{2}\left(\psi^*_{;a}-ieA_a\psi^*\right)\left(\psi_{;b}+ieA_b\psi\right)g^{ab}$$

$$-\frac{1}{2}\frac{m^2}{\hbar^2}\psi^*\psi-\frac{1}{16\pi}F_{ab}F_{cd}g^{ac}g^{bd}$$

Equiv. principle yields from spec. relativity:

electric charge constant

$$L = -\frac{1}{2}\left(\psi^*_{;a} - ieA_a\psi^*\right)\left(\psi_{;b} + ieA_b\psi\right)g^{ab}$$

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Why ψ complex?

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If ψ was real, it would be absent:

$$-ieA_a\psi^*\psi_{;b}g^{ab}$$
$$+ieA_b\psi^*_{,a}\psi g^{ab}$$
$$= ieA_ag^{ab}(\psi^*_{,a}\psi - \psi_{,a}\psi^*)$$
$$= 0 \text{ if } \psi^* = \psi$$

☐ Vary w. resp. to ψ^* ⇒ E.L. eqn:

$$\underbrace{\psi_{;ab}\, g^{ab} - \frac{m^2}{\hbar^2}\psi}_{\text{Klein Gordon part}} + \underbrace{ieA_a g^{ab}\left(\psi_{;b} + ieA_b\psi\right) + ieA_{a;b}\, g^{ab}\psi}_{\psi \text{ is affected by } A} = 0$$

and varying w. resp. to ψ yields the compl. conj. equation.

☐ Vary w. resp. to A_a ⇒ E.L. eqn:

$$\frac{1}{4\pi}\underbrace{F_{ab;c}\, g^{bc}}\, - \underbrace{ie\psi\left(\psi^*_{;a} - ieA_a\psi^*\right) + ie\psi^*\left(\psi_{;a} + ieA_a\psi\right)} = 0$$

Klein Gordon part $\quad\quad$ Ψ is affected by A

and varying w. resp. to Ψ yields the compl. conj. equation.

- Vary w. resp. to A_a ⇒ E.L. eqn:

$$\underbrace{\frac{1}{4\pi} F_{ab;c}\, g^{bc}}_{\text{plain Maxwell part}} \underbrace{- ie\Psi\left(\Psi^*_{;a} - ieA_a\Psi^*\right) + ie\Psi^*\left(\Psi_{;a} + ieA_a\Psi\right)}_{A \text{ is affected by } \Psi, \Psi^*.} = 0$$

Dirac equation: (Brief treatment of basics only of Dirac spinors)

In special relativity: (with units such that $\hbar = 1$)

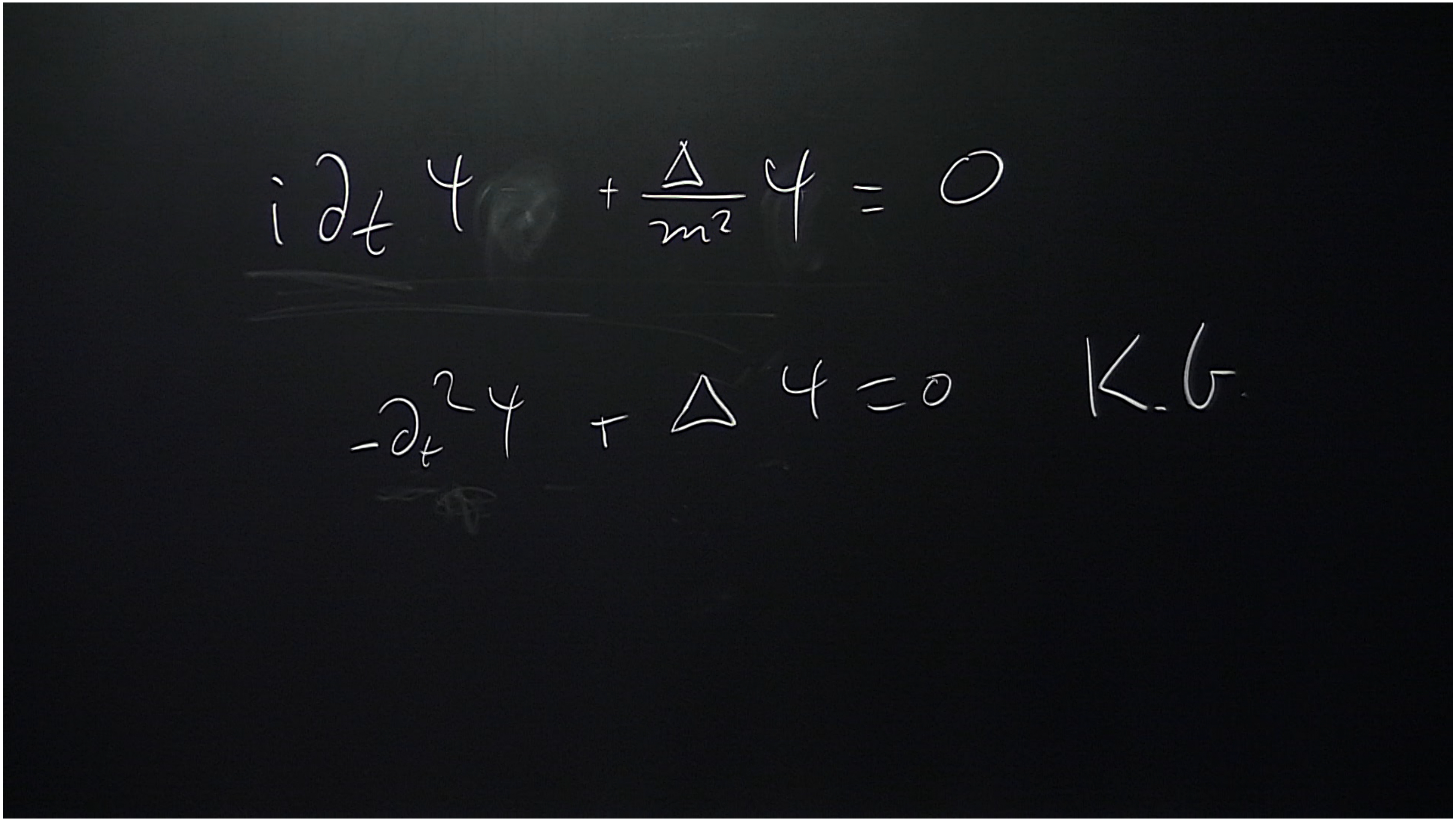
$$\boxed{\left(i\,\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0}$$ "Dirac equation" (D)

where $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$ is a "Spinor"

↑ describes spin $1/2$ particles such as electrons and quarks

and the four 4×4 matrices γ^μ obey:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\eta^{\overset{L}{\mu\nu}} \qquad (*)$$



$$i\partial_t \psi + \frac{\Delta}{2m^2}\psi = 0$$

$$-\partial_t^2 \psi + \Delta \psi = 0$$

K.G.

$$\boxed{\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0}$$ "Dirac equation" (D)

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ is a "Spinor"

↑ describes spin ½ particles such as electrons and quarks

and the four 4×4 matrices γ^μ obey:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \qquad (*)$$

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}^{\mu\nu}$$

□ **Why (*)?** Equation (*) is specifically chosen so that each component of ψ obeys the Klein Gordon equation. Indeed:

$$(D) \Rightarrow \left(-i\gamma^\mu\partial_\mu - m\right)\left(i\gamma^\nu\partial_\nu - m\right)\psi = 0$$

$$\Rightarrow \left(+\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + i\gamma^\mu\partial_\mu m - im\gamma^\nu\partial_\nu + m^2\right)\psi = 0$$

$$\Rightarrow \left(\underbrace{\gamma^\mu\gamma^\nu}_{\text{antisymmetric part not needed, it would drop out.}}\overbrace{\partial_\mu\partial_\nu}^{\text{symmetric under } \mu\leftrightarrow\nu} + m^2\right)\psi = 0$$

$$\Rightarrow \left(\frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\partial_\mu\partial_\nu + m^2\right)\psi = 0$$

$$\overset{(*)}{\Rightarrow} \quad 1 \; \left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\psi = 0$$

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$$\overset{(*)}{\Rightarrow} \quad 1\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\psi = 0$$

which is the Klein Gordon equation in flat space.

In general relativity:

- By choosing an orthonormal tetrad, $\{\theta^i\}$, we achieve

$$g^{\mu\nu} = \eta^{\mu\nu} \quad \forall p \in M$$

i.e. one set of matrices γ^μ obeying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ suffices.

- This motivates:

$$(i\gamma^\mu \nabla_\mu - m)\psi = 0$$

In general relativity:

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- This motivates:

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- But what is the covariant derivative of a spinor?

$$\nabla \psi = ?$$

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- This motivates:

$$(i\gamma^\mu \nabla_{e_\mu} - m)\psi = 0$$

- But what is the covariant derivative of a spinor?

$$\nabla_{e_\mu}\psi = ?$$

Recall: The covariant derivative of a vector yields the infinitesimal Lorentz transformation by which the vector rotates under infinitesimal parallel transport.

Idea: The covariant derivative of a <u>spinor</u> should yield the rotation of the spinor by the same infinitesimal Lorentz transformation.

Recall: Infinitesimal parallel transport of a vector e_ρ in direction e_μ:

$$e_\rho \to e_\rho + \nabla_{e_\mu} e_\rho = e_\rho + \omega^\sigma_\rho(e_\mu)\, e_\sigma$$

Recall: the curvature 1-form takes values that are infinitesimal Lorentz transformations.

Recall intuition why parallel transport yields Lorentz transformation:
Parallel transport preserves the

This is an infinitsimal Lorentz transformation $\Lambda_\rho{}^\sigma$:

$$e_\rho \to \Lambda_\rho{}^\sigma e_\sigma \quad \text{with} \quad \Lambda^\sigma_\rho = \delta^\sigma_\rho + \omega^\sigma_\rho(e_\mu)$$

Idea: The covariant derivative of a <u>spinor</u> should yield the rotation of the spinor by the same infinitesimal Lorentz transformation.

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This is an infinitesimal Lorentz transformation Λ_ρ^σ:

$$e_\rho \to \Lambda_\rho^\sigma e_\sigma \quad \text{with} \quad \Lambda^\sigma_\rho = \delta^\sigma_\rho + \omega^\sigma_\rho(e_\mu)$$

because ω^σ_ρ obeys: $\omega_{\sigma\rho} = -\omega_{\rho\sigma}$. (Which is the defining equation for infinitesimal Lorentz transformations)

(Recall intuition why parallel transport yields Lorentz transformation: Parallel transport preserves the lengths of vectors, i.e. they can at most "rotate" and in 3+1 dim. this is Lorentz transformations.)

Now that we know the inf. Lorentz transf. for any inf. parallel transport:

⤳ **Strategy:** Apply the same inf. Lorentz transformation on spinors for their parallel transport.

To this end: Recall from Special Relativity how an infinitesimal Lorentz transformation acts on a spinor:

- ☐ Assume $\{s_i\}_{i=1}^{4}$ are ON basis in Spinor space, i.e.

$$\psi = \psi^{i}(x)\, s_i$$

these are Spinor indices: $i = 1,2,3,4$

- ☐ How do the s_i transform under Lorentz transformations?

$\rightsquigarrow$ **Strategy:** Apply the same inf. Lorentz transformation on spinors for their parallel transport.

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- How do the s_i transform under Lorentz transformations?
  I.e., what is $\nabla_{e_i} s_j = ?$ (In analogy to $\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a)e_\nu$)

i.e.

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- How do the s_i transform under Lorentz transformations?
  I.e., what is $\nabla_{e_i} s_j = ?$ (In analogy to $\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a) e_\nu$)

- From special relativity it is known that under infinitesimal Lorentz transformations,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

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vectors transform as

$$e_\mu \rightarrow e_\mu + \omega_\mu{}^\nu e_\nu$$

and the Dirac spinors transform as:

$$s_i \rightarrow s_i \underbrace{- \frac{1}{4} \omega_\mu{}^\nu [\gamma^\mu, \gamma_\nu] s_i}$$

Where does $[\gamma^\mu, \gamma^\nu]$ come from?

Recall that e.g. translations in space are generated by momentum operators, $e^{-i\vec{\alpha}\vec{p}} f(x) e^{i\vec{\alpha}\vec{p}} = f(x+\vec{\alpha})$, if they obey the commutation relations $[x_i, p_j] = i\delta_{ij}$.

Similarly, Lorentz transformations are generated by operators $M^{\mu\nu}$:

$$e^{-i\omega_{\mu\nu}M^{\mu\nu}} f \; e^{i\overset{\text{antisym.}}{\omega_{\mu\nu}} \overset{\text{antisym.}}{M^{\mu\nu}}} = \Lambda(f)$$

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if these $M^{\mu\nu}$ obey certain commutation relations. In spinor space, the unique objects that obey these commutation relations are the $M^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$.

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and the Dirac spinors transform as:

$$s_i \to s_i \underbrace{-\frac{1}{4}\omega_\mu{}^\nu [\gamma^\mu, \gamma_\nu] s_i}$$

⇒ under infinitesimal Lorentz transf. the spinor "rotates" by this amount.

$$e^{a\frac{d}{dx}} f(x) = f(x+a)$$

$$x_1 \partial_{x_1} - \partial_{x_1} x_1$$

$$e^{\alpha L_z} f(\vec{x}) = f(R\vec{x})$$

$\uparrow$ rot. by α
about z-axis

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Apply to GR:

If a vector e_μ is infinitesimally parallel transported in the direction of e_a then it obtains an infinitesimal "rotation", namely the infinitesimal Lorentz transformation

$$\omega^\mu{}_\nu(e_a)$$

which is the value of the connection 1-form, i.e.:

$$e_\mu \to e_\mu + \overbrace{\omega_\mu^\nu(e_a)}^{\text{local value of the connection form}} e_\nu$$

→ From this one can immediately read off again the covariant derivative for vectors:

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$$\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a)\, e_\nu$$

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$$s_i \to s_i - \frac{1}{4}\overbrace{\omega(e_a)_\mu{}^\nu}^{\substack{\text{local infinitesimal Lorentz transformation,}\\ \text{i.e., local value of the connection 1-form.}}} [\gamma^\mu, \gamma_\nu]\, s_i$$

☐ Since, under infinitesimal parallel transport:

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□ Since, under infinitesimal parallel transport:

$$s_i \to s_i + \underbrace{\nabla_{e_a}}_{\text{to be determined}} s_i$$

⇒ The covariant derivative of the basis's vectors $\{s_i\}$ of Dirac spinors is:

$$\nabla_{e_a} s_i = -\frac{1}{4}\,\omega_\mu{}^\nu(e_a)\,[\gamma^\mu,\gamma_\nu]\,s_i$$

⇒ For general Dirac spinors $\psi(x) = \psi^i(x)\,s_i$ the Leibniz rule for ∇ yields:

$$\nabla_{e_a}\psi = \nabla_{e_a}\big(\overset{\text{scalar coefficient functions}}{\psi^i(x)}\,s_i\big) = \big(\nabla_{e_a}\psi^i(x)\big)\,s_i + \psi^i(x)\,\nabla_{e_a} s_i$$

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i.e.:

$$\boxed{\nabla_{e_a}\psi = e_a(\psi) - \frac{1}{4}\omega(e_a)^{\nu}_{\mu}\,[\gamma^{\mu},\gamma_{\nu}]\,\psi}$$

function

$$e_a(\psi) = s_i\, e_a(\psi^i)$$

vector field

Dirac equation:

The general relativistic Dirac equation

$$\boxed{\left(i\gamma^\mu \nabla_{e_\mu} - m\right)\psi = 0}$$

now takes this explicit form:

$$i\gamma^\mu \underbrace{e_\mu(\psi)}_{\uparrow} - i\frac{1}{4}\,\omega(e_\mu)^\nu{}_\rho\,\gamma^\mu[\gamma^\rho,\gamma_\nu]\psi - m\psi = 0$$

in a chart, this becomes a directional derivative of ψ.

Remark: The relationship between the Dirac operator $D = i\gamma^\mu\nabla_{e_\mu}$ and the Laplace or d'Alembert operator $\Box$ also becomes:

$$D = d + \delta .$$

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To this end, one re-interprets the Grassmann algebra of differential forms as a so-called Clifford algebra.