

Title: General Relativity for Cosmology - Lecture 10

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Collection: General Relativity for Cosmology (Kempf)

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# GR for Cosmology, Achim Kempf

## Lecture 10

Note Title

Recall:  $\square$  The curvature map,  $R$ , is defined through:

$$R: \xi_1, \xi_2, \xi_3 \rightarrow R(\xi_1, \xi_2)\xi_3 = (\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1, \xi_2]})\xi_3$$

So, " $R$ " can stand for the tensor, the map and this  $R$ !

$\square$  1st Bianchi Identity:

$$\sum R(\xi, \eta)v = \sum (\nabla_{\xi}(\nabla_{\eta}v) - \nabla_{\eta}(\nabla_{\xi}v))$$

Recall:

□ The curvature map,  $R$ , is defined through:

$$R: \xi_1, \xi_2, \xi_3 \rightarrow R(\xi_1, \xi_2)\xi_3 = \underbrace{(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1, \xi_2]})}_{R(\xi_1, \xi_2)}\xi_3$$

So, "R" can stand for the tensor, the map and this R!

□ 1st Bianchi Identity:

$$\sum_{\text{cyclic}} R(\xi, \eta)v = \sum_{\text{cyclic}} \left( \nabla_{\xi}(\nabla_{\eta}v) + \nabla_{\eta}(\nabla_{\xi}v) \right)$$

□ 2nd Bianchi Identity:

$$\sum_{\text{cyclic}} \left( (\nabla_{\xi}R)(\eta, v) + R(\nabla_{\xi}\eta, v) \right) = 0$$

In a chart? (Assuming no torsion, and using  $\frac{\partial}{\partial x^i}$ ,  $dx^i$  bases)

1st Bianchi:  $\sum_{(jke)} R^i{}_{jke} = 0$   
↑ cyclic sum

2nd Bianchi:  $\sum_{(k\ell m)} R^i{}_{jkl;m} = 0$   
↑ cyclic sum

Other useful properties:

(Note: This antisymmetry will be useful because it allows one to view  $R$  as a 2-form, which is (1,1) tensor-valued)

□  $R^i{}_{jke} = -R^i{}_{jek}$

□  $R_{ijke} = -R_{jike}$

□  $D \dots = D \dots$

$\langle R(\xi, \eta)v, \zeta \rangle = \langle R(\xi, \eta)\zeta, v \rangle$   
/ D (e ... ) - / D ( ... )

In a chart? (Assuming no torsion, and using  $\frac{\partial}{\partial x^i}$ ,  $dx^i$  bases)

1st Bianchi:  $\sum_{(jke)} R^i{}_{jke} = 0$   
↑ cyclic sum

2nd Bianchi:  $\sum_{(k\ell m)} R^i{}_{jkl;m} = 0$   
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Other useful properties:

(Note: This antisymmetry will be useful because it allows one to view  $R$  as a 2 form, which is  $(1,1)$  tensor-valued)

□  $R^i{}_{jke} = -R^i{}_{jek}$

□  $R_{ijke} = -R_{jike}$

□  $R_{ijke} = R_{k\ell ij}$

←  $\left( \begin{aligned} \langle R(\xi, \eta)v, S \rangle &= \langle R(\xi, \eta)S, v \rangle \\ \langle R(\xi, \eta)v, S \rangle &= -\langle R(v, S)\xi, \eta \rangle \end{aligned} \right)$

## Contractions of R:

The Ricci Tensor:

$$R_{je} := R^i_{jil}$$

⇒ clearly:  $R_{je} dx^i dx^l \in T_p(M)_2$

The Curvature Scalar:

$$R := g^{je} R_{je}$$

Then, 2nd Bianchi identity implies:

$$\left( R_i{}^k - \frac{1}{2} \delta_i^k R \right)_{;k} = 0$$

⇒ The so-called "Einstein tensor"  $G_i{}^k := R_i{}^k - \frac{1}{2} \delta_i^k R$  obeys:

$$G_i{}^k{}_{;k} = 0$$

(this property was crucial  
guidance for Einstein, as  
we will see)

Recall strategy:

□ Specified  $g \Rightarrow$  specified distances in  $M$   
 $\Rightarrow$  implicitly specified "shape" of  $M$

Then, alternatively:

□ Specified  $\nabla \Rightarrow$  specified parallel transport in  $M$

$\Rightarrow$  specified "shape" of  $M$ , namely:

$\nabla$  specifies Torsion  $T$  and Curvature  $R$ .

Now assume a manifold is specified by giving a metric  $g$ .

There ought to exist a  $\nabla$  which describes the same manifold.

How does  $g$  determine  $\nabla$ ?

Idea: The parallel transport of vectors  $\eta, v$  must be such that their inner product (i.e. their lengths and relative angles) stays constant:

Consider any path  $\gamma$  and any two vector fields  $\eta, v$  that are parallel transported along  $\gamma$ , i.e., for which:

(i.e., autoparallel to  $\gamma$ )

$$\nabla_{\dot{\gamma}} \eta(\gamma(t)) = 0, \quad \nabla_{\dot{\gamma}} v(\gamma(t)) = 0 \quad \text{for all } t.$$

Then, require:  $\frac{d}{dt} (g(\gamma(t))_{bc} \eta^b(\gamma(t)) v^c(\gamma(t))) = 0$

$$\text{i.e.: } 0 = \dot{\gamma}^a (g_{bc} \eta^b v^c)_{;a} = \dot{\gamma}^a (g_{bc;a} \eta^b v^c + g_{bc} \eta^b_{;a} v^c + g_{bc} \eta^b v^c_{;a})$$

$\nabla_{\dot{\gamma}} \langle g, \eta \otimes v \rangle$   
 by  $\nabla$  obeying Leibniz rule  
 because  $\nabla_{\dot{\gamma}} \eta = 0$   
 because  $\nabla_{\dot{\gamma}} v = 0$

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$\nabla_{\dot{\gamma}} \langle g, \eta \otimes v \rangle$   
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i.e.:  $0 = \dot{\gamma}^a (g_{bc} \eta^b v^c)_{;a} = \dot{\gamma}^a (g_{bc;a} \eta^b v^c + g_{bc} \eta^b_{;a} v^c + g_{bc} \eta^b v^c_{;a})$

$\Rightarrow$   $0 = g_{bc;a} \dot{\gamma}^a \eta^b v^c$  for all arbitrary  $\dot{\gamma}, \eta, v$ !

$\Rightarrow$  Compatibility of  $\nabla$  with  $g$  means:

$$\nabla_{\xi} g = 0 \quad \text{for all } \xi$$

$$\text{i.e.: } 0 = \xi^a (g_{bc} \eta^b v^c)_{;a} = \xi^a (g_{bc;a} \eta^b v^c + g_{bc} \eta^b_{;a} v^c + g_{bc} \eta^b v^c_{;a})$$

$\Rightarrow$   $0 = g_{bc;a} \xi^a \eta^b v^c$  for all arbitrary  $\xi, \eta, v$ !

$\Rightarrow$  Compatibility of  $\nabla$  with  $g$  means:

$$\nabla_{\xi} g = 0 \quad \text{for all } \xi$$

Is there a  $\nabla$  for each choice of  $g$ ? Indeed:

Fund. theorem of (pseudo) Riemannian geometry:

For each (pseudo) Riemannian manifold  $(M, g)$  there

$\Rightarrow$   $0 = g_{bcja} \xi^a \eta^b v^c$  for all arbitrary  $\xi, \eta, v$ !

$\Rightarrow$  Compatibility of  $\nabla$  with  $g$  means:

$$\nabla_{\xi} g = 0 \quad \text{for all } \xi$$

Is there a  $\nabla$  for each choice of  $g$ ? Indeed:

Fund. theorem of (pseudo) Riemannian geometry:

For each (pseudo) Riemannian manifold  $(M, g)$  there exists a unique  $\nabla$  that is torsionless and compatible with  $g$ , i.e., which obeys  $\nabla g = 0$ , the Levi-Civita connection.

More generally:  $\forall (M, g)$  and a tensor field  $\mathcal{T}$  with  $\mathcal{T}_{ij}^k = -\mathcal{T}_{ji}^k$  there is a metric-preserving  $\nabla$  whose torsion is  $\mathcal{T}$ .

In a chart: How to obtain the Levi-Civita  $\nabla$  from  $g$ ?

$$\nabla g = 0 \text{ means } g_{\mu\nu,\alpha} - g_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} - g_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} = 0 \quad \text{I}$$

$$\text{i.e. } g_{\alpha\mu,\nu} - g_{\alpha\beta}\Gamma^{\beta}_{\mu\nu} - g_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} = 0 \quad \text{II}$$

$$\text{and } g_{\nu\alpha,\mu} - g_{\nu\beta}\Gamma^{\beta}_{\alpha\mu} - g_{\beta\alpha}\Gamma^{\beta}_{\nu\mu} = 0 \quad \text{III}$$

$$\text{take: } \frac{1}{2}(-\text{I} + \text{II} + \text{III})$$

$$\Rightarrow \frac{1}{2}(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}) = g_{\alpha\beta}\Gamma^{\beta}_{\nu\mu}$$

Thus:  $\Gamma^{\beta}_{\nu\mu} = \frac{1}{2}g^{\alpha\beta}(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha})$

↑ "Levi-Civita" connection or also called "Riemannian" connection.

## Upgrade the math:

□ Make use of arbitrary bases  $e_i, \theta^i$  in (co-) tangent spaces: frames

□ Allow forms to be tensor-valued: obtain, e.g., torsion and curvature forms. Also: connection forms.

⇒ We will obtain powerful, simple equations that relate  $\nabla, g, R, T$ . (Even the Bianchi identities will look simple)

Now: Assume again that  $\nabla$  and  $g$  are still unrelated and  $T \neq 0$ .  
(possibly)

## "Moving frames":

Def: A "moving frame" is a set,  $\{e_i\}_{i=1}^n$ , of contravariant vector fields  $e_i$  which, together, at each point  $p \in M$  form a basis of  $T_p(M)$ .

Def: We denote the dual basis  $\{\theta^i\}_{i=1}^n$ .

$$\text{It obeys: } \theta^i(e_j) = \delta^i_j.$$

Def: For  $n=4$  it may be called **vierbein** or **tetrad**.  
(in arb. dimensions: "vielbein" = many legs)  
german: 4 legs.

Notice: Each co-vector  $\theta^i(x)$  is a 1-form, and  $d\theta^i$  is a 2-form!

Def: Collect them in a "Frame":  $\theta^i \otimes e_i$ , i.e. a (1,0)-tensor valued 1-form

Remark: If we choose e.g.  $\theta^i(x) := dx^i$ , then  $d\theta^i(x) = 0$ .

Remark: A general choice for the  $\theta^i(x)$  can always be written in the form:

$$\theta^i(x) = \lambda(x)^i_j dx^j$$

↑ scalar coefficient functions

Def: We denote the expansion coefficients by functions  $C^i_{jk}$ :

Exercise:

Express the  $C^i_{jk}$  in terms of the  $\lambda^i_j$ .

convention

$$d\theta^i = -\frac{1}{2} C^i_{jk} \theta^j \wedge \theta^k \quad \text{with} \quad C^i_{jk} = -C^i_{kj}$$

coefficient functions depend on choice of frame      basis for space of all 2-forms      the sym. part would drop out

## Coefficients:

▣ Torsion:  $T^i_{\kappa\epsilon} := \langle \theta^i, T(e_\kappa, e_\epsilon) \rangle$

▣ Curvature:  $R^i_{j\kappa\epsilon} := \langle \theta^i, R(e_\kappa, e_\epsilon)e_j \rangle$

▣ Metric:  $g_{i\kappa} := g(e_i, e_\kappa) = \langle e_i, e_\kappa \rangle$

▣ Christoffel:  $\Gamma^i_{\kappa j} e_i := \nabla_{e_\kappa} e_j$

Consider arbitrary change of frame: (has nothing to do with a change of chart!)

▣ assume  $\bar{\theta}^i(x) = A^i_j(x) \theta^j(x)$

▣ then:  $\bar{e}_i(x) = (A^{-1})^j_i(x) e_j(x)$

↖ (because we chose bases that are dual:  $\bar{\theta}^i(\bar{e}_j) = \delta^i_j$ )

Another step towards more abstract formulation:

## Tensor-valued p-forms:

Def: A  $(r,s)$ -tensor-valued p-form  $\phi$  is an anti-symmetric p-multilinear mapping at each  $q \in M$ :

$$\phi : \underbrace{T_q(M)^r \times \dots \times T_q(M)^r}_p \rightarrow T_q(M)^s$$

Def: The p-forms  $\phi_{j_1, \dots, j_p}^{i_1, \dots, i_p} := \phi(\theta^{i_1}, \dots, \theta^{i_p}, e_{j_1}, \dots, e_{j_p})$  are called the component p-forms relative to the basis  $\{e_i\}_{i=1}^n$ .

## Special cases:

- $(r,s)$  tensors are  $(r,s)$  tensor-valued 0-forms.
- p-forms are  $(0,0)$  tensor-valued forms.

## Torsion 2-form:

□ We recall that  $J(\xi, \eta) = -J(\eta, \xi) \Rightarrow$  can define the torsion's  $(1,0)$  tensor-valued 2-form

through its action on 2 vector fields  $\xi, \eta$ :

"torsion 2-form"  $\rightarrow$   $\Theta^i(\xi, \eta) e_i := J(\xi, \eta)$

the 2 form  $\Theta^i$   
fed 2 vectors to  
yield a vector

□ Given a frame:

$$\Theta^i = \frac{1}{2} J^i_{kl} \theta^k \wedge \theta^l$$

using their antisymmetry

## Curvature 2-form:

recall that  
in canonical  
basis:

$$R^i_{jkl} = -R^i_{jlk}$$

□ We recall that also  $R(\xi, \eta) = -R(\eta, \xi)$

⇒ can define curvature's (1,1) tensor-valued 2-form:

"curvature 2-form" →  $\underbrace{\Omega^i_j(\xi, \eta)}_{\text{tangent vector}} e_i := R(\xi, \eta) \underbrace{e_j}_{\text{tangent vector}}$

*numbers*

Recall:  $R: \xi \eta e_i \rightarrow \nabla_\xi \nabla_\eta e_i - \nabla_\eta \nabla_\xi e_i - \nabla_{[\xi, \eta]} e_i$

□ Given a frame  $\{\theta^i\}_{i=1}^n$ :

$$\Omega^i_j = \frac{1}{2} R^i_{jke} \theta^k \wedge \theta^e$$

## The connection as a form?

□ Nontrivial because:

1. Christoffels  $\Gamma^i_{\kappa j} e_i := \nabla_{e_\kappa} e_j$   
are not tensors to start with!

2.  $\Gamma^i_{\kappa j}$  is not antisym. in any indices,  
so can't be a 2-form (but can be 1-form):

□ Define the connection 1-forms  $\omega^i_j$ :  $\omega^i_j := \Gamma^i_{\kappa j} \theta^\kappa$

Thus:

$$\underbrace{\nabla_\xi}_{\text{vector}} e_j = \underbrace{\omega^i_j(\xi)}_{\text{vector}} e_i$$

scalars

(because  $\nabla_{\xi^\kappa e_\kappa} e_j = \xi^\kappa \nabla_{e_\kappa} e_j$ )

□ Proposition: cov. deriv. for covectors reads

$$\nabla_{\xi} \theta^i = -\omega^i_{\ j}(\xi) \theta^j$$

Proof:  $0 = \nabla_{\xi} \langle \theta^i, e_j \rangle = \langle \nabla_{\xi} \theta^i, e_j \rangle + \langle \theta^i, \nabla_{\xi} e_j \rangle$

$= \langle \nabla_{\xi} \theta^i, e_j \rangle + \langle \theta^i, \omega^k_{\ j}(\xi) e_k \rangle \quad (*)$

$= \omega^i_{\ j}(\xi)$  because  $\langle \theta^i, e_k \rangle = \delta^i_k$

$\Rightarrow$  indeed:

$$\nabla_{\xi} \theta^i = -\omega^i_{\ j}(\xi) \theta^j$$

contrast with  $\langle \cdot, e_j \rangle$   
to verify that this is Eq. (\*)



Connection 1-forms are non-tensorial:

Proposition: Under change of frame  $\bar{\theta}^i(x) = A^i_j(x) \theta^j(x)$   
the transformation is:

$$\bar{\omega}^a_b = \underbrace{A^a_i}_{1\text{-form}} \underbrace{\omega^i_j}_{1\text{-form}} \underbrace{A^{-1j}_b}_{\text{matrix inverse}} - \underbrace{(dA)^a_i}_{1\text{-form}} \underbrace{(A^{-1})^i_b}_{\text{functions}}$$

Proof:  $-\bar{\omega}(\xi)^a_b \bar{\theta}^b = \nabla_\xi \bar{\theta}^a = \nabla_\xi (A^a_b \theta^b) \stackrel{\text{Leibniz rule}}{=} (dA^a_b(\xi)) \theta^b + A^a_b \nabla_\xi \theta^b$

$$= dA^a_b(\xi) \theta^b - A^a_b \omega(\xi)^b_c \theta^c$$

$$= dA^a_b(\xi) A^{-1c}_b \bar{\theta}^c - A^a_b \omega(\xi)^b_c A^{-1c}_d \bar{\theta}^d$$

true for all  $\bar{\theta} \Rightarrow$  proposition above. ✓

# The "absolute exterior differential" $D$ :

(It generalizes both  $\nabla$  and  $d$ )

□ Proposition: (proof, see e.g. Straumann: check tensorial behaviour under frame change)

For every  $(r,s)$  tensor-valued  $p$ -form  $\phi$  there exists a unique

$(r,s)$  tensor-valued  $(p+1)$  form  $D\phi$  whose components relative to  $\{\theta^i\}$  are:

$$\begin{aligned}
 (D\phi)_{j_1 \dots j_s}^{i_1 \dots i_r} &= \underbrace{d\phi_{j_1 \dots j_s}^{i_1 \dots i_r}}_{p+1 \text{ form}} + \underbrace{\omega^{\ell i_1}}_{1 \text{ form}} \wedge \underbrace{\phi_{j_1 \dots j_s}^{\ell i_2 \dots i_r}}_{p \text{ form}} + \dots \\
 &\quad - \omega^{\ell j_1} \wedge \phi_{\ell i_2 \dots i_r}^{i_1 \dots i_s} - \dots
 \end{aligned}
 \tag{*}$$

▣ Proposition:  $D$  is an anti-derivation: <sup>degree of  $\phi$</sup>

$$D(\phi \wedge \psi) = D\phi \wedge \psi + (-1)^p \phi \wedge D\psi$$

▣ Special cases:

- An ordinary  $p$ -form is  $(0,p)$  tensor-valued.

In this case, clearly:

$$D = d$$

- An ordinary tensor field is a tensor-valued 0-form. In this case:

$$D = \nabla$$

Exercise: Verify

Hint: Choose frame  $\theta^i = dx^i$ , use  $\omega^i_j = \Gamma^i_{\kappa j} \theta^\kappa$ ,

then show (\*) implies indeed:

$$\phi_{i_1 \dots i_p j_1 \dots j_k}^{i_1 \dots i_p} = \phi_{i_1 \dots i_p j_1 \dots j_k}^{i_1 \dots i_p} + \Gamma^i_{\kappa j} \phi_{i_1 \dots i_p j_1 \dots j_k}^{i_1 \dots i_p} + \dots - \Gamma^e_{\kappa j} \phi_{i_1 \dots i_p j_1 \dots j_k}^{i_1 \dots i_p} - \dots$$

# How are $\omega, g, \Theta, \Omega$ related now?

Proposition: (Exercise: check)

An affine connection  $\nabla$  is metric, if and only if  $Dg = 0$ , i.e., iff:

$$\underbrace{dg_{ik} - \omega_{ik} - \omega_{ki}}_{(0,2)\text{ tensor-valued 1-form}} = 0$$

They express torsion and curvature in terms of the connection

## Theorem: "The Cartan structure equations"

In special case of frame  $\theta^i = dx^i$ :

$$J^i_{kj} = \Gamma^i_{kj} - \Gamma^i_{jk}$$

1.)

$$\Theta^i = d\theta^i + \omega^i_j \wedge \theta^j \quad \text{i.e.} \quad \Theta^i = D\theta^i$$

= 0 for metric connection

Torsion  $\Theta = \Theta^i_j$  is (1,0) tensor-valued 2-form

(The frame,  $\theta = \theta^i_j$ , is a (1,0) tensor-valued 1-form) notice the upper index clear

$$R^i_{jkl} = \Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^s_{jk}\Gamma^i_{sl} - \Gamma^s_{jl}\Gamma^i_{sk}$$

2.)

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$$

Proof of 2.:

$$\Omega^i_j(\xi, \eta) e_i = \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\xi, \eta]} e_j$$

$$= \nabla_\xi (\omega^i_j(\eta) e_i) - \nabla_\eta (\omega^i_j(\xi) e_i) - \omega^i_j([\xi, \eta]) e_i$$

$$= \underbrace{\xi(\omega^i_j(\eta)) - \eta(\omega^i_j(\xi)) - \omega^i_j([\xi, \eta])}_{\text{Leibniz}} e_i$$

$$+ \underbrace{(\omega^i_j(\eta) \omega^k_i(\xi) - \omega^i_j(\xi) \omega^k_i(\eta))}_{\text{Leibniz}} e_k$$

$$= d\omega^i_j(\xi, \eta) e_i + (\omega^i_k \wedge \omega^k_j)(\xi, \eta) e_i$$

true for all  $\xi, \eta, e_i \Rightarrow \checkmark$

## Use of the Cartan Structure equations?

- Allow proof of simple formulation of the Bianchi identities:

1st Bianchi:  $D\Theta^i = \Omega^i_j \wedge \theta^j$

2nd Bianchi:  $D\Omega^i_j = 0$

- Thus, for metric connection, i.e. when  $dg_{ik} = \omega_{ik} + \omega_{ki}$  and  $\Theta^i = 0$  (same as  $\nabla g = 0$ , and  $\Gamma_{ij} = \Gamma_{ji}$ )

then:

$$\Omega^i_j \wedge \theta^j = 0$$

$$D\Omega^i_j = 0$$

## Proposition:

- In the case of metric connection, the Cartan equations yield for arbitrary bases:

$$\Gamma_{ki}^l = \frac{1}{2} \left( C_{ki}^l - g_{is} g^{sj} C_{kj}^s - g_{ks} g^{sj} C_{ij}^s \right) + \frac{1}{2} g^{ij} (g_{ij,k} + g_{jki} - g_{kij})$$

$C_{ki}^l = 0$  in canonical frame  $\{dx^i\}$

Recall:

$$d\theta^i = -\frac{1}{2} C^i_{jk} \theta^j \wedge \theta^k$$

convention

coefficient functions depend on choice of frame

basis for space of all 2-forms

- In this case, also:

$$R^i_{jab} = \Gamma^i_{bj,a} - \Gamma^i_{aj,b} + \Gamma^i_{ae} \Gamma^e_{bj} - \Gamma^i_{be} \Gamma^e_{aj} - \Gamma^i_{ej} C^e_{ab}$$

absent in canonical frame

□ In the case of metric connection, the Cartan equations yield for arbitrary bases:

$$\Gamma^{\ell}_{ki} = \frac{1}{2} \left( C^{\ell}_{ki} - g_{is} g^{\ell j} C^s_{kj} - g_{ks} g^{\ell j} C^s_{ij} \right) + \frac{1}{2} g^{\ell j} (g_{ijs,k} + g_{jki,i} - g_{kij,s})$$

$C^{\ell}_{ki} = 0$  in canonical frame  $\{dx^i\}$

Recall:

$$d\theta^i = -\frac{1}{2} C^i_{jk} \theta^j \wedge \theta^k$$

*convention*  
*coefficient functions depend on choice of frame*

*basis for space of all 2-forms*

□ In this case, also:

$$R^i_{jab} = \Gamma^i_{bj,a} - \Gamma^i_{aj,b} + \Gamma^i_{ae} \Gamma^e_{bj} - \Gamma^i_{be} \Gamma^e_{aj} - \Gamma^i_{ej} C^e_{ab}$$

absent in canonical frame