Title: General Relativity for Cosmology - Lecture 10

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Lecture 10

Note Title

# Recall:

- The curvature map, $R$, is defined through:

$$R: \xi_1, \xi_2, \xi_3 \to R(\xi_1, \xi_2)\xi_3 = \underbrace{\left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}\right)}_{R(\xi_1,\xi_2)}\xi_3$$

So, "R" can stand for the tensor, the map and this R!

- 1st Bianchi Identity:

$$\sum R(\xi,\eta)\nu = \sum\left(T(T(\xi,\eta),\nu) + (\nabla_\xi T)(\eta,\nu)\right)$$

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Note Title

## Recall:

- The curvature map, $R$, is defined through:

$$R: \xi_1, \xi_2, \xi_3 \to R(\xi_1, \xi_2)\xi_3 = \left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1,\xi_2]}\right)\xi_3$$

$R(\xi_1, \xi_2)$

So, "$R$" can stand for the tensor, the map and this $R$!

- 1st Bianchi Identity:

$$\sum_{\text{cyclic}} R(\xi,\eta)\nu = \sum_{\text{cyclic}} \left(T(T(\xi,\eta),\nu) + (\nabla_\xi T)(\eta,\nu)\right)$$

- 2nd Bianchi Identity:

$$\sum_{\text{cyclic}} \left((\nabla_\xi R)(\eta,\nu) + R(T(\xi,\eta),\nu)\right) = 0$$

## In a chart? (Assuming no torsion, and using $\frac{\partial}{\partial x^i}$, $dx^i$ bases)

**1st Bianchi:** $\sum_{(jke)} R^i{}_{jke} = 0$

↑ cyclic sum

**2nd Bianchi:** $\sum_{(kem)} R^i{}_{jkl;m} = 0$

↑ cyclic sum

## Other useful properties:

(Note: This antisymmetry will be useful because it allows one to view $R$ as a 2-form, which is (1,1) tensor-valued)

- $R^i{}_{jke} = -R^i{}_{jek}$ ←
- $R_{ijke} = -R_{jike}$ ($\langle R(\xi,\eta)v, s\rangle = \langle R(\xi,\eta)s, v\rangle$)
- $R$ [illegible]

## In a chart? (Assuming no torsion, and using $\frac{\partial}{\partial x^i}$, $dx^i$ bases)

**1st Bianchi:** $\sum_{(jke)} R^i{}_{jke} = 0$

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## Other useful properties:

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- $R^i{}_{jke} = -R^i{}_{jek}$ ←
- $R_{ijke} = -R_{jike}$ $\quad \left(\langle R(\xi,\eta)v, s\rangle = \langle R(\xi,\eta)s, v\rangle\right.$
- $R_{ijke} = R_{keij}$ $\quad \left.\langle R(\xi,\eta)v, s\rangle = -\langle R(v,s)\xi, \eta\rangle\right)$

## Contractions of $R$:

**The Ricci Tensor:** $R_{jl} := R^{i}{}_{jil}$

$\Rightarrow$ clearly: $R_{jl} dx^j \otimes dx^l \in T^*_p(M)_2$

**The Curvature Scalar:** $R := g^{jl} R_{jl}$

**Then, 2nd Bianchi identity implies:**

$$\left(R_i{}^k - \tfrac{1}{2}\delta_i^k R\right)_{;k} = 0$$

$\Rightarrow$ The so-called **"Einstein tensor"** $G_i{}^k := R_i{}^k - \tfrac{1}{2}\delta_i{}^k R$ obeys:

$$G_i{}^k{}_{;k} = 0$$

(this property was crucial guidance for Einstein, as we will see)

Recall strategy:

- Specified $g$ $\Rightarrow$ specified distances in $M$
  $\Rightarrow$ implicitly specified "shape" of $M$

Then, alternatively:

- Specified $\nabla$ $\Rightarrow$ specified parallel transport in $M$
  $\Rightarrow$ specified "shape" of $M$, namely:
  $\nabla$ specifies Torsion $T$ and Curvature $R$.

Now assume a manifold is specified by giving a metric $g$.

There ought to exist a $\nabla$ which describes the same manifold.

How does $g$ determine $\Gamma$?

**Idea:** The parallel transport of vectors $\eta, v$ must be such that their inner product (i.e. their lengths and relative angles) stays constant:

Consider any path $\gamma$ and any two vector fields $\eta, v$ that are parallel transported (i.e., autoparallel to $\gamma$) along $\gamma$, i.e., for which:

$$\nabla_{\dot\gamma}\,\eta(\gamma(t)) = 0\,, \quad \nabla_{\dot\gamma}\, v(\gamma(t)) = 0 \quad \text{for all } t\,.$$

**Then, require:**

$$\frac{d}{dt}\Big(g(\gamma(t))_{bc}\;\eta^b(\gamma(t))\,v^c(\gamma(t))\Big) = 0$$

$$\| \quad \nabla_{\dot\gamma}\langle g, \eta\otimes v\rangle \quad \|$$

$$\text{i.e.:}\quad 0 = \dot\gamma^a\big(g_{bc}\,\eta^b v^c\big)_{;a} \overset{\text{by } \nabla \text{ obeying Leibniz rule}}{=} \dot\gamma^a\Big(g_{bc;a}\,\eta^b v^c + g_{bc}\,\underbrace{\eta^b{}_{;a}}_{0 \text{ because } \nabla_{\dot\gamma}\eta = 0}\, v^c + g_{bc}\,\eta^b\,\underbrace{v^c{}_{;a}}_{0 \text{ because } \nabla_{\dot\gamma} v = 0}\Big)$$

**Then, require:** $\frac{d}{dt}\Big( g(\gamma(t))_{bc}\, \eta^b(\gamma(t))\, v^c(\gamma(t)) \Big) = 0$

$\| \nabla_{\dot\gamma} \langle g, \eta \otimes v \rangle$

by $\nabla$ obeying Leibniz rule

$\| \dot\gamma^a \big( g_{bc} \eta^b v^c \big)_{;a} = \dot\gamma^a \big( g_{bc;a} \eta^b v^c + g_{bc} \underbrace{\eta^b{}_{;a}}_{0 \text{ because } \nabla_{\dot\gamma}\eta = 0} v^c + g_{bc} \eta^b \underbrace{v^c{}_{;a}}_{0 \text{ because } \nabla_{\dot\gamma} v = 0} \big)$

i.e.: $0 = g_{bc;a}\, \dot\gamma^a \eta^b v^c$

$\Rightarrow$ $0 = g_{bc;a}\, \dot\gamma^a \eta^b v^c$ for all arbitrary $\dot\gamma, \eta, v$ !

$\Rightarrow$ Compatibility of $\nabla$ with $g$ means:

$$\boxed{\nabla_\xi g = 0 \quad \text{for all } \xi}$$

i.e.: $0 = \dot{x}^a (g_{bc}\eta^b v^c)_{;a} = \dot{x}^a (g_{bc;a}\eta^b v^c + g_{bc}\eta^b{}_{;a} v^c + g_{bc}\eta^b v^c{}_{;a})$ *

$\Rightarrow$ $0 = g_{bc;a}\dot{x}^a \eta^b v^c$ for all arbitrary $\dot{x}, \eta, v$ !

$\Rightarrow$ Compatibility of $\nabla$ with $g$ means:

$$\nabla_\xi g = 0 \quad \text{for all } \xi$$

Is there a $\nabla$ for each choice of $g$? Indeed:

Fund. theorem of (pseudo) Riemannian geometry:

For each (pseudo) Riemannian manifold $(M, g)$ there

$\Rightarrow$ $0 = g_{bc;a}\, j^a \eta^b \nu^c$ for all arbitrary $j, \eta, \nu$ !

$\Rightarrow$ Compatibility of $\nabla$ with $g$ means:

$$\nabla_\xi g = 0 \quad \text{for all } \xi$$

Is there a $\nabla$ for each choice of $g$? Indeed:

**Fund. theorem of (pseudo) Riemannian geometry:**

For each (pseudo) Riemannian manifold $(M, g)$ there exists a unique $\nabla$ that is torsionless and compatible with $g$, i.e., which obeys $\nabla g = 0$, the Levi-Civita connection.

**More generally:** $\forall (M, g)$ and a tensor field $T$ with $T^k_{ij} = -T^k_{ji}$ there is a metric-preserving $\nabla$ whose torsion is $T$.

**In a chart:** How to obtain the Levi-Civita $\nabla$ from $g$?

$\nabla g = 0$ means $g_{\mu\nu,\alpha} - g_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} - g_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} = 0$ I

i.e. $g_{\alpha\mu,\nu} - g_{\alpha\beta}\Gamma^{\beta}_{\mu\nu} - g_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} = 0$ II

and $g_{\nu\alpha,\mu} - g_{\nu\beta}\Gamma^{\beta}_{\alpha\mu} - g_{\beta\alpha}\Gamma^{\beta}_{\nu\mu} = 0$ III

take: $\frac{1}{2}(-\mathrm{I} + \mathrm{II} + \mathrm{III})$

$$\Rightarrow \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right) = g_{\alpha\beta}\Gamma^{\beta}_{\nu\mu}$$

**Thus:** $$\Gamma^{\beta}_{\nu\mu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)$$

I "Levi-Civita" connection or also called "Riemannian" connection.

## Upgrade the math:

- Make use of arbitrary bases $e_i$, $\theta^i$ in (co-) tangent spaces : frames

- Allow forms to be tensor-valued : obtain, e.g., torsion and curvature forms. Also: connection forms.

$\Rightarrow$ We will obtain powerful, simple equations that relate $\nabla$, $g$, $R$, $T$. (Even the Bianchi identities will look simple)

**Now:** Assume again that $\nabla$ and $g$ are still unrelated and $T \neq 0$. (possibly)

# "Moving frames":

**Def:** A "moving frame" is a set, $\{e_i\}_{i=1}^{n}$, of contravariant vector fields $e_i$ which, together, at each point $p \in \mathcal{M}$ form a basis of $T_p(\mathcal{M})$.

**Def:** We denote the dual basis $\{\theta^i\}_{i=1}^{n}$.

It obeys: $\theta^i(e_j) = \delta^i_j$.

**Def:** For $n = 4$ it may be called vierbein (german: 4 legs.) or tetrad.

(in arb. dimensions: "vielbein" = many legs)

**Notice:** Each co-vector $\theta^i(x)$ is a 1-form, and $d\theta^i$ is a 2-form!

**Def:** Collect them in a "Frame": $\theta^i \otimes e_i$, i.e. a (1,0)-tensor valued 1-form

**Remark:** If we choose e.g. $\theta^i(x) := dx^i$, then $d\theta^i(x) = 0$.

**Remark:** A general choice for the $\theta^i(x)$ can always be written in the form:

$$\theta^i(x) = \lambda(x)^i_j \, dx^j$$

($\lambda(x)^i_j$: scalar coefficient functions)

**Def:** We denote the expansion coefficients by functions $C^i_{jk}$:

**Exercise:**
Express the $C^i_{jk}$ in terms of the $\lambda^i_j$.

$$d\theta^i = \underbrace{-\frac{1}{2}}_{\text{convention}} C^i_{jk} \, \theta^j \wedge \theta^k \quad \text{with} \quad C^i_{jk} = -C^i_{kj}$$

$C^i_{jk}$: coefficient functions, depend on choice of frame

$\theta^j \wedge \theta^k$: basis for space of all 2-forms

$C^i_{jk} = -C^i_{kj}$: the sym. part would drop out

## Coefficients:

- Torsion: $T^i_{kl} := \langle \theta^i, T(e_k, e_l) \rangle$
- Curvature: $R^i_{jkl} := \langle \theta^i, R(e_k, e_l) e_j \rangle$
- Metric: $g_{ik} := g(e_i, e_k) = \langle e_i, e_k \rangle$
- Christoffel: $\Gamma^i_{kj} e_i := \nabla_{e_k} e_j$

## Consider arbitrary change of frame:

(has nothing to do with a change of chart!)

- assume $\bar{\theta}^i(x) = A^i_j(x)\, \theta^j(x)$
- then: $\bar{e}_i(x) = (A^{-1})_i{}^j(x)\, e_j(x)$

(because we chose bases that are dual: $\bar{\theta}^i(\bar{e}_j) = \delta^i_j$)

Another step towards more abstract formulation:

# Tensor-valued p-forms:

**Def:** A $(r,s)$-tensor-valued p-form $\phi$ is an anti-symmetric p-multilinear mapping at each $q \in \mathcal{U}$:

$$\phi : \underbrace{T_q(\mathcal{U})' \times \dots \times T_q(\mathcal{U})'}_{p \text{ factors}} \longrightarrow T_q(\mathcal{U})^r_s$$

**Def:** The p-forms $\phi_{j_1 \dots j_s}^{i_1 \dots i_r} := \phi(\theta^{i_1}, \dots \theta^{i_r}, e_{j_1}, \dots, e_{j_s})$ are called the component p-forms relative to the basis $\{e_i\}_{i=1}^n$.

## Special cases:

- $(r,s)$ tensors are $(r,s)$ tensor-valued 0-forms.
- p-forms are $(0,0)$ tensor-valued forms.

# Torsion 2-form:

- We recall that $T(\xi,\eta) = -T(\eta,\xi) \Rightarrow$ can define the torsion's (1,0) tensor-valued 2-form through its action on 2 vector fields $\xi,\eta$:

"torsion 2-form" → $\Theta^i(\xi,\eta)\, e_i := T(\xi,\eta)$

the 2 form $\Theta^i$ fed 2 vectors to yield a vector

- Given a frame:

using their antisymmetry

$$\Theta^i = \frac{1}{2} T^i_{kl}\, \theta^k \wedge \theta^l$$

## Curvature 2-form:

(recall that in canonical basis: $R^i{}_{jkl} = -R^i{}_{jlk}$)

- We recall that also $R(\xi,\eta) = -R(\eta,\xi)$

$\Rightarrow$ can define curvature's (1,1) tensor-valued 2-form:

"curvature 2-form" $\longrightarrow$ $\Omega^i{}_j(\xi,\eta)e_i := R(\xi,\eta)e_j$

($\Omega^i{}_j(\xi,\eta)$: number; $e_i$: tangent vector; $R(\xi,\eta)e_j$: tangent vector)

Recall: $R: \xi\eta e_j \to \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\eta,\xi]} e_j$

- Given a frame $\{\theta^i\}_{i=1}^n$:

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl}\, \theta^k \wedge \theta^l$$

# The connection as a form?

- Nontrivial because:

  1. Christoffels $\Gamma^i_{kj} e_i := \nabla_{e_k} e_j$ are not tensors to start with!

  2. $\Gamma^i_{kj}$ is not antisym. in any indices, so can't be a 2-form (but can be 1-form):

- Define the connection 1-forms $\omega^i_j$: $\boxed{\omega^i_j := \Gamma^i_{kj}\,\theta^k}$

**Thus:**

$$\underbrace{\nabla_\xi e_j}_{\text{vector}} = \overbrace{\omega^i_j(\xi)}^{\text{scalars}} \underbrace{e_i}_{\text{vector}}$$

(because $\nabla_{\xi^k e_k} e_j = \xi^k \nabla_{e_k} e_j$)

**Proposition:** cov. deriv. for covectors reads

$$\nabla_\xi \theta^i = -\omega^i{}_j(\xi)\,\theta^j$$

**Proof:**

$$0 = \nabla_\xi \underbrace{\langle \theta^i, e_j \rangle}_{=\delta^i_j} \overset{\text{Leibniz rule and using that } \nabla \text{ and contractions commute}}{=} \langle \nabla_\xi \theta^i, e_j \rangle + \langle \theta^i, \nabla_\xi e_j \rangle$$

$$= \langle \nabla_\xi \theta^i, e_j \rangle + \underbrace{\langle \theta^i, \omega^k{}_j(\xi) e_k \rangle}_{= \omega^i{}_j(\xi) \text{ because } \langle \theta^i, e_k \rangle = \delta^i_k} \qquad (\star)$$

$\Rightarrow$ indeed:

$$\boxed{\nabla_\xi \theta^i = -\omega^i{}_\ell(\xi)\,\theta^\ell}$$

contract with $\langle\ , e_j\rangle$ to verify that this is Eq. $(\star)$

✓

## Connection 1-forms are non-tensorial:

**Proposition:** Under change of frame $\bar{\theta}^i(x) = A^i{}_j(x)\,\theta^j(x)$

the transformation is:

$$\underbrace{\bar{\omega}^a{}_b}_{\text{1-form}} = \underbrace{\overset{\text{functions}}{A^a{}_i}\,\overset{\text{1-form}}{\omega^i{}_j}\,A^{-1\,j}{}_b}_{\text{1-form}} - \underbrace{\underbrace{(\overset{\text{function}}{dA})^a{}_i}_{\text{1-form}}\,\underbrace{(\overset{\text{matrix inverse.}}{A^{-1}})^i{}_b}_{\text{functions}}}_{\text{1-form}}$$

**Proof:**

$$-\bar{\omega}(\xi)^a{}_b\,\bar{\theta}^b = \nabla_\xi \bar{\theta}^a = \nabla_\xi (A^a{}_b\theta^b) \overset{\text{Leibniz rule}}{=} \big(\overbrace{dA^a{}_b(\xi)}^{=\xi(A^a{}_b)}\big)\,\theta^b + A^a{}_b\,\nabla_\xi\theta^b$$

$$= dA^a{}_b(\xi)\,\theta^b - A^a{}_b\,\omega(\xi)^b{}_c\,\theta^c$$

$$= dA^b{}_b(\xi)\,A^{-1\,b}{}_c\,\bar{\theta}^c - A^a{}_b\,\omega(\xi)^b{}_c\,A^{-1\,c}{}_d\,\bar{\theta}^d$$

true for all $\bar{\theta}$ $\Rightarrow$ proposition above. ✓

# The "absolute exterior differential" D:

(It generalizes both $\nabla$ and $d$)

**Proposition:** (proof, see e.g. Straumann: check tensorial behaviour under frame change)

For every $(r,s)$ tensor-valued $p$-form $\phi$ there exists a unique $(r,s)$ tensor-valued $(p+1)$ form $D\phi$ whose components relative to $\{\theta^i\}$ are:

$$(D\phi)^{i_1\dots i_r}_{j_1\dots j_s} = \underbrace{d\phi^{i_1\dots i_r}_{j_1\dots j_s}}_{p+1\text{-form}} + \overbrace{\omega^{i_1}{}_{\ell}}^{1\text{-form}} \wedge \overbrace{\phi^{\ell i_2\dots i_r}_{j_1\dots j_s}}^{p\text{-form}} + \dots - \omega^{\ell}{}_{j_1} \wedge \phi^{i_1,\dots,i_r}_{\ell,\dots,j_s} - \dots \qquad (*)$$

**Proposition:** $D$ is an anti-derivation: ($p$ = degree of $\phi$)

$$D(\phi \wedge \psi) = D\phi \wedge \psi + (-1)^p \phi \wedge D\psi$$

**Special cases:**

- An ordinary $p$-form is $(0,0)$ tensor-valued. In this case, clearly:

$$D = d$$

- An ordinary tensor field is a tensor-valued 0-form. In this case:

$$D = \nabla$$

**Exercise:** Verify

**Hint:** Choose frame $\theta^i = dx^i$, use $\omega^i{}_j = \Gamma^i{}_{kj}\theta^k$, then show (*) implies indeed:

$$\phi^{i\dots}_{j\dots g_s j_K} = \phi^{i\dots}_{j\dots g_s j_K} + \Gamma^i{}_{k_1}\phi^{k_1\dots}_{j\dots g_s} + \dots - \Gamma^{k_1}_{g_s j}\phi^{i\dots}_{k_1 j_2\dots g_s} - \dots$$

# How are $\omega, g, \Theta, \Omega$ related now?

**Proposition:** (Exercise: check)

An affine connection $\nabla$ is metric, if and only if $Dg = 0$, i.e., iff:

$$\underbrace{dg_{ik}}_{\text{(0,2) tensor-valued 1-form}} - \omega_{ik} - \omega_{ki} = 0$$

**Theorem:** "The Cartan structure equations" — They express torsion and curvature in terms of the connection

In special case of frame $\theta^i = dx^i$:

$T^i_{kj} = \Gamma^i_{kj} - \Gamma^i_{jk}$

1.) $\Theta^i = d\theta^i + \omega^i_j \wedge \theta^j$ i.e. $\Theta^i = D\theta^i$

($\Theta^i = 0$ for metric connection; Torsion $\Theta = \theta^i e_i$ is (1,0) tensor-valued 2-form)

($\theta = \theta^i e_i$ is a (1,0) tensor-valued 1-form — notice the upper index)

$R^i_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \Gamma^s_{lj}\Gamma^i_{ks} - \Gamma^s_{kj}\Gamma^i_{ls}$

2.) $\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$

## Proof of 2.:

$$\Omega^i{}_j(\xi,\eta)e_i = \nabla_\xi \nabla_\eta e_j - \nabla_\eta \nabla_\xi e_j - \nabla_{[\xi,\eta]} e_j$$

$$= \nabla_\xi \left(\omega^i{}_j(\eta) e_i\right) - \nabla_\eta \left(\omega^i{}_j(\xi) e_i\right) - \omega^i{}_j([\xi,\eta]) e_i$$

$$= \left(\xi\left(\omega^i{}_j(\eta)\right) - \eta\left(\omega^i{}_j(\xi)\right) - \omega^i{}_j([\xi,\eta])\right) e_i$$

Leibnit

$$+ \left(\omega^i{}_j(\eta)\, \omega^k{}_i(\xi) - \omega^i{}_j(\xi)\, \omega^k{}_i(\eta)\right) e_k$$

Exercise: Fill in all steps.

$$= d\omega^i{}_j(\xi,\eta) e_i + \left(\omega^i{}_k \wedge \omega^k{}_j\right)(\xi,\eta) e_i$$

true for all $\xi, \eta, e_i$ $\Rightarrow$ ✓

# Use of the Cartan Structure equations?

- Allow proof of simple formulation of the Bianchi identities:

  1st Bianchi: $D\Theta^i = \Omega^i{}_j \wedge \theta^j$

  2nd Bianchi: $D\Omega^i{}_j = 0$

  ← i.e. "Riemannian", i.e. "Levi Civita", without torsion.

- Thus, for metric connection, i.e. when $dg_{ik} = \omega_{ik} + \omega_{ki}$ and $\Theta^i = 0$ (same as $\nabla g = 0$, and $\Gamma_{ij} = \Gamma_{ji}$)

  then:

$$\Omega^i{}_j \wedge \theta^j = 0$$
$$D\Omega^i{}_j = 0$$

## Proposition :

- In the case of metric connection, the Cartan equations yield for arbitrary bases :

$C^k_{ij} = 0$ in canonical frame $\{dx^i\}$

$$\Gamma^l_{ki} = \frac{1}{2}\left(C^l_{ki} - g_{is}g^{lj}C^s_{kj} - g_{ks}g^{lj}C^s_{ij}\right) + \frac{1}{2}\,g^{lj}\left(g_{ij,k} + g_{jk,i} - g_{ki,j}\right)$$

Recall:

$$d\theta^i = -\frac{1}{2}C^i_{jk}\,\theta^j \wedge \theta^k$$

(convention; $C^i_{jk}$ : coefficient functions depend on choice of frame; $\theta^j \wedge \theta^k$ : basis for space of all 2-forms)

- In this case, also:

$$R^i_{jab} = \Gamma^i_{bj,a} - \Gamma^i_{aj,b} + \Gamma^i_{al}\Gamma^l_{bj} - \Gamma^i_{bl}\Gamma^l_{aj} - \Gamma^i_{lj}C^l_{ab}$$

($C^l_{ab}$ : absent in canonical frame)

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$C^k_{ij} = 0$ in canonical frame $\{dx^i\}$

$$\Gamma^l_{ki} = \frac{1}{2}\left(C^l_{ki} - g_{is}g^{lj}C^s_{kj} - g_{ks}g^{lj}C^s_{ij}\right) + \frac{1}{2}\,g^{lj}\left(g_{ij,k} + g_{jk,i} - g_{ki,j}\right)$$

Recall:

$$d\theta^i = -\frac{1}{2}C^i_{jk}\,\theta^j \wedge \theta^k$$

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($C^l_{ab}$: absent in canonical frame)