Title: General Relativity for Cosmology - Lecture 9

Speakers: Achim Kempf

Collection: General Relativity for Cosmology (Kempf)

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GR for Cosmology, Achim Kempf — Lecture 9

Note Title

Recall: So far, we have 2 ways to capture shape:

- Specified g $\Rightarrow$ specified distances in M
  $\Rightarrow$ implicitly specified "shape" of M

(Notice (for essay): See also my newpaper 1510.02725)

Then, new:

- Specified ∇ $\Rightarrow$ specified parallel transport in M
  $\Rightarrow$ implicitly specified "shape" of M

Question:

How does ∇ determine "shape"? Through:

Torsion & Curvature!

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How does ∇ determine "shape"? Through:

Torsion & Curvature!

Recall:

$$\bar{\Gamma}^{\bar{r}}_{\bar{a}\bar{b}} = \frac{\partial \bar{x}^{\bar{r}}}{\partial x^k} \frac{\partial x^i}{\partial \bar{x}^{\bar{a}}} \frac{\partial x^j}{\partial \bar{x}^{\bar{b}}} \Gamma^k_{ij} + \frac{\partial \bar{x}^{\bar{r}}}{\partial x^k} \frac{\partial^2 x^k}{\partial \bar{x}^{\bar{a}} \partial \bar{x}^{\bar{b}}}$$

Recall:

$$\bar{\Gamma}^{r}_{ab} = \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \Gamma^{k}_{ij} + \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial^{2} x^{k}}{\partial \bar{x}^{a} \partial \bar{x}^{b}}$$

Notice: The antisymmetric part of Γ transforms tensorially!

$$\left.\begin{aligned} \Gamma^{k}_{(sym)ij} &:= \frac{1}{2}\left(\Gamma^{k}_{ij} + \Gamma^{k}_{ji}\right) \\ \Gamma^{k}_{(asym)ij} &:= \frac{1}{2}\left(\Gamma^{k}_{ij} - \Gamma^{k}_{ji}\right) \end{aligned}\right\} \quad \Gamma^{k}_{ij} = \Gamma^{k}_{(sym)ij} + \Gamma^{k}_{(asym)ij}$$

$$\Rightarrow \bar{\Gamma}^{r}_{(asym)ab} = \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \Gamma^{k}_{(asym)ij} \quad !$$

In General Relativity: one assumes torsionless ∇, i.e.: $T = 0$.

Idea: "(Extended) equivalence principle:"

Christoffel Γ will express gravitational and psuedo forces.
Therefore, we require that around each $p \in M$ there exists a chart so that $\Gamma(p) = 0$ (i.e. no such forces in free fall).

This rules out the existence of torsion:

Why? The torsion is a tensor.

$\Rightarrow$ It transforms linearly with invertible Jacobian matrices

$$\bar{T}^i_{jk}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} T^a_{bc}(x)$$

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$\Rightarrow$ If T_{ij} vanishes in one cds, it vanishes in all cds.

Proposition:

Proposition: Vice versa, if $T^i_{jk}(x) = 0 \ \forall x \in \mathcal{M}$, then there is for every $p \in \mathcal{M}$ a chart with $\Gamma^i_{jk}(p) = 0$:

Recall: $\xi \in T'(\mathcal{M})$ is autoparallel to a path $\gamma : t \to x(t)$ if

$$\nabla_{\dot\gamma} \xi = 0$$

Meaning: ξ is parallel transported along the path γ in $\mathcal{M}$.

Explicitly: $\frac{d\xi^k}{dt} + \Gamma^k_{ij} \frac{dx^i}{dt} \xi^j = 0$

Geodesics: A curve $\gamma : t \to x(t)$ is called a geodesic if $\dot\gamma$ is autoparallel along γ, i.e. if

Meaning: The path γ in $\mathcal{M}$

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Geodesics: A curve $\gamma : t \to x(t)$ is called a geodesic if $\dot\gamma$ is autoparallel along γ, i.e. if

$$\nabla_{\dot\gamma} \dot\gamma = 0$$

Meaning: The path γ in M is such that the path's tangent vectors are parallel translates of each other.

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$\Rightarrow$ In charts, geodesics $x^\mu(t)$ obey:

$$\frac{d^2x^k}{dt^2} + \Gamma(x)^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \qquad (*)$$

Theory of ordinary differential equations:

$\Rightarrow$ Given $p = \gamma(0)$, each initial condition $\xi = \dot\gamma(0)$ belongs to a unique geodesic γ_ξ of nonzero length.

subscript indicates initial condition vector

Notice: If $\gamma_\xi(t)$ solves $(*)$ then $\gamma_\xi(\lambda t)$ also solves $(*)$ and for $\lambda \in \mathbb{R}$:

The exponential map is defined through:

$$\exp_p : T_p'(M) \to M$$

(really from a neighborhood of 0 in $T_p'(M)$ to a neighborhood of p in M.)

$$\exp_p : \quad \xi \to \exp_p(\xi) := \gamma_\xi(1)$$

where γ is the geodesic with $\gamma_\xi(0) = p$, $\dot{\gamma}_\xi(0) = \xi$.

Observe:

From (G) we obtain:

$$\gamma_\xi(\lambda) = \gamma_{\lambda\xi}(1) = \exp_p(\lambda\xi) \qquad (E)$$

"Geodesic" or "Riemann normal" coordinates:

- $\exp_p$ is a diffeomorphism from a neighborhood of $0 \in T_p'(M) \simeq \mathbb{R}^n$ (isomorphic) into a neighborhood of the point $p \in M$.

$\Rightarrow$ $\exp_p$ provides a chart around p:

- Choose a basis, say $e_1, e_2, \dots, e_n$ of $T_p'(M)$, then:

$$\xi = \xi^i e_i$$

- Through $\exp_p$, the ξ^i become the coordinates of points in a neighborhood of $p \in M$:

$$(\xi^1, \dots, \xi^n) \to \exp_p(\xi^i e_i) \in M$$

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Geodesic eqn. at p: $\frac{d^2x^k}{dt^2} + \Gamma(p)^k{}_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$

Thus: $\left(\Gamma_{sym}{}^k{}_{ij}(p) + \Gamma_{asym}{}^k{}_{ij}(p)\right)\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$

$= 0$ because of type $(\text{antisymmetric})_{ij}\,(\text{symmetric})^{ij}$

$\Rightarrow \Gamma_{sym}{}^k{}_{ij}(p) = 0$ in geodesic cds.

$\Rightarrow$ **Indeed:** If the torsion vanishes, $\Gamma_{asym}{}^k{}_{ij}(p) = \frac{1}{2}T^k{}_{ij}(p) = 0$ then for each $p \in M$ there exists a chart in which the entire gravity and pseudo force field vanishes at p:

$$\Gamma^k{}_{ij}(p) = 0$$

Note:
Quantum fluctuations may induce torsion!

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Note:
Quantum fluctuations may induce torsion!

Then, the **Torsion map** is defined as:

$$\mathcal{T} : T_p'(M) \times T_p'(M) \to T_p'(M)$$

This will be the amount by which an infinitesimal parallelogram spanned by ξ_1 and ξ_2 does not close.

$$\mathcal{T} : \xi_1, \xi_2 \to \mathcal{T}(\xi_1, \xi_2) := \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1, \xi_2]$$

for proof it's a tensor, see Straumann

- It is used to define the **Torsion tensor**, J,

$$J \in T_p'^{2}(M)$$

through:

$$J(\omega, \xi_1, \xi_2) := \langle \underbrace{\omega}_{\in T_p'(M)}, \underbrace{\mathcal{T}(\xi_1, \xi_2)}_{\in T_p(M)} \rangle \in \mathbb{R}$$

we could also write: $= \omega(\mathcal{T}(\xi_1, \xi_2))$

contraction yields a number

feeding 1 covector & 2 vectors to a (1,2) tensor yields a number

Input: DVI - 1920x1080p@60Hz
Output: SDI - 1920x1080i@60Hz

Compare with prior definition:

- Choose canonical bases $\omega := dx^k$, $\xi_1 := \frac{\partial}{\partial x^i}$, $\xi_2 := \frac{\partial}{\partial x^j}$:

- $$T^k_{ij} := dx^k\left(T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right)$$

$$= \left\langle dx^k, T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right\rangle \quad \text{(more convenient notation)}$$

$$= \left\langle dx^k, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]\right\rangle$$

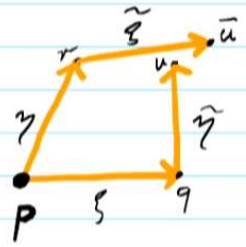
Recall: $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$

$\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right) f = 0 \quad \forall f$

$$= \left\langle dx^k, \Gamma^r_{ij} \frac{\partial}{\partial x^r} - \Gamma^r_{ji} \frac{\partial}{\partial x^r} \right\rangle = \Gamma^r_{ij} \delta^k_r - \Gamma^r_{ji} \delta^k_r$$

- $\Rightarrow$ $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$

compare with the reverse. (In flat space: $x^\mu + \eta^\mu + \xi^\mu = x^\mu + \xi^\mu + \eta^\mu$.)



$\xi, \eta \in T_p'$

$\tilde{\xi} \in T_r'$

$\tilde{\eta} \in T_q'$

Recall parallel transport: $\nabla_{\dot{\gamma}} v = 0$

$$\frac{dv^k}{dt} + \Gamma^k_{ij} \frac{dx^i}{dt} v^j = 0$$

$\tilde{\xi}(r) = ?$

$$\tilde{\xi}^k(x^i + \eta^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i)$$

Now use $v := \xi$, $\frac{dx^i}{dt} = \eta^i$:

$$= \xi^k(x^i) - \Gamma(x)^k_{ij}\, \eta^i \xi^j$$

$\Rightarrow$ Cds. of $\bar{u}$: $x^a + \eta^a + \xi^a - \Gamma(x)^a_{ij}\, \eta^i \xi^j$

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$$\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right) f = 0 \quad \forall f$$

Recall:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

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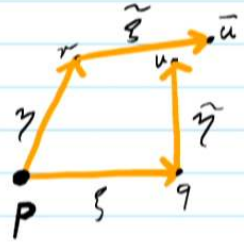
$$\Rightarrow \boxed{T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}}$$

Geometric meaning of torsion? Parallelograms would not close!

Travel from p infinitesimally in ξ and then η direction, and compare with the reverse. (In flat space: $x^\mu + \eta^\mu + \xi^\mu = x^\mu + \xi^\mu + \eta^\mu$.)



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Now use $v := \xi$, $\frac{dx^i}{dt} = \eta^i$:

$$= \xi^k(x^i) - \Gamma(x)^k_{ij}\, \eta^i \xi^j$$

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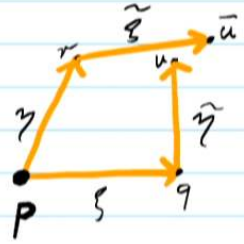

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Now use $v := \xi$, $\frac{dx^i}{dt} = \eta^i$:

$$= \xi^k(x^i) - \Gamma(x)^k_{ij}\, \eta^i \xi^j$$

Analogously obtain: Cds. of u: $x^a + \xi^a + \eta^a - \Gamma(x)^a_{ij}\,\xi^i\eta^j$

Torsion!

$\Rightarrow$ Cd. distance from u to $\bar{u}$ is: $(\Gamma(x)^a_{ji} - \Gamma(x)^a_{ij})\,\eta^i\xi^j = T^a_{ji}\,\eta^i\xi^j$ ✓

Comment: We had:

$$\tilde{\xi}^k(x^i + \eta^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i) = \xi^k(x^i) - \Gamma(x)^k_{ij}\,\eta^i\xi^j$$

this is also: $= \xi^k(x^i) - \left(\eta^i\xi^k_{,i} + \Gamma(x)^k_{ij}\,\eta^i\xi^j\right) + \eta^i\xi^k_{,i}$

$$= \xi^k(x^i) - \eta^i\xi^k_{;i} + \eta^i\xi^k_{,i}$$

Thus: cd distance from u to $\bar{u}$ is:

$\Rightarrow$ Cd. distance from u to $\bar{u}$ is: $\left(\Gamma(x)^a_{j i} - \Gamma(x)^a_{i j}\right)\eta^i \xi^j = T^a_{j i}\eta^i \xi^j$ ✓

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$$= \xi^k(x^i) - \eta^i \xi^k_{;i} + \eta^i \xi^k_{,i}$$

Thus: cd distance from u to $\bar{u}$ is:

$$\left(\cancel{x^a} + \cancel{\eta^a} + \cancel{\xi^a} - \eta^i \xi^k_{;i} + \eta^i \xi^k_{,i}\right) - \left(\cancel{x^a} - \cancel{\xi^a} - \cancel{\eta^a} + \xi^i \eta^k_{;i} - \eta^i \xi^k_{,i}\right) = T^a_{j i}\eta^i \xi^j$$

Recall that indeed: $T : \eta, \xi \to T(\eta, \xi) = \nabla_\eta \xi - \nabla_\xi \eta - [\eta, \xi]$

Curvature:

Assume ξ_1, ξ_2 and ξ_3 are tangent vectors at $p \in M$.

- The curvature map, R, is defined through:

$$R: \xi_1, \xi_2, \xi_3 \to R(\xi_1, \xi_2)\xi_3 = \left(\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} - \nabla_{[\xi_1, \xi_2]}\right)\xi_3 \in T_p'(M)$$

($R(\xi_1, \xi_2)$: an operator, or map, acting on ξ_3)

- It defines the curvature tensor, R,

(R can be fed one covector and 3 vectors to yield a number)

$$R \in T_3'(M)$$

through:

$$= \omega\left(R(\xi_1, \xi_2)\xi_3\right)$$

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- It defines the curvature tensor, R,

$R \in T_3'(M)$ ← can be fed one covector and 3 vectors to yield a number

through:

$$= \overbrace{\omega\left(R(\xi_1, \xi_2)\xi_3\right)}$$

$$R^i{}_{jkl} = \langle dx^i, R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^j}\rangle$$

$$= \langle dx^i, \left(\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^l}} - \nabla_{\frac{\partial}{\partial x^l}}\nabla_{\frac{\partial}{\partial x^k}} - \nabla_{\underbrace{\left[\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right]}_{=0}}\right)\frac{\partial}{\partial x^j}\rangle$$

$$= \langle dx^i, \nabla_{\frac{\partial}{\partial x^k}}\Gamma^s_{lj}\frac{\partial}{\partial x^s} - \nabla_{\frac{\partial}{\partial x^l}}\Gamma^s_{kj}\frac{\partial}{\partial x^s}\rangle$$

$$= \langle dx^i, \left(\overbrace{\Gamma^s_{lj,k} + \Gamma^r_{lj}\Gamma^s_{kr}} - \overbrace{\Gamma^s_{kj,l} - \Gamma^r_{kj}\Gamma^s_{lr}}\right)\frac{\partial}{\partial x^s}\rangle$$

$$= \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \underbrace{\Gamma^s_{lj}\Gamma^i_{ks} - \Gamma^s_{kj}\Gamma^i_{ls}}$$

(at origin of geodesic cds they vanish.)

Curvature tensor's meaning?

Intuition:

- Contains derivatives of Γ $\Rightarrow$
- expresses variation in gravitational forces
  $\Rightarrow$
- expresses the strength and direction of "tidal forces".

Geometry:

-

Proposition: (Ricci Identity)

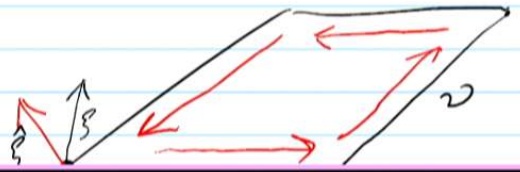
Assume the torsion vanishes and that ξ is a vector field. Then:

$$\xi^a{}_{;cd} - \xi^a{}_{;dc} = R^a{}_{cdb}\,\xi^b$$

(here: $\xi^a{}_{;cd} := \xi^a{}_{;c;d}$ etc.)

Remark: (a bit messy to derive because need Taylor expansion, see, e.g., text by Stewart or Einstein)

It implies that for parallel transport along infinitesimal parallelogram:



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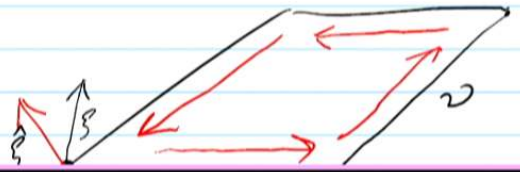

Consider an arbitrary $f(M)$-linear map:

(e.g. Torsion or Curvature map)

$$K : \underbrace{\xi_1 \times \xi_2 \times \dots \times \xi_r}_{\text{tangent vectors}} \longrightarrow \underbrace{K(\xi_1, \dots, \xi_r)}_{\text{tangent vector}}$$

i.e. at each $p \in M$:

$$K : T_p(M)^r \longrightarrow T_p(M)^1$$

- We can view K as a tensor $\tilde{K} \in T_p(M)^1_r$,

(as we did for R and T)

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- Now let the usual derivative of the tensor $\tilde{K}$ define the derivative of the map K:

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$$\tilde{K}(\omega, \xi_1, \ldots, \xi_r) := \langle \omega, K(\xi_1, \ldots, \xi_r) \rangle$$

- Now let the usual derivative of the tensor $\tilde{K}$ define the derivative of the map K:

$$\langle \omega, (\nabla_\xi K)(\xi_1, \ldots, \xi_r) \rangle := \nabla_\xi \tilde{K}(\omega, \xi_1, \ldots \xi_r)$$

$\nabla_\xi K$: new concept: covariant derivative of a map $K: T_p(M)^r \to T_p(M)'$

$\nabla_\xi \tilde{K}$: usual cov. derivative of a $(1, r)$ tensor when fed one covector & r vectors

Using ∇ for map:

1st Bianchi Identity:

$$\sum_{\text{cyclic}} R(\xi,\eta)v = \sum_{\text{cyclic}} \Big(T(T(\xi,\eta),v) + (\nabla_\xi T)(\eta,v) \Big)$$

2nd Bianchi Identity:

$$\sum_{\text{cyclic}} \Big((\nabla_\xi R)(\eta,v) + R(T(\xi,\eta),v) \Big) = 0$$

with obvious simplification in case $T = 0$.

↙ or like the homogeneous Maxwell equations

cyclic

Indeed:

$$\overbrace{(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi)\,\nu - \nabla_{[\xi,\eta]}\nu + \text{cyclic}}$$

skip by 1 cyclically skip by 1 cyclically

$$= \nabla_\xi(\nabla_\eta \nu - \nabla_\nu \eta) - \nabla_{[\eta,\nu]}\xi + \text{cyclic}$$

Exercise: Prove that without torsion: $\boxed{\nabla_\eta \nu - \nabla_\nu \eta = [\eta,\nu]}$ (easy!)

$$= \underbrace{\nabla_\xi[\eta,\nu] - \nabla_{[\eta,\nu]}\xi}_{\text{because again } \nabla_a b - \nabla_b a = [a,b]} + \text{cyclic}$$

$$= [\xi,[\eta,\nu]] + \text{cyclic}$$

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cyclic

Indeed: $(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi)\nu - \nabla_{[\xi,\eta]}\nu + \text{cyclic}$

ship by 1 cyclically ↓ ship by 1 cyclically ↓

$= \nabla_\xi(\nabla_\eta \nu - \nabla_\nu \eta) - \nabla_{[\eta,\nu]}\xi + \text{cyclic}$

Exercise: Prove that (without torsion): $\nabla_\eta \nu - \nabla_\nu \eta = [\eta, \nu]$ (easy!)

$= \nabla_\xi [\eta,\nu] - \nabla_{[\eta,\nu]}\xi + \text{cyclic}$

// because again $\nabla_a b - \nabla_b a = [a,b]$

$= [\xi, [\eta,\nu]] + \text{cyclic}$