Title: General Relativity for Cosmology - Lecture 8

Speakers: Achim Kempf

Collection: General Relativity for Cosmology (Kempf)

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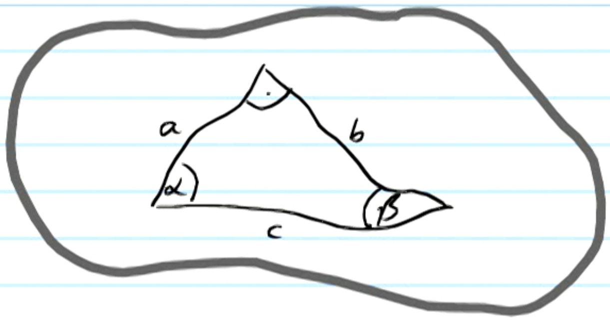
GR for Cosmology, Achim Kempf — Lecture 8

te Title

# How to describe the "shape" of a manifold?

## Historically:

E.g., on a potato-shaped surface:




$$a^2 + b^2 \neq c^2$$

$$\alpha + \beta + 90^\circ \neq 180^\circ$$

Helmholtz & Gauß already considered checking for curvature of space this way.

## Recall:

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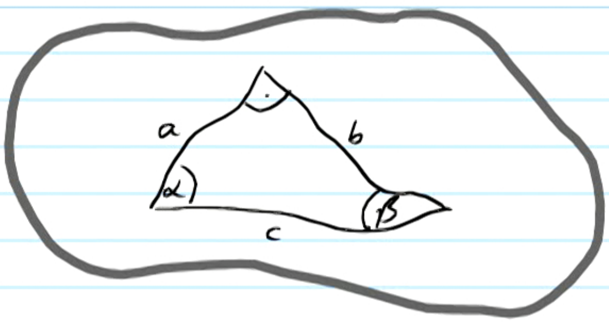


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# Recall:

Defined $g_{\mu\nu}(x)$

$\Rightarrow$ infinitesimal distances $\Rightarrow$ finite distances $\Rightarrow$ shape

# Alternative idea:

A manifold's shape, i.e., its curvature, also reveals itself in the nontriviality of the parallel transport of vectors on the manifold:

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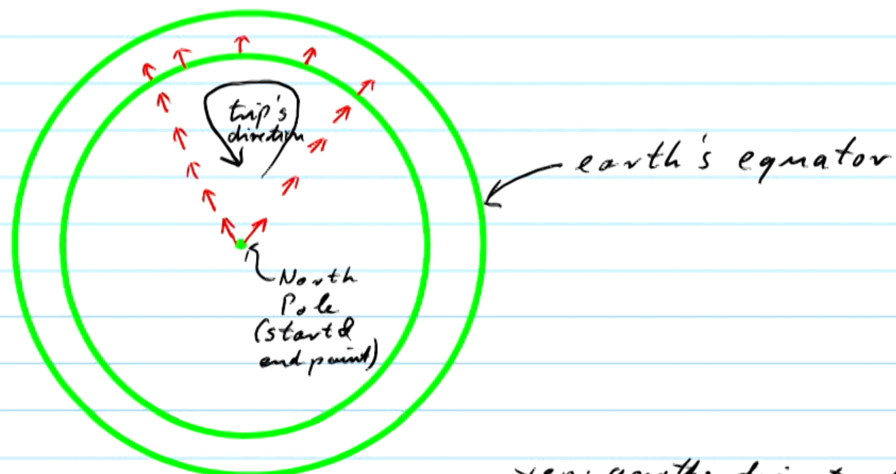
# Alternative idea:

A manifold's shape, i.e., its curvature, also reveals itself in the nontriviality of the parallel transport of vectors on the manifold:

## Example:

- start with a vector at North Pole.
- parallel transport down to some lower latitude, along that latitude and then back to pole.
- vector will arrive at pole rotated, in spite of having only been parallel transported!




## This motivates:

yep: another derivative!

Try to define local shape through "derivative" of vectors

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nontriviality of the parallel transport of vectors on the manifold:

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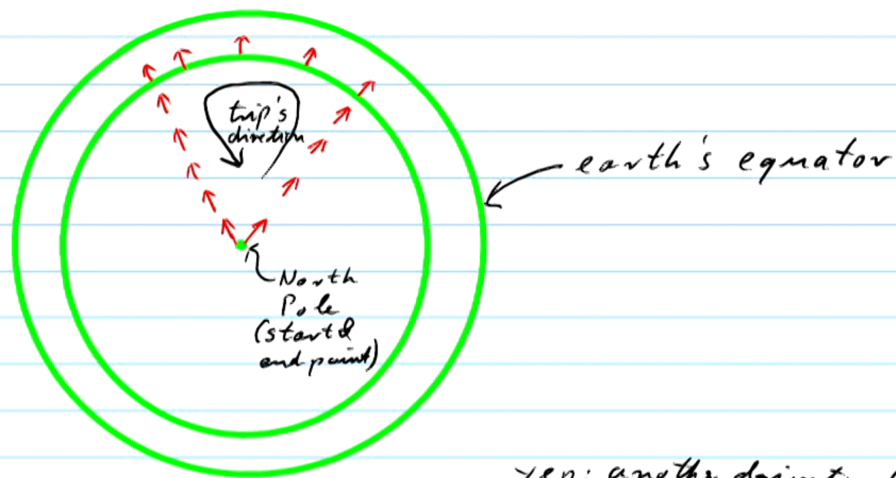


**This motivates:**

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Try to define local shape through "derivative" of vectors with respect to parallel transport!

**Recall:** Lie derivative insensitive to $g$. Indeed, for $\xi_1 = \frac{\partial}{\partial x^1}, \xi_2 = \frac{\partial}{\partial x^2}$,

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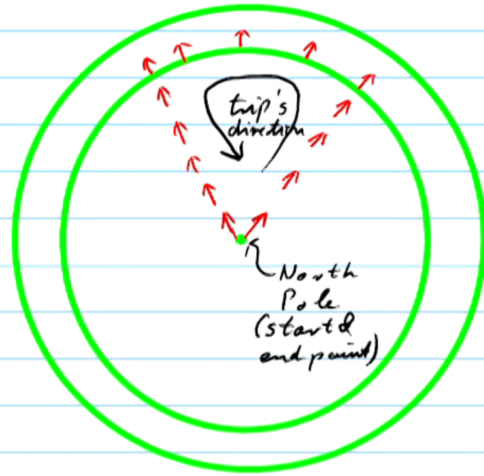


# This motivates:

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Try to define local shape through "derivative" of vectors with respect to parallel transport!

**Recall:** Lie derivative insensitive to $g$. Indeed, for $\xi_1 = \frac{\partial}{\partial x^1}$, $\xi_2 = \frac{\partial}{\partial x^2}$, we have $[L_{\xi_1}, L_{\xi_2}] = L_{[\xi_1, \xi_2]} = 0 \Rightarrow$ No shape info from $L_\xi$!

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# The Covariant Differentiation, $\nabla$:

**Definition:** Any linear map of tangent vector fields

$$\nabla : T^1(M) \times T^1(M) \to T^1(M)$$

$$\nabla : \quad \eta, \xi \to \nabla_\xi \eta$$

obeying

$$\text{(I)} \qquad \nabla_{f\xi}\eta = f\,\nabla_\xi \eta, \quad \forall f \in \mathcal{F}(M)$$

$$\text{(II)} \qquad \nabla_\xi(f\eta) = \overbrace{\xi(f)}^{\nabla_\xi f \,=}\eta + f\,\nabla_\xi \eta \qquad \text{(Leibniz rule)}$$

is called a covariant derivative or affine connection.

**Note:**

For now, let us assume a metric has not (yet) been

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(II) $\nabla_\xi(f\eta) = \xi(f)\,\eta + f\,\nabla_\xi\eta$ (Leibniz rule)

is called a covariant derivative or affine connection.

**Note:**

For now, let us assume a metric has not (yet) been specified, so we are free to choose $\nabla$, and this choice defines the shape of M!

$\nabla$ in a chart: Choose as bases for $T_x(M)$, e.g.: $\left\{\frac{\partial}{\partial x^i}\right\}$

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## $\nabla$ in a chart:

- Choose as bases for $T_x(M)$, e.g.: $\left\{\frac{\partial}{\partial x^i}\right\}$

- Given a covariant derivative $\nabla$, consider its action on basis vectors, such as, e.g.,: $\xi = \frac{\partial}{\partial x^i}$, $\eta = \frac{\partial}{\partial x^j}$:

Recall: $L_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$

$$\boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k{}_{ij}(x) \frac{\partial}{\partial x^k}}$$

The $\Gamma^k{}_{ij}$ are called "Christoffel symbol" or "Connection coefficients"

Thus, via the axioms:

function, vector

$$\nabla_\xi \eta = \nabla_{\xi^i \frac{\partial}{\partial x^i}} \left(\eta^j \frac{\partial}{\partial x^j}\right) \overset{(I)}{=} \xi^i \nabla_{\frac{\partial}{\partial x^i}} \left(\eta^j \frac{\partial}{\partial x^j}\right)$$

$\leftarrow$ recall: $\nabla_\alpha f = \alpha(f) \;\; \forall \alpha \in T_p^1(M),\ f \in F_p(M)$

$$\overset{(II)}{=} \xi^i \left( \eta^j{}_{,i} \frac{\partial}{\partial x^j} + \eta^j \Gamma^k{}_{ij}(x) \frac{\partial}{\partial x^k} \right)$$

$$= \left( \xi^i \eta^k{}_{,i} + \xi^i \eta^j \Gamma^k{}_{ij} \right) \frac{\partial}{\partial x^k}$$

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Recall: $L_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$

$$\boxed{\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k{}_{ij}(x) \frac{\partial}{\partial x^k}}$$

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**Thus, via the axioms:**

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← recall: $\nabla_\alpha f = \alpha(f) \quad \forall \alpha \in T_p^1(M),\ f \in F_p(M)$

$$\overset{(II)}{=} \xi^i \left( \eta^j{}_{,i} \frac{\partial}{\partial x^j} + \eta^j \Gamma^k{}_{ij}(x) \frac{\partial}{\partial x^k} \right)$$

$$= \left( \xi^i \eta^k{}_{,i} + \xi^i \eta^j \Gamma^k{}_{ij} \right) \frac{\partial}{\partial x^k}$$

**Notation:** $\eta^k{}_{;i} := \eta^k{}_{,i} + \eta^j \Gamma^k{}_{ij}$

↑ semi-colon for covariant derivatives

Thus: $\nabla_\xi \eta = \xi^i \eta^k{}_{;i} \frac{\partial}{\partial x^k}$ $(*)$

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**Important:** the $\Gamma^k{}_{ij}$ transform non-tensorially when $x \to \bar{x}$:

On one hand:

because $\frac{\partial}{\partial \bar{x}}$ is tangent vector

$$\nabla_{\partial/\partial\bar{x}^a} \frac{\partial}{\partial \bar{x}^b} = \bar{\Gamma}^c{}_{ab} \frac{\partial}{\partial \bar{x}^c} = \bar{\Gamma}^c{}_{ab} \frac{\partial x^k}{\partial \bar{x}^c}\frac{\partial}{\partial x^k} \qquad (I)$$

On the other hand:

$$\nabla_{\partial/\partial\bar{x}^a} \frac{\partial}{\partial \bar{x}^b} = \nabla_{\underbrace{\frac{\partial x^i}{\partial \bar{x}^a}}_{\in \mathcal{F}(M)} \frac{\partial}{\partial x^i}} \left( \frac{\partial x^j}{\partial \bar{x}^b}\frac{\partial}{\partial x^j}\right)$$ use axiom (b) =>

$$= \frac{\partial x^i}{\partial \bar{x}^a}\, \nabla_{\frac{\partial}{\partial x^i}} \left( \underbrace{\frac{\partial x^j}{\partial \bar{x}^b}}_{\in \mathcal{F}(M)}\frac{\partial}{\partial x^j}\right)$$ use Leibniz rule (c) =>

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**On one hand:**

$$\nabla_{\partial/\partial\bar{x}^a}\frac{\partial}{\partial\bar{x}^b} = \bar{\Gamma}^c{}_{ab}\frac{\partial}{\partial\bar{x}^c} = \bar{\Gamma}^c{}_{ab}\frac{\partial x^k}{\partial\bar{x}^c}\frac{\partial}{\partial x^k} \qquad (\mathrm{I})$$

**On the other hand:**

$$\nabla_{\partial/\partial\bar{x}^a}\frac{\partial}{\partial\bar{x}^b} = \nabla_{\underbrace{\frac{\partial x^i}{\partial\bar{x}^a}}_{\in\mathcal{F}(M)}\frac{\partial}{\partial x^i}}\left(\frac{\partial x^j}{\partial\bar{x}^b}\frac{\partial}{\partial x^j}\right)$$

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use Leibniz rule (c) =>

$$= \frac{\partial x^i}{\partial\bar{x}^a}\left[\left(\frac{\partial}{\partial x^i}\frac{\partial x^j}{\partial\bar{x}^b}\right)\frac{\partial}{\partial x^j} + \frac{\partial x^j}{\partial\bar{x}^b}\Gamma^k{}_{ij}\frac{\partial}{\partial x^k}\right]$$

$$= \left(\frac{\partial}{\partial\bar{x}^a}\frac{\partial x^j}{\partial\bar{x}^b}\right)\frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial\bar{x}^a}\frac{\partial x^j}{\partial\bar{x}^b}\Gamma^k{}_{ij}\frac{\partial}{\partial x^k} \qquad (\mathrm{II})$$

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# On the other hand:

$$\nabla_{\partial/\partial \bar{x}^a} \frac{\partial}{\partial \bar{x}^b} = \nabla_{\underbrace{\frac{\partial x^i}{\partial \bar{x}^a}}_{\in \mathcal{F}(M)} \frac{\partial}{\partial x^i}} \left( \frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial}{\partial x^j} \right) \qquad \text{use axiom (b)} \Rightarrow$$

$$= \frac{\partial x^i}{\partial \bar{x}^a} \nabla_{\frac{\partial}{\partial x^i}} \left( \underbrace{\frac{\partial x^j}{\partial \bar{x}^b}}_{\in \mathcal{F}(M)} \frac{\partial}{\partial x^j} \right) \qquad \text{use Leibniz rule (c)} \Rightarrow$$

$$= \frac{\partial x^i}{\partial \bar{x}^a} \left[ \left( \frac{\partial}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k{}_{ij} \frac{\partial}{\partial x^k} \right]$$

$$= \left( \frac{\partial}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \right) \frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial x^j}{\partial \bar{x}^b} \Gamma^k{}_{ij} \frac{\partial}{\partial x^k} \qquad \text{(II)}$$

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Compare I, II $\Rightarrow$

$$\bar{\Gamma}^c{}_{ab}\frac{\partial x^k}{\partial \bar{x}^c} = \frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b} + \frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k{}_{ij} \qquad \left( \cdot \frac{\partial \bar{x}^r}{\partial x^k} \Rightarrow \right)$$

$$\Rightarrow \quad \boxed{\bar{\Gamma}^r{}_{ab} = \underbrace{\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b}} + \underbrace{\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k{}_{ij}}}$$

This term is indep. of $\Gamma$

$\Rightarrow$ $\Gamma$ can be zero in one coordinate system and nonzero in another!

Only this term would be there, if the $\Gamma^k{}_{ij}$ were tensor coefficients in the $\frac{\partial}{\partial x^i}$, $dx^j$ bases.

(Can be shown to be equivalent)

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$$\Rightarrow \quad \boxed{\bar\Gamma^r{}_{ab} = \underbrace{\frac{\partial \bar x^r}{\partial x^k}\frac{\partial^2 x^k}{\partial \bar x^a \partial \bar x^b}} + \underbrace{\frac{\partial \bar x^r}{\partial x^k}\frac{\partial x^i}{\partial \bar x^a}\frac{\partial x^j}{\partial \bar x^b}\Gamma^k{}_{ij}}}$$

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(Can be shown to be equivalent)

**Physicists' definition of $\nabla$:** Any set of $n^3$ functions $\Gamma^r_{ab}(x)$ which transform this way are defining a cov. derivative $\nabla$.

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# The "absolute" covariant derivative $\nabla$:

Consider the covariant derivative but:
without choosing a direction vector $\xi$:

$$\nabla : T_x(M)^1 \to T_x(M)^1_1$$

$$\nabla : \eta = \eta^i(x)\frac{\partial}{\partial x^i} \to \nabla\eta(x) = \eta^k{}_{;i}(x)\, dx^i \otimes \frac{\partial}{\partial x^k}$$

(I.e. feed the open covariant slot of $\nabla\eta$ with contravariant $\xi$.)

**Indeed:** The contraction of $\nabla\eta$ with $\xi$ yields:

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without choosing a direction vector $\xi$:

$$\nabla : T_x(M)^1 \to T_x(M)^1_1$$

$$\nabla : \eta = \eta^i(x)\frac{\partial}{\partial x^i} \to \nabla\eta(x) = \eta^k{}_{;i}(x)\, dx^i \otimes \frac{\partial}{\partial x^k}$$

(I.e. feed the open covariant slot of $\nabla\eta$ with contravariant $\xi$.)

**Indeed:** The contraction of $\nabla\eta$ with $\xi$ yields:

$$\nabla\eta(\xi) = \eta^k{}_{;i}\, \underbrace{dx^i(\xi)}\, \frac{\partial}{\partial x^k} = \eta^k{}_{;i}\,\xi^i \frac{\partial}{\partial x^k} = \nabla_\xi \eta \qquad \text{ok with } (*)$$

$$dx^i(\xi) = \xi(x^i) = \xi^j \frac{\partial}{\partial x^j} x^i = \xi^j \delta^i_j = \xi^i$$

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**Indeed:** The contraction of $\nabla\eta$ with $\xi$ yields:

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We defined $\nabla$ algebraically. Now, extract the

**Geometric meaning of $\nabla$:** (If $\nabla$ describes infinitesimal parallel transport it should also describe finite parallel transport)

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We defined $\nabla$ algebraically. Now, extract the

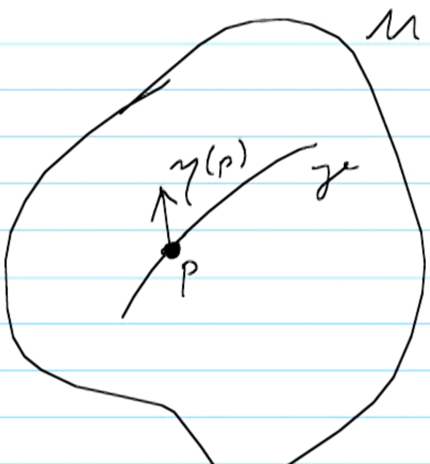
## Geometric meaning of $\nabla$:

(If $\nabla$ describes infinitesimal parallel transport it should also describe finite parallel transport)

**Definition:** Assume $\nabla$ is given. Choose a path $\gamma: \mathbb{R} \to M$.

Then, a tangent vector field $\eta$ is called auto-parallel along $\gamma$, if

$$\nabla_{\dot\gamma}\, \eta = 0$$




i.e. if $\eta$ doesn't change under parallel transport along the path $\gamma$.

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In a chart,

$$\eta = \eta^i(x)\frac{\partial}{\partial x^i}$$

and

$$\gamma : [a,b] \to M$$

$$\gamma : t \to x^i(t)$$

and the tangent vector:

$$\dot{\gamma}(x(t)) = \frac{dx^k}{dt}\frac{\partial}{\partial x^k}$$

Thus: $\nabla_{\dot{\gamma}}\eta = \nabla_{\frac{dx^k}{dt}\frac{\partial}{\partial x^k}}\left(\eta^i\frac{\partial}{\partial x^i}\right) = \frac{dx^k}{dt}\nabla_{\frac{\partial}{\partial x^k}}\left(\eta^i\frac{\partial}{\partial x^i}\right)$

$$= \frac{dx^k}{dt}\left(\frac{\partial \eta^i}{\partial x^k}\frac{\partial}{\partial x^i} + \eta^i\Gamma^j{}_{ki}\frac{\partial}{\partial x^j}\right)$$

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and

$$\gamma : [a,b] \to \mathcal{M}$$

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$$= \underbrace{\frac{dx^k}{dt}\left(\frac{\partial\eta^i}{\partial x^k}\right.}\left.\frac{\partial}{\partial x^i} + \eta^i\Gamma^j{}_{ki}\frac{\partial}{\partial x^j}\right)$$

$$= \left(\frac{d\eta^i}{dt} + \frac{dx^k}{dt}\eta^j\Gamma^i{}_{kj}\right)\frac{\partial}{\partial x^i} \overset{!}{=} 0$$

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$\Rightarrow$ $\eta$ autoparallel to $\gamma$ means:

$$\frac{d\eta^i}{dt} + \eta^j \frac{dx^k}{dt} \Gamma^i{}_{kj} = 0$$

I.e. this is the condition for the vectors of $\eta$ being parallel translates of each other, along $\gamma$.

## Conclusion:

This is 1st order ODEs for $\eta$. Thus:

Initial condition $\eta(\gamma(0)) \Rightarrow$ solution $\eta(\gamma(t))$ exists at least locally

$\Rightarrow$ **Proposition:**

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## Conclusion:

This is 1st order ODEs for $\eta$. Thus:

Initial condition $\eta(\gamma(0)) \Rightarrow$ solution $\eta(\gamma(t))$ exists at least locally

## Proposition:

Given a path $\gamma : [t, s] \to \mathcal{M}$, the autoparallel transport of a tangent vector $\eta$ at $\gamma(t)$ to $\gamma(s)$ is unique.

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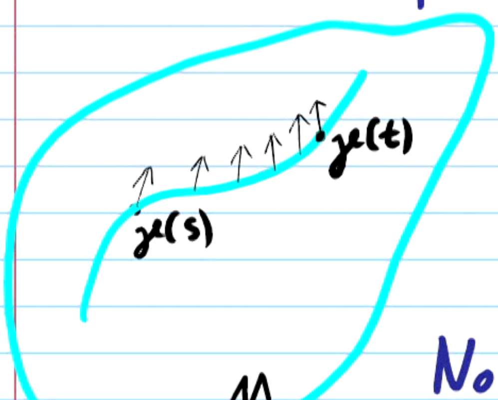
I.e., the path $\gamma$ defines a parallel transport map $\tau$:

$$\tau(t,s):\ T'_{\gamma(t)} \to T'_{\gamma(s)}$$

$$\tau(t,s):\ \eta(\gamma(t)) \to \eta(\gamma(s))$$

- Q: Can one use $\tau$ to obtain $\nabla$ as a Newton-Leibniz limit?

- Proposition: (for the proof, see e.g. the text by Straumann)




$$\nabla_{\dot{\gamma}}\, \eta(\gamma(t)) = \frac{d}{ds}\Big|_{s=t} \tau(s,t)\big(\eta(\gamma(s))\big)$$

Note: Since we can choose paths with arbitrary $\dot{\gamma}$

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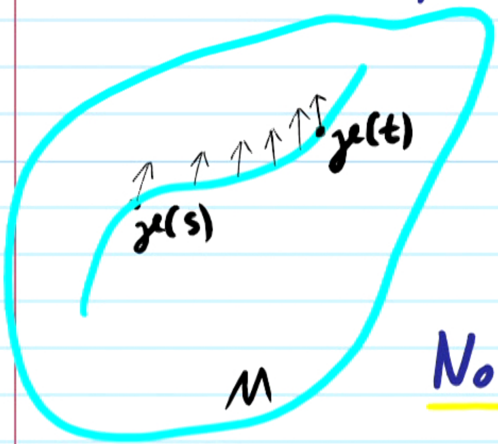
**Proposition:** (for the proof, see e.g. the text by Straumann)




$$\nabla_{\dot\gamma}\,\eta(\gamma(t)) = \frac{d}{ds}\Big|_{s=t}\tau(s,t)\big(\eta(\gamma(s))\big)$$

**Note:** Since we can choose paths with arbitrary $\dot\gamma$ this equation can be used as a geometric definition of $\nabla$.

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## $\nabla$ for arbitrary tensors:

- The parallel transport map $\tau(s,t)$ transports tangent vectors $\eta$ from $\gamma(s)$ to $\gamma(t)$.

- **Definition:** $\tau(s,t)$ also parallel transports the dual vectors $\omega$, namely so that contraction is conserved:

$$\boxed{\underbrace{\tau(\omega)}_{\text{parallel transported } \omega}\Big(\underbrace{\tau(\eta)}_{\text{parallel transported } \eta}\Big) = \omega(\eta)} \qquad (C)$$

- Extension of $\tau$ to tensor products:

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tangent vectors $\eta$ from $\gamma(s)$ to $\gamma(t)$.

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$$\boxed{\underbrace{\tau(\omega)}_{\substack{\text{parallel}\\ \text{transported } \omega}}\Big(\underbrace{\tau(\eta)}_{\substack{\text{parallel}\\ \text{transported } \eta}}\Big) = \omega(\eta)} \qquad (C)$$

- Extension of $\tau$ to tensor products:

$$\tau(S_1 \otimes S_2) := \tau(S_1) \otimes \tau(S_2) \qquad (T)$$

$S_1$ and $S_2$ are tensors of arbitrary rank.

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## Definition:

arbitrary tensor, arb. point $\in M$, arb. tangent vector

$$\nabla_\xi S(p) := \nabla_{\dot\gamma} S(\gamma(t))\Big|_{t=0}$$

$$:= \frac{d}{dt}\Big|_{t=0} \tau(t,0)\big(S(\gamma(t))\big)$$

here, $\gamma$ is any path through $p$ obeying:

$$\dot\gamma(0) = \xi(p)\ , \quad \gamma(0) = p$$

Exercise:

Show that when $S$ is a scalar function $S \in \mathcal{F}(M)$, then:

$$\nabla_\xi S = \xi(S) = \xi^i \frac{\partial}{\partial x^i} S$$

## Absolute covariant derivative:

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# Absolute covariant derivative:

(for abs. derivative one is not specifying the direction.)

$$(\nabla S')(\underbrace{\eta_1,\dots,\eta_p,\omega_1,\dots,\omega_r,\xi}_{\text{fed to } \nabla S \text{ which is } (r,p) \text{ tensor}}) := \nabla_\xi S'(\eta_1,\dots,\eta_p,\omega_1,\dots,\omega_r)$$

## Properties of $\nabla$:

* $\nabla$ is a derivation: (because $\nabla$ inherits the Leibniz rule from $\frac{d}{}$)

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# Properties of $\nabla$:

* $\nabla$ is a derivation: (because $\nabla$ inherits the Leibniz rule from $\frac{d}{ds}$)

$$\nabla_\xi(S_1\otimes S_2) = \frac{d}{ds}\Big|_{s=t} \overset{\tau(s,t)}{\tau}\Big(\overset{S_1(\gamma(s))}{S_1}\otimes \overset{S_2(\gamma(s))}{S_2}\Big) = \frac{d}{ds}\Big|_{s=t}\tau(S_1)\otimes\tau(S_2)$$

$$= \left[\frac{d}{ds}\Big|_{s=t}\tau(S_1)\right]\otimes \tau(S_2)\Big|_{s=t} + \tau(S_1)\Big|_{s=t}\otimes \frac{d}{ds}\Big|_{s=t}\tau(S_2)$$

$$= (\nabla_\xi S_1)\otimes S_2 + S_1\otimes\nabla_\xi S_2 \qquad (A)$$

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tangent vectors $\eta$ from $\gamma(s)$ to $\gamma(t)$.

- **Definition:** $\tau(s,t)$ also parallel transports the dual vectors $\omega$, namely so that contraction is conserved:

$$\boxed{\underbrace{\tau(\omega)}_{\substack{\text{parallel}\\ \text{transported } \omega}}\Big(\underbrace{\tau(\eta)}_{\substack{\text{parallel}\\ \text{transported } \eta}}\Big) = \omega(\eta)} \qquad (C)$$

- Extension of $\tau$ to tensor products:

$$\tau(S_1 \otimes S_2) := \tau(S_1) \otimes \tau(S_2) \qquad (T)$$

$S_1$ and $S_2$ are tensors of arbitrary rank.

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# Action of $\nabla$ on tensors in a chart?

- Recall: $\boxed{\nabla_\xi \frac{\partial}{\partial x^i} = \xi^l \Gamma^k{}_{li} \frac{\partial}{\partial x^k}}$

- Problem: Find $\nabla_\xi dx^i = ?$

→ • Consider $\eta \otimes \omega$.

tangent vector field ($\eta$)

cotangent vector field ($\omega$)

• Differentiate:

$$\nabla_\xi(\eta \otimes \omega) \overset{(A)}{=} (\nabla_\xi \eta) \otimes \omega + \eta \otimes \nabla_\xi \omega$$

Same strategy will be used below

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- Recall: $\boxed{\nabla_\xi \frac{\partial}{\partial x^i} = \xi^l \Gamma^k{}_{li} \frac{\partial}{\partial x^k}}$

- Problem: Find $\nabla_\xi dx^i = ?$

tangent vector field

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$\rightsquigarrow$ • Consider $\eta \otimes \omega$.

• Differentiate:

$$\nabla_\xi(\eta \otimes \omega) \overset{(A)}{=} (\nabla_\xi \eta) \otimes \omega + \eta \otimes \nabla_\xi \omega$$

• Contract: (use that $\nabla_\xi$ and contraction commute)

Same strategy will be used below for general tensors.

(by exercise)

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**Problem**: Find $\nabla_\xi dx^i = ?$

tangent vector field

cotangent vector field

- Consider $\eta \otimes \omega$.
- Differentiate:

$$\nabla_\xi(\eta \otimes \omega) \overset{(A)}{=} (\nabla_\xi \eta) \otimes \omega + \eta \otimes \nabla_\xi \omega$$

- Contract: (use that $\nabla_\xi$ and contraction commute)

(by exercise above)

$$\xi(\omega(\eta)) = \nabla_\xi(\underbrace{\omega(\eta)}_{\text{scalar function}}) = \omega(\nabla_\xi \eta) + (\nabla_\xi \omega)(\eta)$$

Same strategy will be used below for general tensors.

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will be used below for general tensors.

(by exercise above)

- **Contract:** (use that $\nabla_\xi$ and contraction commute)

$$\xi(\omega(\eta)) = \nabla_\xi(\underbrace{\omega(\eta)}_{\text{scalar function}}) = \omega(\nabla_\xi \eta) + (\nabla_\xi \omega)(\eta)$$

(i.e. $\xi(\omega(\eta)) = \omega(\nabla_\xi \eta) + (\nabla_\xi \omega)(\eta)$)

$\Rightarrow$ An expression for $\nabla_\xi(\omega)(\eta)$:

$$(\nabla_\xi \omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_\xi \eta) \qquad (*)$$

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⇒ An expression for $\nabla_\xi(\omega)(\eta)$:

$$(\nabla_\xi \omega)(\eta) = \xi(\omega(\eta)) - \omega(\nabla_\xi \eta) \qquad (*)$$

Now: Choose $\omega := dx^j$ and $\eta := \frac{\partial}{\partial x^i}$

$$\Rightarrow \quad (\nabla_\xi dx^j)\left(\frac{\partial}{\partial x^i}\right) = \underbrace{\xi\Big(\overbrace{\left\langle dx^j, \frac{\partial}{\partial x^i}\right\rangle}^{\delta^j_i \,=\, \text{const.}}\Big)}_{=0} - \left\langle dx^j, \nabla_\xi \frac{\partial}{\partial x^i}\right\rangle$$

$$= -\left\langle dx^j, \xi^l \Gamma^k_{li} \frac{\partial}{\partial x^k}\right\rangle$$

$$= -\xi^l \Gamma^j_{li}$$

Notation: $\langle \omega, \xi \rangle = \omega(\xi)$ (inner product, contraction)

$$\Rightarrow \quad \boxed{\nabla_\xi dx^j = -\xi^l \Gamma^j_{li}\, dx^i}$$

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**Now:** Choose $\omega := dx^j$ and $\eta := \frac{\partial}{\partial x^i}$

$\left(\begin{array}{l}\text{Notation:} \\ \langle \omega, \xi \rangle = \omega(\xi) \\ \text{(inner product, contraction)}\end{array}\right)$

$$\Rightarrow (\nabla_\xi dx^j)\left(\frac{\partial}{\partial x^i}\right) = \xi\Big(\underbrace{\big(\overbrace{\langle dx^j, \tfrac{\partial}{\partial x^i}\rangle}^{\delta^j_i = \text{const.}}\big)}_{=0}\Big) - \langle dx^j, \nabla_\xi \tfrac{\partial}{\partial x^i}\rangle$$

$$= -\langle dx^j, \xi^l \Gamma^k_{li} \tfrac{\partial}{\partial x^k}\rangle$$

$$= -\xi^l \Gamma^j_{li}$$

$$\Rightarrow \boxed{\nabla_\xi dx^j = -\xi^l \Gamma^j_{li}\, dx^i}$$

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**For general tensors:** (by exactly same strategy as above but applied to multiple tensor products, we obtain:)

$$\nabla_\xi S(\eta_1, \ldots, \eta_r, \omega_1, \ldots, \omega_s)$$ (as in Eq. (*) above)

$$= \xi\big(S(\eta_1 \ldots \eta_r, \omega_1 \ldots \omega_s)\big)$$

$$- S(\nabla_\xi \eta_1, \eta_2, \ldots, \eta_r, \omega_1, \ldots \omega_s) - \ldots$$

$$- S(\eta_1, \nabla_\xi \eta_2, \ldots)$$

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**For general tensors:** (by exactly same strategy as above but applied to multiple tensor products, we obtain:)

$$\nabla_\xi S(\eta_1,\dots,\eta_r,\omega_1,\dots,\omega_s) \qquad \text{(as in Eq. } (*) \text{ above)}$$

$$= \xi\big(S(\eta_1\dots\eta_r,\omega_1\dots\omega_s)\big)$$

$$- S(\nabla_\xi \eta_1,\eta_2,\dots,\eta_r,\omega_1,\dots\omega_s) - \dots$$

$$- S(\eta_1,\dots,\nabla_\xi \eta_r,\omega_1,\dots\omega_s)$$

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that $\nabla_\xi S'$ reads

$$\nabla_\xi S' = \xi^k S^{i_1 \dots i_p}_{j_1 \dots j_q ; k} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}$$

with:

$$S^{i_1 \dots i_p}_{j_1 \dots j_q ; k} := S^{i_1 \dots i_p}_{j_1 \dots j_p , k} + \Gamma^{i_1}_{k \ell} S^{\ell i_2 \dots i_p}_{j_1 \dots j_q}$$

$$+ \dots + \Gamma^{i_p}_{k \ell} S^{i_1 \dots \ell}_{j_1 \dots j_q}$$

$$- \Gamma^{\ell}_{k j_1} S^{i_1 \dots i_p}_{\ell \dots j_q}$$

$$- \dots - \Gamma^{\ell}_{k j_q} S^{i_1 \dots i_p}_{j_1 \dots \ell}$$

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