Title: Two applications of homotopy theory to physics

Speakers: Dan Freed

Series: Colloquium

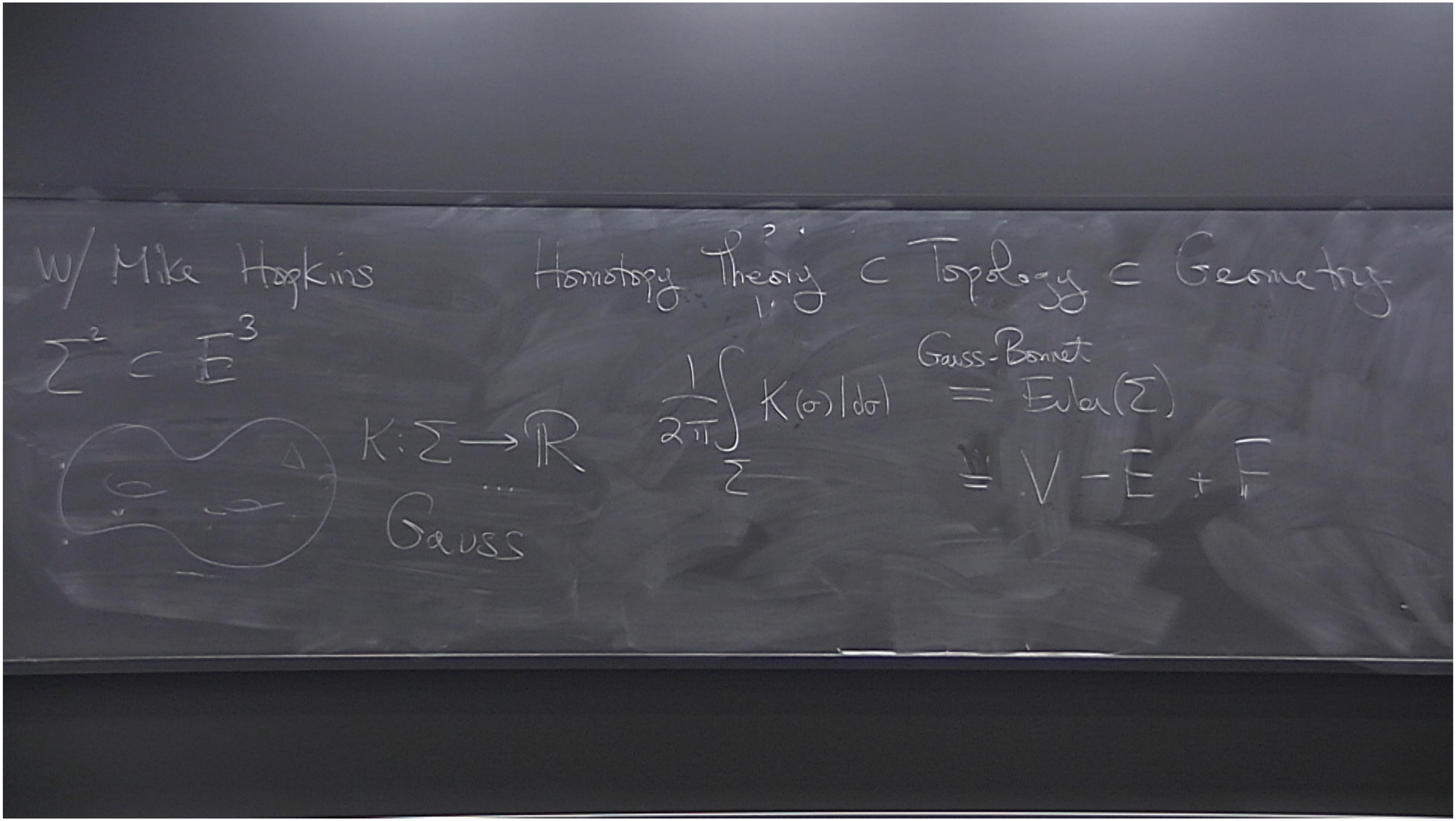
Date: September 18, 2019 - 2:00 PM

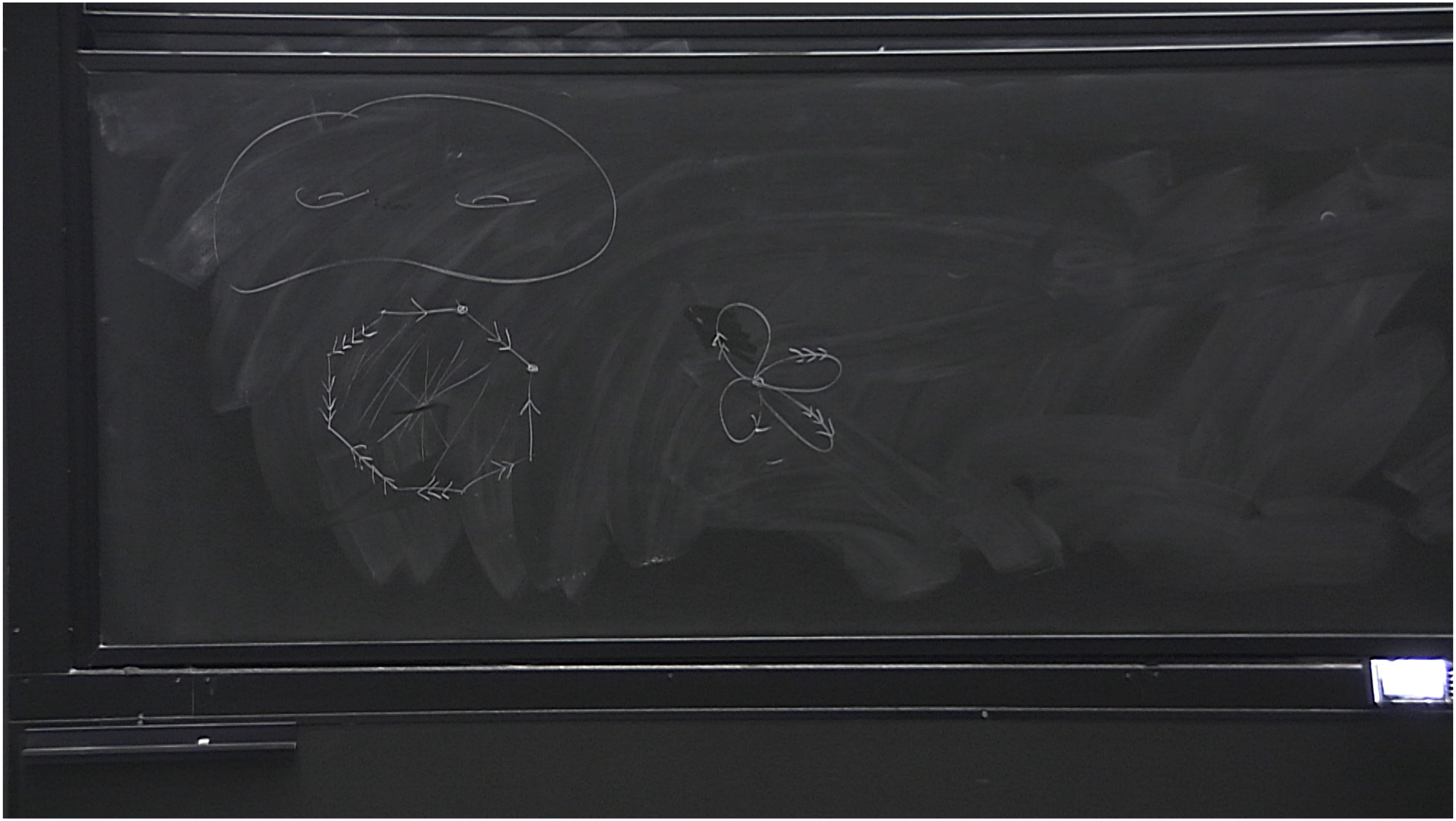
URL: http://pirsa.org/19090112

Abstract: Topology illuminates properties of geometric spaces which are independent of scale. Scale-independent features of physical systems play an important role, for example when deducing the large-scale behavior from a small-scale description. After an introduction to basic topological ideas, I will discuss two joint results with Mike Hopkins, one an application to string theory and the other an application to condensed matter theory.

w/ Mike Hopkins

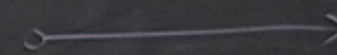
Homotopy Theory $\subset$ Topology $\subset$ Geometry

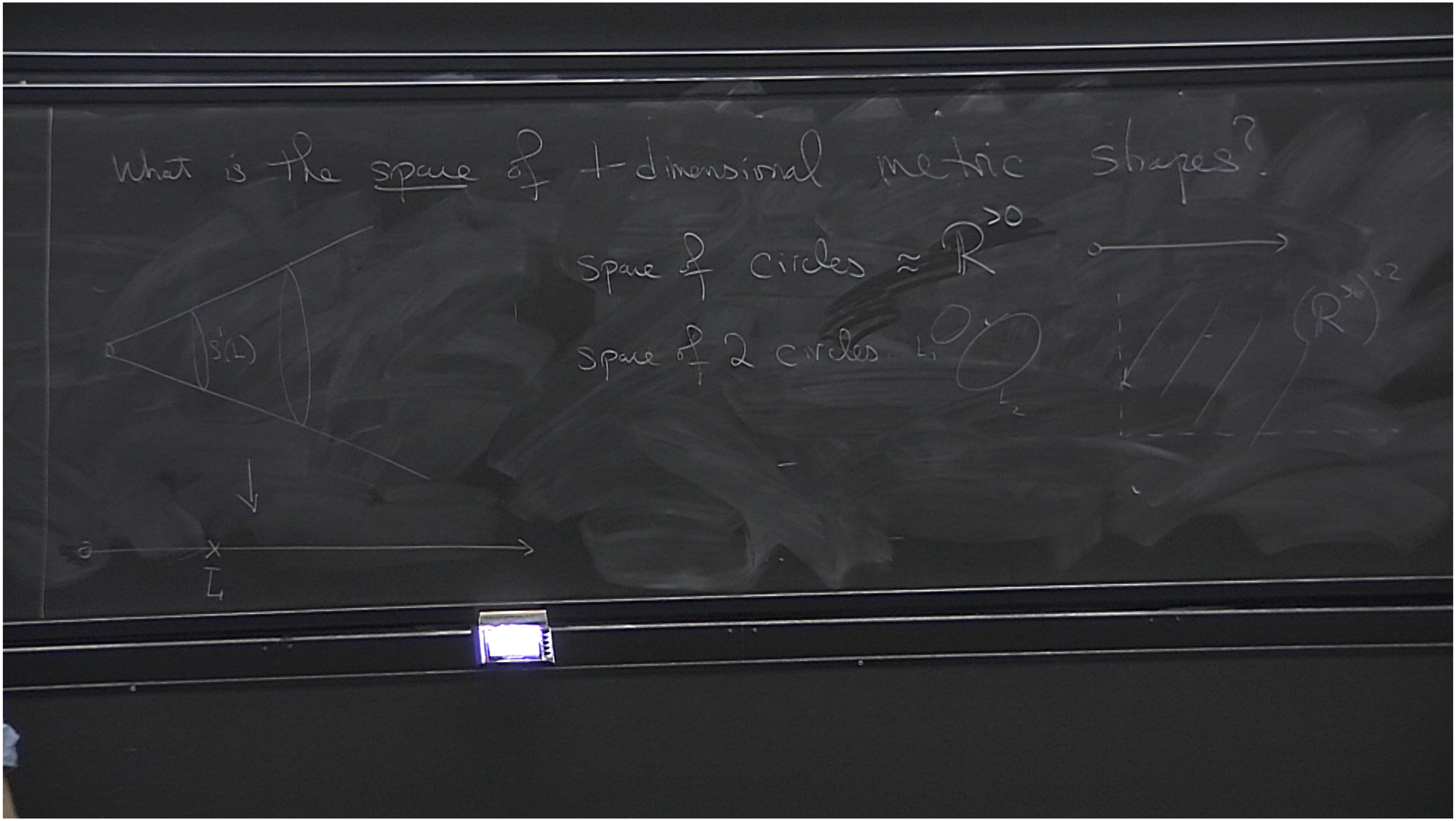

W/ Mike Hopkins
Homotopy Theory ⊂ Topology ⊂ Geometry
$\Sigma^2 \subset E^3$
Gauss-Bonnet
$\frac{1}{2\pi}\int_\Sigma K(\sigma)|d\sigma| = \mathrm{Euler}(\Sigma)$
$= V - E + F$
$K : \Sigma \to \mathbb{R}$
Gauss

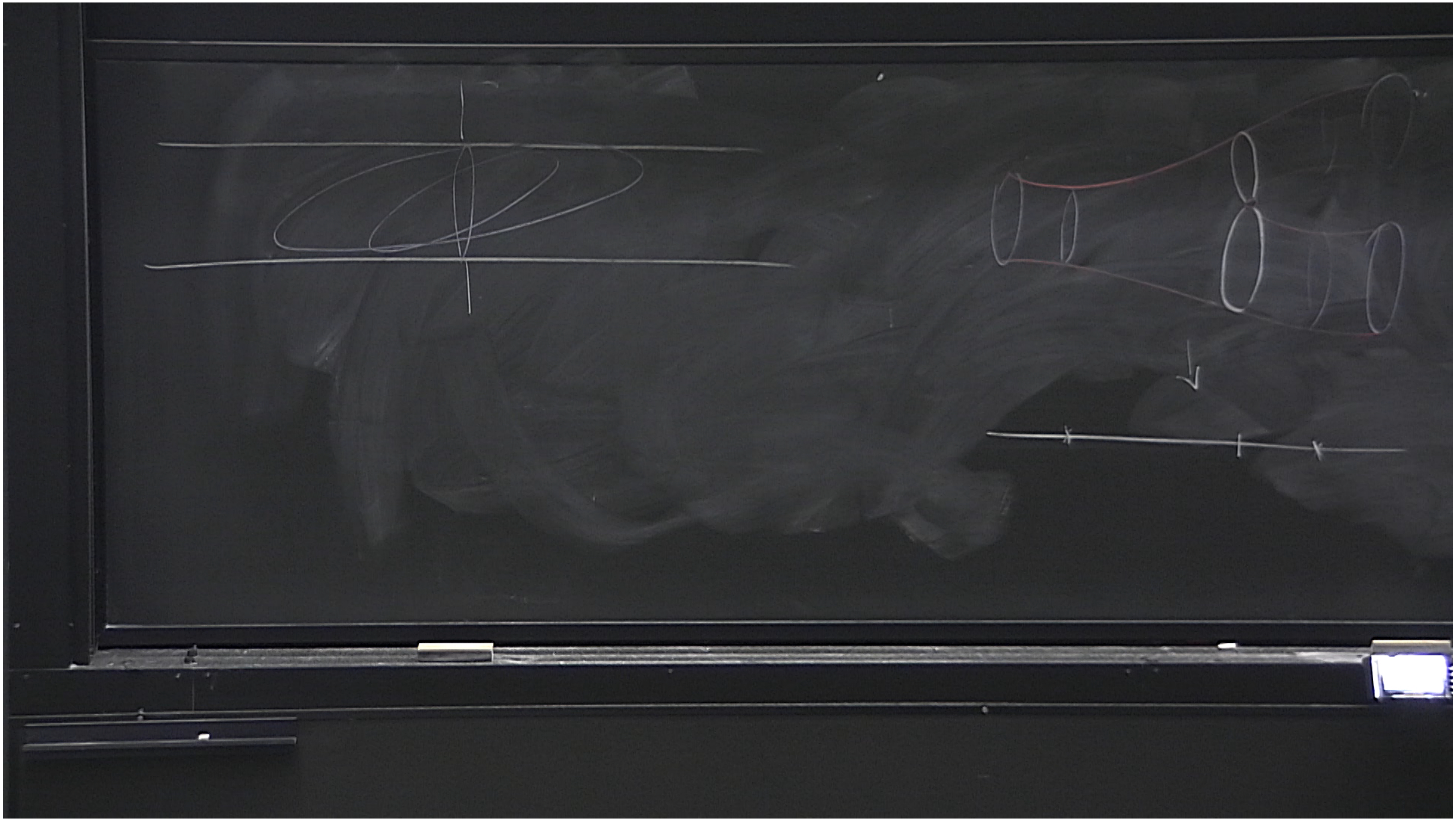


What is the space of 1-dimensional metric shapes?

space of circles $\approx \mathbb{R}^{>0}$




What is the space of 1-dimensional metric shapes?
space of circles ≈ R^{>0}
space of 2 circles
S^1(L)
L_1
L_2
(R^{>0})^{×2}



8

$\pi_0 \mathcal{M} \xrightarrow{\cong} \mathbb{Z}^{\geq 0}$

Gap condition

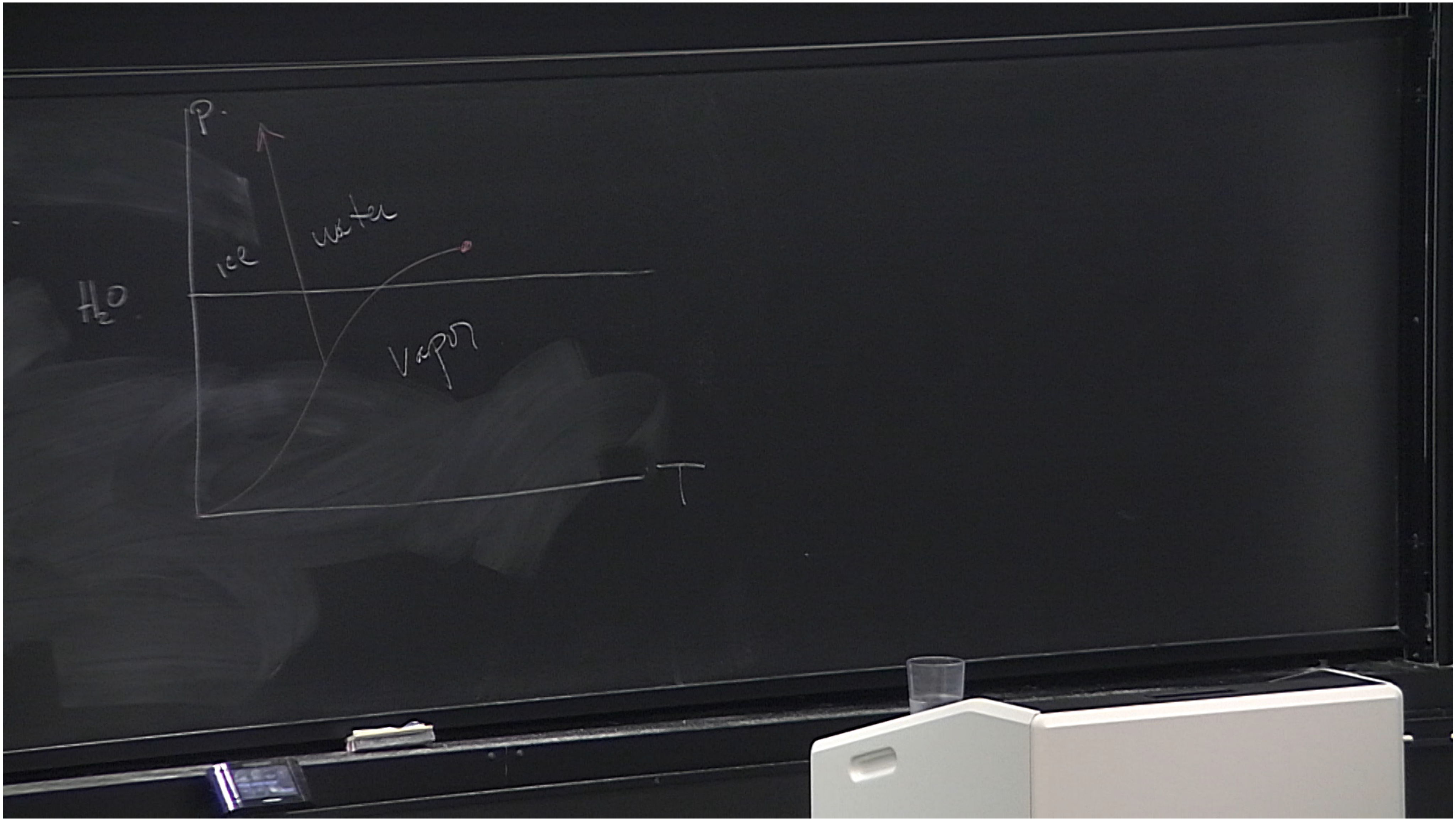
Q: Classify gapped / invertible phases of matter.

Moduli space of QM systems.

Fix $<$ dimension, symmetry type

Compute: $\pi_0 \mathcal{M}(n, H)$

Htpy type of $\mathcal{M}(n, H)$


P
ice
water
H_2O
vapor
T

Move to a problem in field thy.

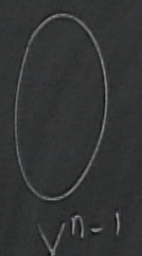
1. Phase determined by long dist. behavior
2. Long dist behavior, if gapped, is well-approximated by a topological* field thy.

<u>Q</u>: Compute $\pi_0 \mathcal{M}(n, H)$

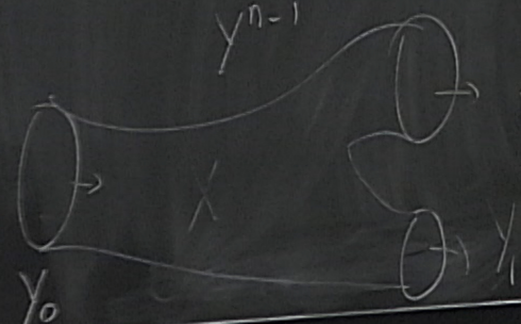
field thy

Top field thy

$$F : \left(\mathrm{Bord}_{\langle n-1,\, n\rangle}, \amalg\right) \longrightarrow \left(\mathrm{Vect}_{\mathbb{C}}, \otimes\right)$$



$\longmapsto F(Y)$ vector space.



$$\longmapsto \left(F(X) : F(Y_1) \longrightarrow F(Y_2)\right)$$

- Locality ~ Extended
- Unitarity
- Invertible.

$$
\begin{tikzcd}
\mathrm{Bord}_n \arrow[r, "F"] \arrow[d] \arrow[dr, dashed] & \mathcal{C} \\
|\mathrm{Bord}_n| \arrow[r, "F"] & \mathcal{C}^\times \arrow[u, hook]
\end{tikzcd}
$$

∞-loop spaces $\{\, |\mathrm{Bord}_n|, \mathcal{C}^\times$

② Anomalies. 1) An <u>anomaly</u> is* an invertible $(n+1)$-dim'l field thy.

2) F has anomaly α if it is a map

$F : 1 \longrightarrow \alpha$.

	F
$\underline{1}$	α

Main Theorem

$\mathcal{M}'_{\text{top}}(n, H_n) :=$ moduli space of reflection positive invertible n-dimensional extended topological field theories with symmetry group H_n

Theorem (F.-Hopkins): There is a 1:1 correspondence

$$\pi_0 \mathcal{M}'_{\text{top}}(n, H_n) \cong [MTH, \Sigma^{n+1} I\mathbb{Z}]_{\text{tor}}$$

spacetime dimension

symmetry group

Input Data

Thom spectrum

Stabilized symmetry group

Anderson dual to sphere spectrum

torsion subgroup

Main Theorem

$\mathcal{M}'_{\cancel{\text{top}}}(n, H_n) :=$ moduli space of reflection positive invertible n-dimensional extended ~~topological~~ field theories with symmetry group H_n

Theorem (F.-Hopkins): There is a 1:1 correspondence

$$\pi_0\mathcal{M}'_{\text{top}}(n, H_n) \cong [MTH, \Sigma^{n+1}I\mathbb{Z}]_{\text{tor}}$$

Conjecture (F.-Hopkins): There is a 1:1 correspondence

$$\pi_0\mathcal{M}'(n, H_n) \cong [MTH, \Sigma^{n+1}I\mathbb{Z}]$$

Time-reversal invariance of M-theory

Parity invariance question (Witten): Can we consistently formulate 11-dimensional M-theory on *unoriented* manifolds?

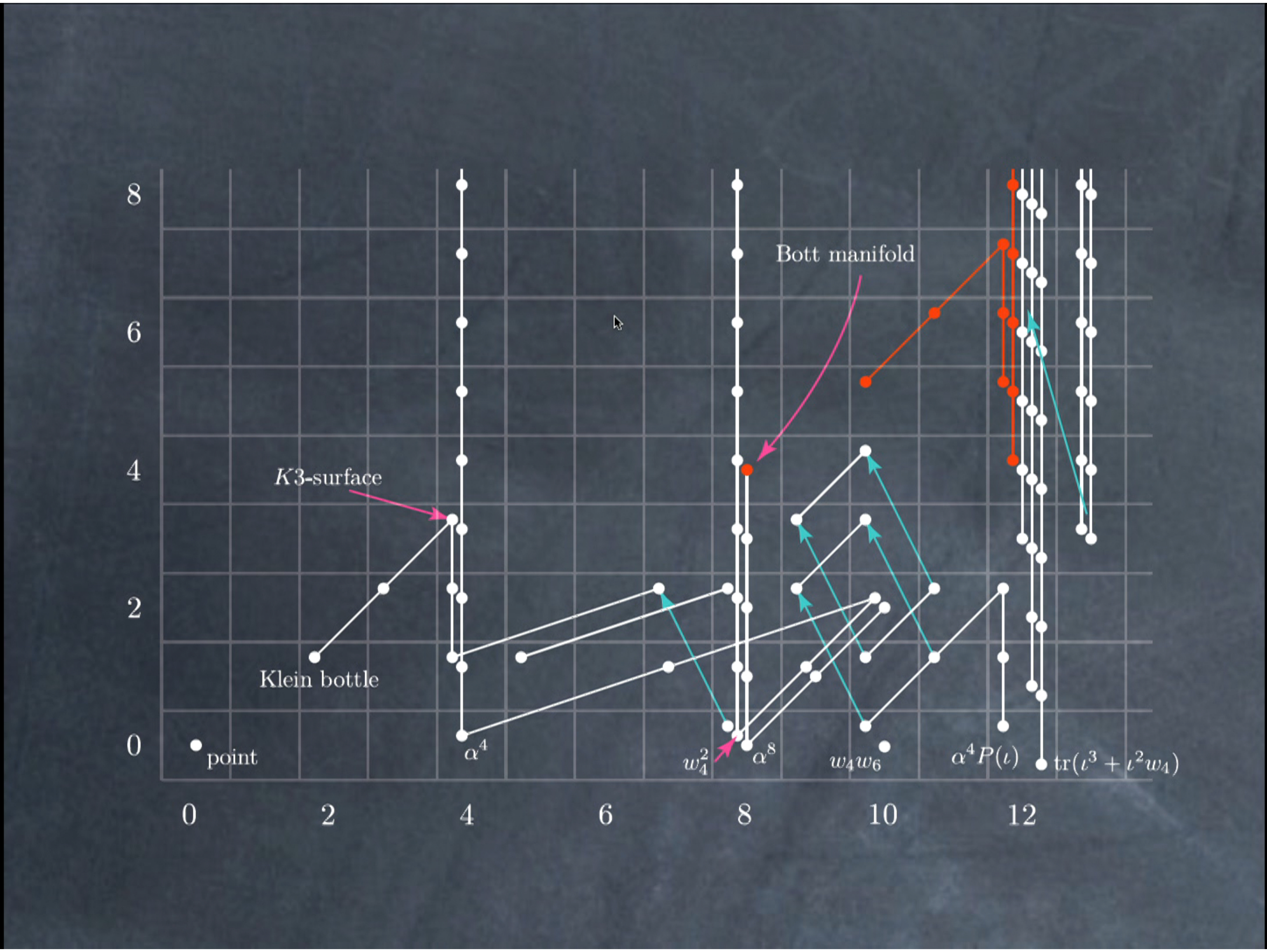
The anomaly α is an invertible 12-dimensional *topological* invertible field theory: $\alpha = \alpha_{RS} \otimes \alpha_C$ where RS = Rarita-Schwinger and $C = C$-field

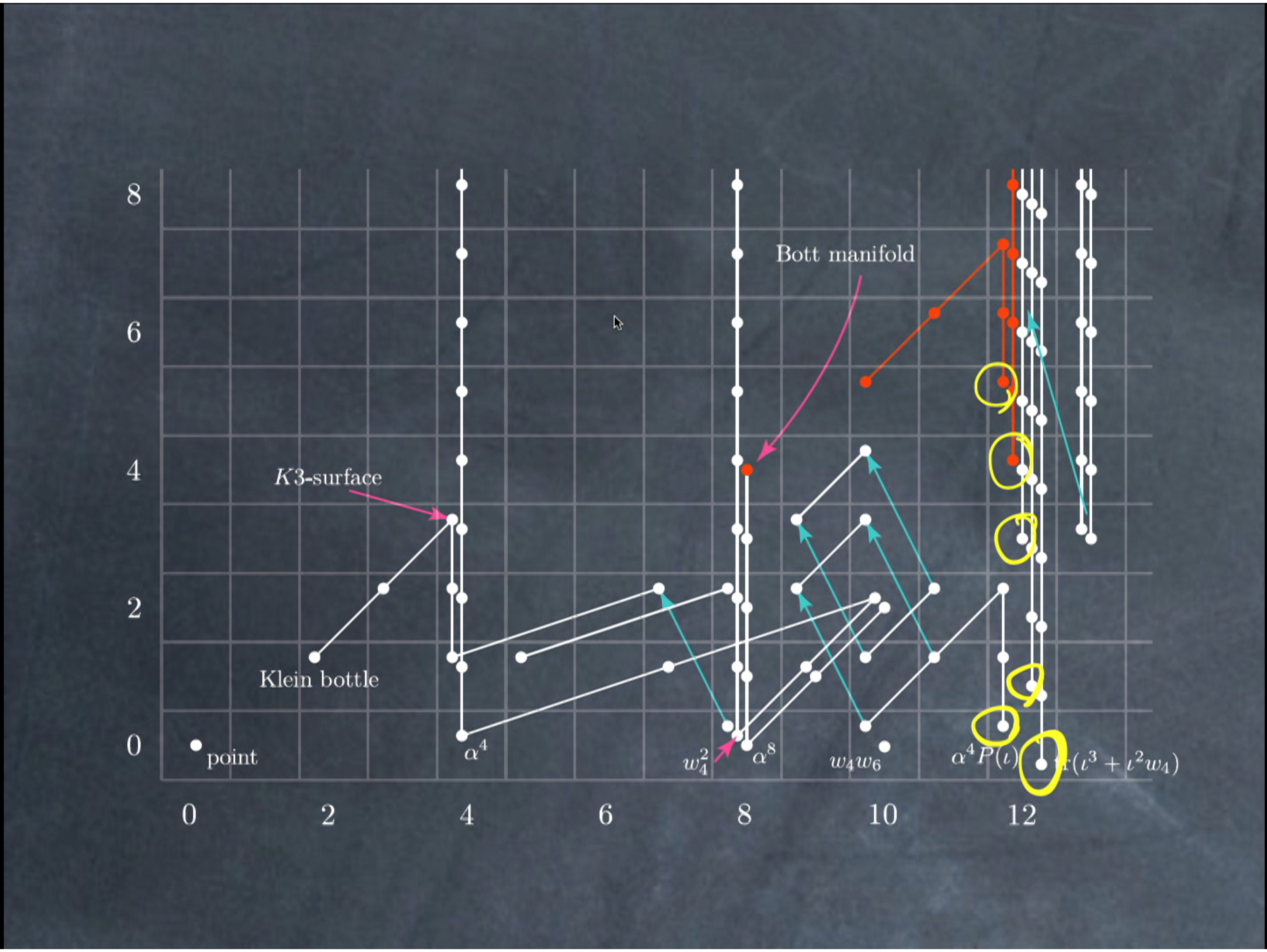
Theorem (F.-Hopkins): The total anomaly α is trivializable

Theorem (F.-Hopkins): The following six $\mathfrak{m}_c$-manifolds generate the group $\pi_{12} M\mathfrak{m}_c \otimes \mathbb{Z}_2$:

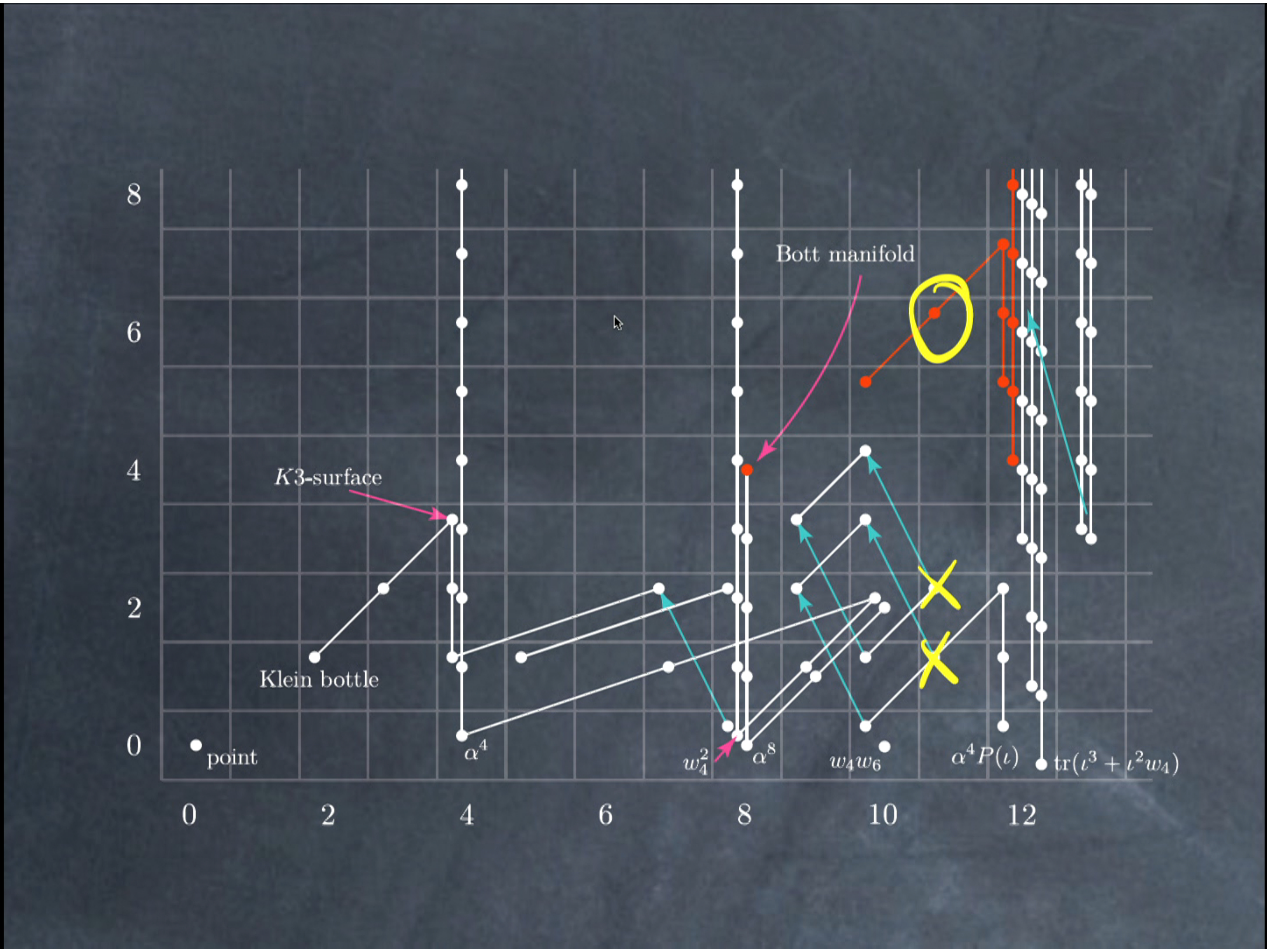
$$(W_0', \tilde{c}_0'), \quad (W_0'', 0), \quad (W_1, \lambda)$$
$$(K \times \mathbb{HP}^2, \lambda), \quad (\mathbb{RP}^4, \tilde{c}'_{\mathbb{RP}^4}) \times B, \quad (\mathbb{RP}^4 \# \mathbb{RP}^4, 0) \times B.$$

Conjecture (F.-Hopkins): There are two trivializations of α


8
6
4
2
0
0
2
4
6
8
10
12
Bott manifold
$K3$-surface
Klein bottle
point
α^4
w_4^2
α^8
$w_4 w_6$
$\alpha^4 P(\iota)$
$\mathrm{tr}(\iota^3 + \iota^2 w_4)$



Bott manifold
K3-surface
Klein bottle
point
α^4
w_4^2
α^8
$w_4 w_6$
$\alpha^4 P(\iota)$
$(\iota^3 + \iota^2 w_4)$
0
2
4
6
8
10
12



Bott manifold
K3-surface
Klein bottle
point
α^4
w_4^2
α^8
$w_4 w_6$
$\alpha^4 P(\iota)$
$\mathrm{tr}(\iota^3 + \iota^2 w_4)$

Invertible topological phases of matter

$\mathcal{M}(n, H_n)$ moduli space of gapped invertible $(n-1)$-dimensional lattice systems with symmetry type H_n

Two principles:
- The deformation class of a quantum system is determined by its low energy behavior
- The low energy physics of a *gapped* system is well-approximated by a *topological** field theory

We approximate discrete nonrelativistic lattice models with continuous relativistic field theories. A mathematical justification would be nice...

We compute using the bordism formula

$$\pi_0\mathcal{M}'(n, H_n) \cong [MTH, \Sigma^{n+1}I\mathbb{Z}]$$

Computations

Class DIII (Pin^+):

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^+) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^+) \longrightarrow$	$\mathrm{coker}\,\Phi$
4	$16\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/16\mathbb{Z}$	0
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

- FF_n is the group of free fermion theories (KO group)
- $TP_n(H) = \pi_0\mathcal{M}'(n, H_n)$ is group of topological phases (Main Thm)
- Φ is the map described above (essentially ABS)
- The FF_n groups are well-known. Many TP_n appear in the condensed matter literature (together with Φ) via other methods

Interacting fermionic topological insulators/superconductors in 3D

Chong Wang and T. Senthil
Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
(Dated: June 1, 2015)

Symmetry Protected Topological (SPT) phases are a minimal generalization of the concept of topological insulators to interacting systems. In this paper we describe the classification and properties of such phases for three dimensional(3D) electronic systems with a number of different symmetries. For symmetries representative of all classes in the famous 10-fold way of free fermion topological insulators/superconductors, we determine the stability to interactions. By combining with results on *bosonic* SPT phases we obtain a classification of electronic 3D SPT phases for these symmetries. In cases with a normal $U(1)$ subgroup we show that this classification is complete. We describe the non-trivial surface and bulk properties of these states. In particular we discuss interesting correlated surface states that are not captured in a free fermion description. We show that in many, but not all cases, the surface can be gapped while preserving symmetry if it develops intrinsic topological order.

CONTENTS

I. INTRODUCTION

Much of our current understanding of topological insulators/superconductors is informed by models of free fermions and their associated band structure[1]. Within this description there is a very mature understanding of the possible such phases in diverse dimensions. A classification of these free fermion topological phases exists[2] yielding results that depend on the global symmetry and the spatial dimensionality. A defining characteristic of such phases is the presence of non-trivial surface states that are protected by the global symmetry.

The free fermion description is clearly the appropriate starting point to discuss the possibility of topological insulators/superconductors in weakly correlated materials. In recent years however attention has turned toward materials with strong electron correlations as possible platforms for similar phenomena. These include the mixed valence compound[3] SmB_6, and iridium oxides on py-

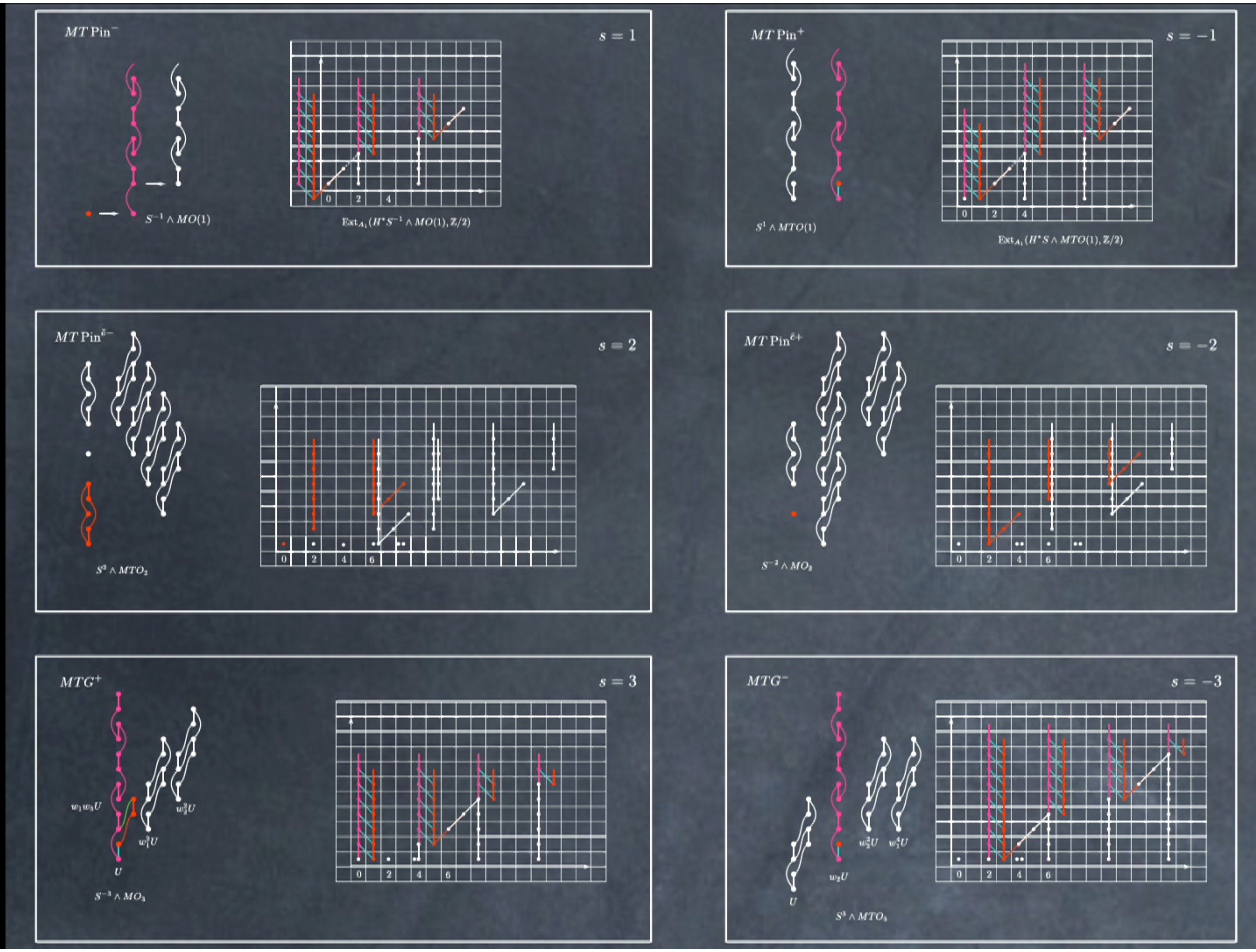
IV. $\mathbb{Z}_2^T$ WITH $\mathcal{T}^2 = -1$: DIII CLASS

In this section we apply the results obtained in Sec.III to superconductors with only time-reversal symmetry (the DIII class). This was recently discussed in Ref. 22 using powerful Walker-Wang methods. We reproduce part of the results there in a physically simpler and constructive approach[34] following the ideas of Ref. 21 and the previous section.

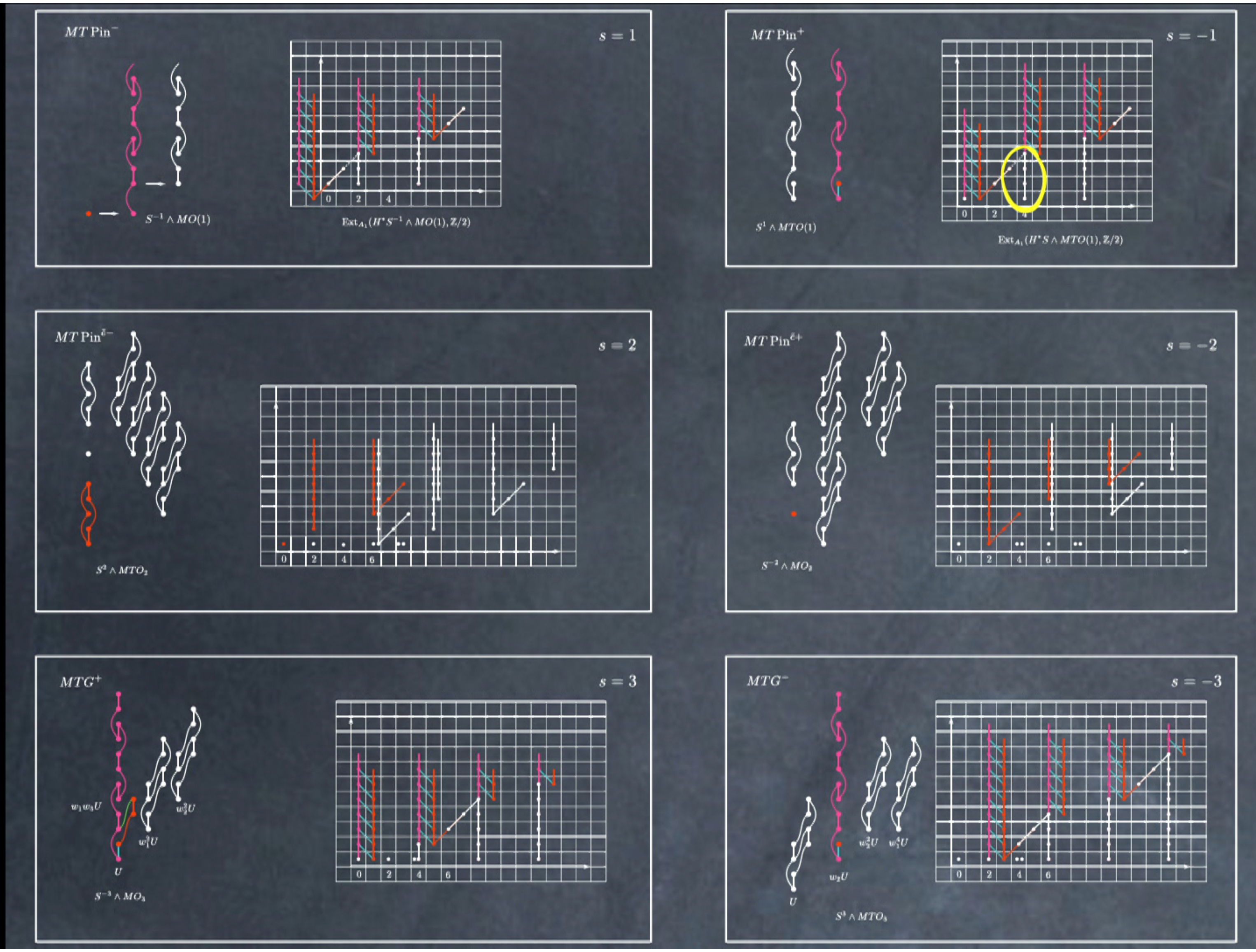
At free fermion level, the DIII class superconductors in 3D are classified by $\mathbb{Z}$, with an integer index ν signifying the number of gapless Majorana cones on the surface protected by time-reversal symmetry:

$$H = \sum_{i=1}^{\nu} \chi_i^\dagger (p_x \sigma_x + p_y \sigma_z) \chi_i. \tag{17}$$

If ν is even ($\nu = 2n$), one can group the Majorana cones into n Dirac cones $\psi_i = \chi_{2i-1} + i\chi_{2i}$, and the theory looks exactly the same as Eq. (9). The $U(1)$ symmetry $\psi \to e^{i\theta}\psi$ is now an emergent symmetry at low energy. We can instead consider the $U(1)$ as a microscopic symmetry, apply the results in Sec.III to obtain interacting gapped surface states, and then break the $U(1)$ symmetry explicitly by adding fermion pairing term. A similar strategy was useful in the Walker-Wang approach[22]. For the $n = 8$ ($\nu = 16$) state, the resulting surface is trivially gapped, and further breaking the $U(1)$ symmetry does not introduce anything nontrivial. Hence the $\mathbb{Z}$ classification from band theory reduces to $\mathbb{Z}_{16}$ with interaction. For the $n = 4$ ($\nu = 8$) state, the resulting surface is topologically ordered, but all the quasi-particles are charge-neutral under the $U(1)$, hence breaking $U(1)$ symmetry does not affect anything either. These establish the $\nu = 16$ state as a trivial one, and the $\nu = 8$ state as equivalent to a boson SPT, which are consistent with the results in Ref. 22. The $n = 2$ ($\nu = 4$) and $n = 1$ ($\nu = 2$) states, however, have surface topological orders involving the $U(1)$ symmetry non-trivially, hence need more careful examination.



MT Pin⁻
s = 1
S⁻¹ ∧ MO(1)
Ext_{A₁}(H*S⁻¹ ∧ MO(1), ℤ/2)
MT Pin⁺
s = −1
S¹ ∧ MTO(1)
Ext_{A₁}(H*S ∧ MTO(1), ℤ/2)
s = 2
S² ∧ MTO₂
s = −2
S⁻² ∧ MO₂
MTG⁺
s = 3
U
S⁻³ ∧ MO₃
MTG⁻
s = −3
U
S³ ∧ MTO₃



$MT\,\mathrm{Pin}^-$
$s = 1$
$S^{-1} \wedge MO(1)$
$\mathrm{Ext}_{A_1}(H^*S^{-1} \wedge MO(1), \mathbb{Z}/2)$
$MT\,\mathrm{Pin}^+$
$s = -1$
$S^1 \wedge MTO(1)$
$\mathrm{Ext}_{A_1}(H^*S \wedge MTO(1), \mathbb{Z}/2)$
$s = 2$
$S^2 \wedge MTO_2$
$s = -2$
$S^{-2} \wedge MO_2$
MTG^+
$s = 3$
U
$S^{-3} \wedge MO_3$
MTG^-
$s = -3$
U
$S^3 \wedge MTO_3$

HOMOTOPY THEORETIC CLASSIFICATION OF SYMMETRY PROTECTED PHASES

JONATHAN A. CAMPBELL

ABSTRACT. We classify a number of symmetry protected phases using Freed-Hopkins' homotopy theoretic classification. Along the way we compute the low-dimensional homotopy groups of a number of novel cobordism spectra

1. INTRODUCTION AND OUTLINE

1.1. **Introduction.** Recently, symmetry protected topological phases (SPTs) have received a great deal of attention. Not only are they interesting phases of matter outside of the Landau symmetry breaking classification, but their realizations in nature would have applications to, for example, quantum computation. Very roughly, two systems are in the same SPT phase if their Hamiltonians are gapped, have an action by a group G, and can be smoothly deformed into one another equiv-

7.9. **$\mathbf{Z}/2m_f$ charge superconductor.** We now compute $M(\mathrm{Spin} \times_{\mathbf{Z}/2} \mathbf{Z}/2m)$ for any m. We first note that the computation for m odd is uninteresting since the extension is trivially split in that case. We are thus reduced to computing $M(\mathrm{Spin} \times_{\mathbf{Z}/2} \mathbf{Z}/2^n)$. The computation proceeds in much the same way as above, but with one technical difference.

Let $G = M(\mathrm{Spin} \times_{\mathbf{Z}/2} \mathbf{Z}/2^n)$. As before, this fits into an extension

$$\mathbf{Z}/2 \to G \to SO \times \mathbf{Z}/2^{n-1}$$

and we get a corresponding pullback diagram

$$\begin{tikzcd} BG \arrow[r] \arrow[d] & B\,\mathrm{Spin} \arrow[d] \\ BSO \times B\mathbf{Z}/2^{n-1} \arrow[r, "(\mathrm{Id},2\xi)"'] & BSO \end{tikzcd}$$

where again, 2ξ is twice the sign representation. This gives us the equivalence

$$MG := M(\mathrm{Spin} \times_{\mathbf{Z}/2} \mathbf{Z}/2^n) \simeq M\,\mathrm{Spin} \wedge (B\mathbf{Z}/2^{n-1})^{2\xi}.$$

We do not have an analogue for Atiyah's succinct identification of the homotopy type of $(B\mathbf{Z}/2^{n-1})^{2\xi}$, but we need only know $H^*(B(\mathbf{Z}/2^n)^{2\xi}; \mathbf{Z}/2)$ as an $\mathcal{A}(1)$-module. For this, the Thom isomorphism suffices. The $\mathcal{A}(1)$-module structure of $H^*((B\mathbf{Z}/2^n)^{2\xi}; \mathbf{Z}/2)$ will be the same as $H^*(B\mathbf{Z}/2^n)$ but missing the bottom two cells.

First, we need the $\mathcal{A}(1)$ structure of the cohomology $H^*(BC_{2^n}; \mathbf{Z}/2)$. We note that by standard computations [23, p.251], $H^*(BC_{2n}; \mathbf{Z}/2^n) \cong \mathbf{Z}/2^n[\alpha, \beta]/(\alpha^2 =$

Invertible phases on a particular space

Motivated by ideas of Kitaev, we deduce

Ansatz: Let Y be a locally compact topological space equipped with the action of a compact Lie group G. Then the group of invertible topological phases on Y of symmetry type (H,ρ) is the Borel-Moore equivariant *homology* group $E^{hG}_{0,BM}(Y)$, where $E = E_{(H,\rho)} = \Sigma^2 I\mathbb{Z}^{MTH}$

Topological crystalline phases are a special case.

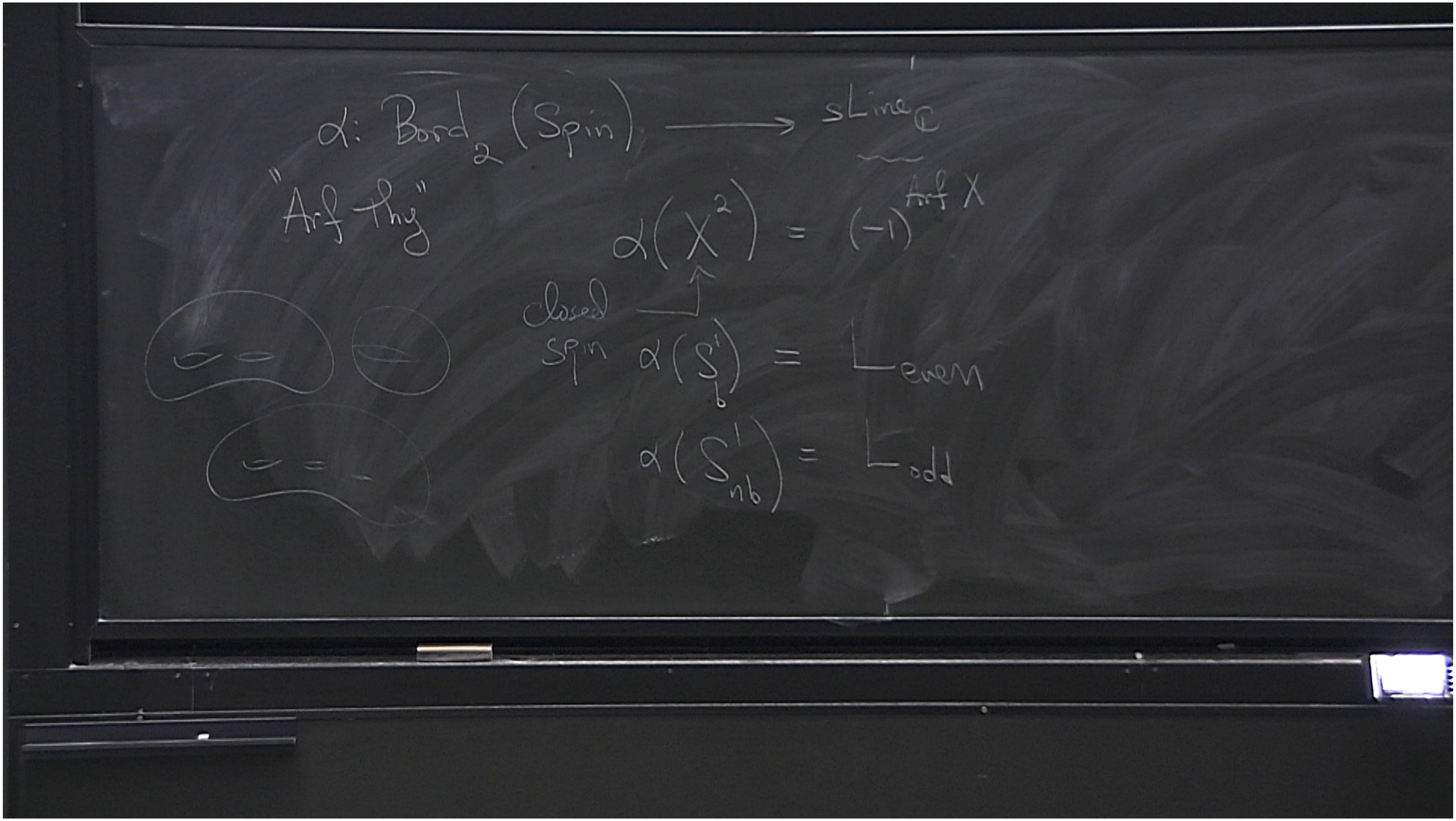
References

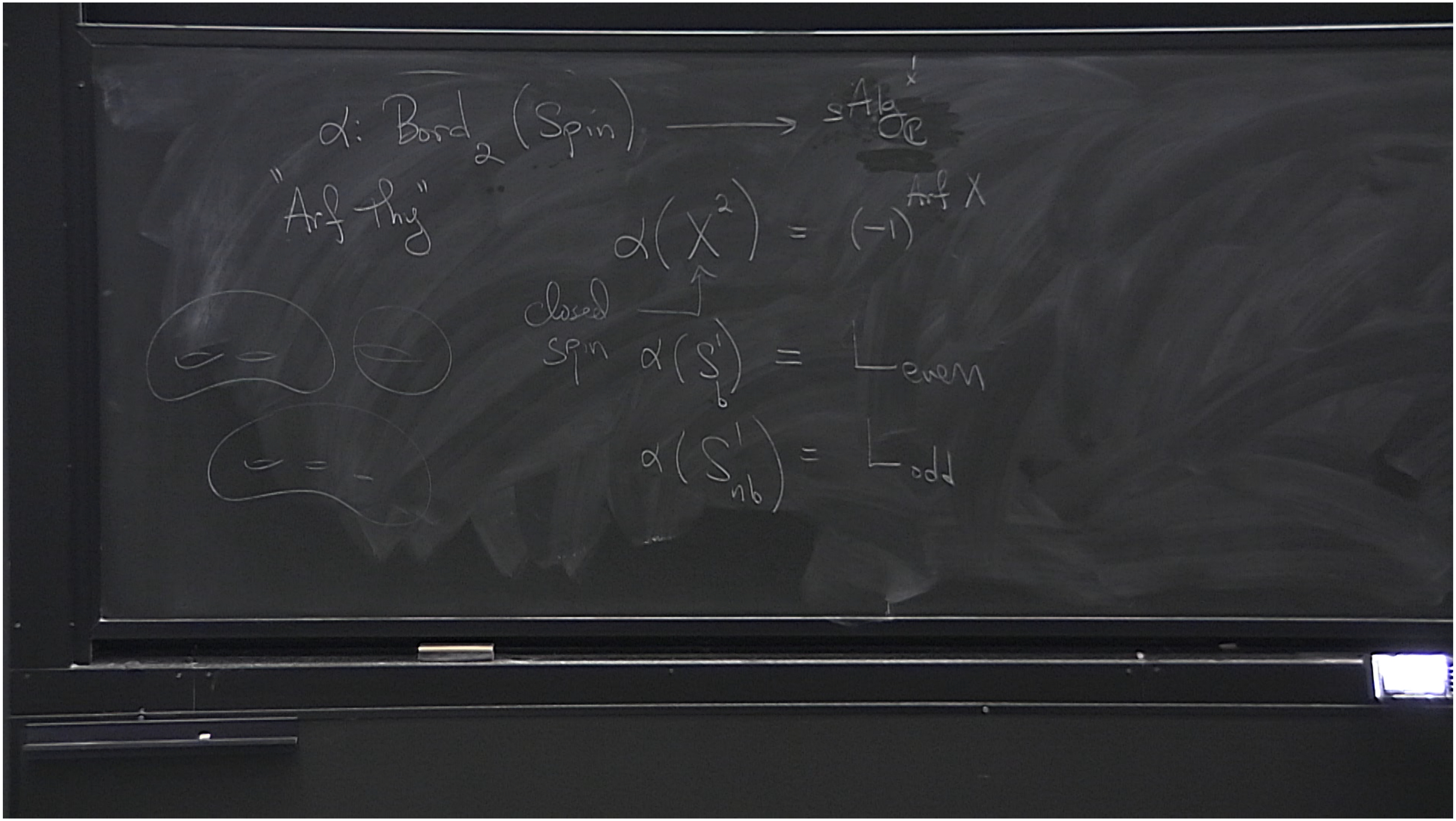
(all joint with Mike Hopkins)

`arXiv:1908.09916` (*M-theory anomaly cancellation*)

`arXiv:1604.06527` (*Reflection positivity and invertible topological phases*)

`arXiv:1901.06419` (*Invertible phases of matter with spatial symmetry*)


$\alpha: \mathrm{Bord}_2(\mathrm{Spin}) \longrightarrow \mathrm{sLine}_{\mathbb{C}}$
"Arf thy"
$\alpha(X^2) = (-1)^{\mathrm{Arf}\, X}$
closed spin
$\alpha(S^1_b) = L_{even}$
$\alpha(S^1_{nb}) = L_{odd}$


$\alpha : \mathrm{Bord}_2(\mathrm{Spin}) \longrightarrow s\mathrm{Alg}_{\mathbb{C}}^{\times}$
"Arf Thy"
$\alpha(X^2) = (-1)^{\mathrm{Arf}\, X}$
closed spin
$\alpha(S^1_b) = L_{even}$
$\alpha(S^1_{nb}) = L_{odd}$