Title: PSI 2019/2020 - Relativity (Kubiznak) - Lecture 6

Speakers: David Kubiznak

Collection: PSI 2019/2020 - Relativity (Kubiznak)

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OUR GOAL: WRITE EINSTEIN EQS

$$\boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}}$$

NEED TO UNDERSTAND DIFF. GEOMETRY

STAGE $\equiv$ MANIFOLD

ACTORS = TENSORS

DEF: A TENSOR OF T

IS A MULTILINEAR M

$$T : \underbrace{T_p^*M \times \dots \times T_p^*M}_{k\text{-TIMES}}$$

CAN WRITE

$$T = \underbrace{T^{\alpha_1 \dots \alpha_k}{}_{\beta_1 \dots \beta_\ell}}_{\text{COMPONENTS}} \frac{\partial}{\partial x}$$

DEF: A TENSOR OF TYPE (k,l), RANK $(k+l)$ IS A MULTILINEAR MAP T:

$$T: \underbrace{T_p^*M \times \dots \times T_p^*M}_{k\text{-TIMES}} \times \underbrace{T_pM \times \dots \times T_pM}_{l\text{-TIMES}}$$

CAN WRITE

$$T = \underbrace{T^{\alpha_1 \dots \alpha_k}{}_{\beta_1 \dots \beta_l}}_{\text{COMPONENTS}} \underbrace{\frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_k}} \otimes dx^{\beta_1} \otimes \dots \otimes dx^{\beta_l}}_{\text{BASIS}}$$

COMPONENTS = TENSOR EVALUATED ON BASIS

EG. $T^{\alpha \dots}{}_{\beta \dots} = T\left(dx^\alpha, \frac{\partial}{\partial x^\beta}\right)$

INDEED: $T\left(dx^\alpha, \frac{\partial}{\partial x^\beta}\right) =$

$$= \left(T^\gamma{}_\delta \frac{\partial}{\partial x^\gamma} \otimes dx^\delta\right)\left(dx^\alpha, \frac{\partial}{\partial x^\beta}\right)$$

$$= T^\gamma{}_\delta \, dx^\alpha\left(\frac{\partial}{\partial x^\gamma}\right) \otimes dx^\delta\left(\frac{\partial}{\partial x^\beta}\right)$$

$$= T^\alpha{}_\beta$$

OPERATIONS WITH TENSORS (TENSOR ALGEBRA)

i) IF SAME TYPE ⇒ T+S TENSOR

ii) TENSOR PRODUCT ⊗ ... CREATES BIGGER TENSORS:

iii) CONTRACTION ... SMALLER TENSORS BY "EATING CO-VARIANT & NORMAL INDICES"

iv) TENSORS ARE INVARIANT OBJECTS!

DEF: A TENSOR OF TYPE (k,l), RANK $(k+l)$

IS A MULTILINEAR MAP T:

$$T: \underbrace{T_p^*M \times \ldots \times T_p^*M}_{k\text{-TIMES}} \times \underbrace{T_pM \times \ldots \times T_pM}_{l\text{-TIMES}} \to \mathbb{R}$$

CAN WRITE

$$T = \underbrace{T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_l}}_{\text{COMPONENTS}} \underbrace{\frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_k}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_l}}_{\text{BASIS}}$$

COMPONENTS = TENSOR EVALUATED ON BASIS

EG. $T^{\alpha}{}_{\beta} = T\left(dx^\alpha, \frac{\partial}{\partial x^\beta}\right)$

INDEED: $T\left(dx^\alpha, \frac{\partial}{\partial x^\beta}\right) =$

$$= \left(T^{\gamma}{}_{\delta} \frac{\partial}{\partial x^\gamma} \otimes dx^\delta\right)\left(dx^\alpha, \frac{\partial}{\partial x^\beta}\right)$$

$$= T^{\gamma}{}_{\delta}\, dx^\alpha\left(\frac{\partial}{\partial x^\gamma}\right) \otimes dx^\delta\left(\frac{\partial}{\partial x^\beta}\right)$$

$$= T^{\alpha}{}_{\beta}$$

DEF: A TENSOR OF TYPE (k,l), RANK $(k+l)$ IS A MULTILINEAR MAP T:

$$T: \underbrace{T_p^*M \times \dots \times T_p^*M}_{k\text{-TIMES}} \times \underbrace{T_pM \times \dots \times T_pM}_{l\text{-TIMES}} \to \mathbb{R}$$

CAN WRITE

$$T = \underbrace{T^{\alpha_1 \dots \alpha_k}{}_{\beta_1 \dots \beta_l}}_{\text{COMPONENTS}} \underbrace{\frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_k}} \otimes dx^{\beta_1} \otimes \dots \otimes dx^{\beta_l}}_{\text{BASIS}}$$

COMPONENTS = TENSOR EVALUATED ON BASIS

EG. $T^{\alpha}{}_{\beta} = T\left(dx^\alpha, \frac{\partial}{\partial x^\beta}\right)$

INDEED: $T\left(dx^\alpha, \frac{\partial}{\partial x^\beta}\right) =$

$$= \left(T^\gamma{}_\delta \frac{\partial}{\partial x^\gamma} \otimes dx^\delta\right)\left(dx^\alpha, \frac{\partial}{\partial x^\beta}\right)$$

$$= T^\gamma{}_\delta \underbrace{dx^\alpha\left(\frac{\partial}{\partial x^\gamma}\right)}_{\delta^\alpha_\gamma} \otimes \underbrace{dx^\delta\left(\frac{\partial}{\partial x^\beta}\right)}_{\delta^\delta_\beta}$$

$$= T^\alpha{}_\beta$$

Ex: $T \dots (1,1)$ $\quad T = T^{\alpha}{}_{\beta}\, \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta}$

$S \dots (0,2)$ $\quad S = S_{\gamma\delta}\, dx^{\gamma} \otimes dx^{\delta}$

$$T \otimes S = T^{\alpha}{}_{\beta}\, \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta} \otimes S_{\gamma\delta}\, dx^{\gamma} \otimes dx^{\delta}$$

$$= \underbrace{T^{\alpha}{}_{\beta}\, S_{\gamma\delta}}_{(T\otimes S)^{\alpha}{}_{\beta\gamma\delta}}\, \underbrace{\frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta} \otimes dx^{\gamma} \otimes dx^{\delta}}_{\text{BASIS}}$$

Ex: $T \ldots (1,1)$ $\quad T = T^{\alpha}{}_{\beta} \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta}$

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$$T \otimes S = T^{\alpha}{}_{\beta} \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta} \otimes S_{\gamma\delta}\, dx^{\gamma} \otimes dx^{\delta}$$

$$= \underbrace{T^{\alpha}{}_{\beta} S_{\gamma\delta}}_{(T\otimes S)^{\alpha}{}_{\beta\gamma\delta}} \underbrace{\frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta} \otimes dx^{\gamma} \otimes dx^{\delta}}_{\text{BASIS}} \quad \ldots (1,3) \ldots$$

EX: $T \ldots (1,1)$ $\quad T = T^{\alpha}{}_{\beta} \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta}$

$S \ldots (0,2)$ $\quad S = S_{\gamma\delta}\, dx^{\gamma} \otimes dx^{\delta}$

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$$= \underbrace{T^{\alpha}{}_{\beta} S_{\gamma\delta}}_{(T\otimes S)^{\alpha}{}_{\beta\gamma\delta}} \frac{\partial}{\partial x^{\alpha}} \otimes \underbrace{dx^{\beta} \otimes dx^{\gamma} \otimes dx^{\delta}}_{\text{BASIS}} \quad \ldots (1,3) \ldots$$

$$T_{CONT} = T^{\alpha}{}_{\beta}\, dx^{\beta}\left(\frac{\partial}{\partial x^{\alpha}}\right) = T^{\alpha}{}_{\alpha} \ \ldots \text{ SCALAR}$$

FAT BASIS

COMPONENTS DO TRANSFORM:

$$T'^{\alpha\ldots}{}_{\mu\ldots}(x') = \frac{\partial x'^{\alpha}}{\partial x^{\sigma}}\cdots\frac{\partial x^{\varkappa}}{\partial x'^{\mu}}\cdots T^{\sigma\ldots}{}_{\varkappa\ldots}$$

COMPONENTS DO TRANSFORM:

$$\underbrace{T'^{\alpha\ldots}{}_{\mu\ldots}(x') = \frac{\partial x'^{\alpha}}{\partial x^{\sigma}}\cdots\frac{\partial x^{\varkappa}}{\partial x'^{\mu}}\cdots T^{\sigma\ldots}{}_{\varkappa\ldots}}$$

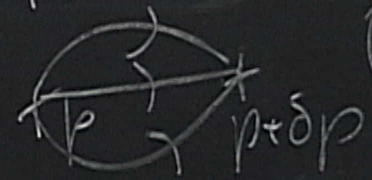
c) CONNECTION

AIM: DIFFERENTIATE TENSORS (THE RESULT BETTER BE A TENSOR AS WELL)

REMIND: $\left.\frac{df}{dt}\right|_{t_0} = \lim_{s\to 0}\frac{f(t_0+s)-f(t_0)}{s}$

2 PROBLEMS WHEN ON A MANIFOLD:

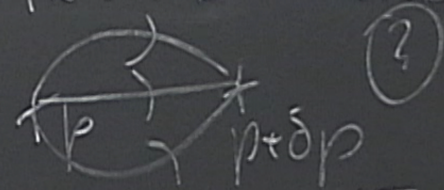
1) WHAT IS $\underbrace{p}_{to}+\underbrace{\delta p}_{S}$? HO DO WE GO THERE ? ?



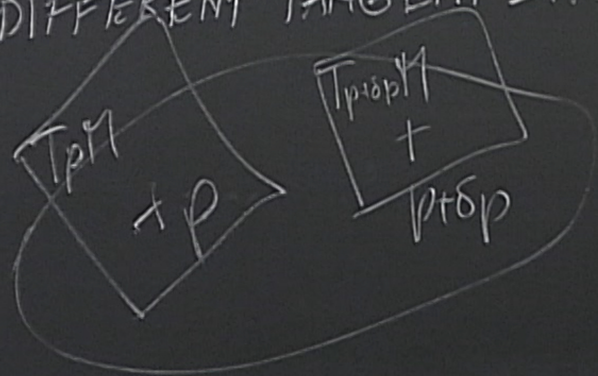
2)

2 PROBLEMS WHEN ON A MANIFOLD:

1) WHAT IS $\underbrace{p}_{\text{to}} + \underbrace{\delta p}_{S}$? HOW DO WE GO THERE ?



2) VECTOR AT p AND A VECTOR AT $p+\delta p$ LIVE IN DIFFERENT TANGENT SPACES



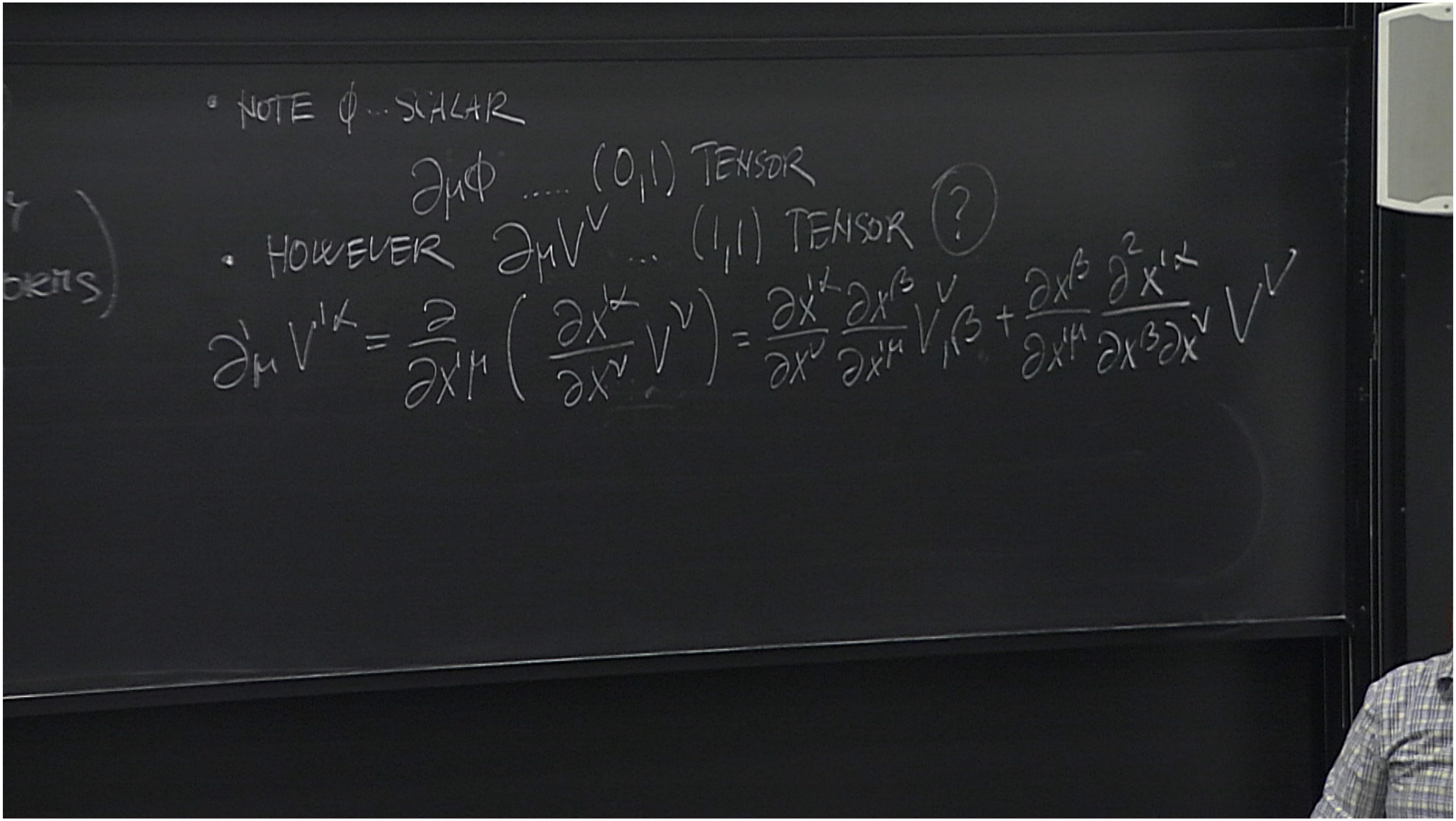
$\beta \to 0$ S

3 STAND POSS

- LIE DERIVATIVE (NEED VECTOR FIELD U)
- EXTERIOR DERIVATIVE (WORKS ONLY FOR A SUBSET OF TENSORS... DIFF. FORMS)
- COVARIANT DERIVATIVE (SUPER UGLY)

EXTRA OBJECT.. CONNECTION $\Gamma^{\alpha}_{\beta\gamma}$

NOT A TENSOR


• NOTE ϕ ... SCALAR
$\partial_\mu \phi$ ..... (0,1) TENSOR
• HOWEVER $\partial_\mu V^\nu$ ... (1,1) TENSOR ?
$\partial'_\mu V'^\alpha = \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\alpha}{\partial x^\nu} V^\nu\right) = \frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x^\beta}{\partial x'^\mu} V^\nu{}_{,\beta} + \frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial^2 x'^\alpha}{\partial x^\beta \partial x^\nu} V^\nu$

• NOTE ϕ ... SCALAR

$\partial_\mu \phi$ ..... (0,1) TENSOR

• HOWEVER $\partial_\mu V^\nu$ ... (1,1) TENSOR ?

$$\partial'_\mu V'^\alpha = \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x'^\alpha}{\partial x^\nu} V^\nu\right) = \underbrace{\frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x^\beta}{\partial x'^\mu} V^\nu{}_{,\beta}}_{\text{OK FOR (1,1) TENSOR}} + \underbrace{\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial^2 x'^\alpha}{\partial x^\beta \partial x^\nu} V^\nu}_{\text{PROBLEM!}}$$

RIEMANN (1830)

$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\alpha\mu} V^\alpha}$$

IN ORDER $\nabla_\mu V^\nu \ldots$ (1,1) TENSOR WE REQUIRE

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$$\Gamma'^\alpha{}_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\chi}\frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial x^z}{\partial x'^\gamma}\Gamma^\chi{}_{\delta z} -$$

RIEMANN (1830)

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IN ORDER $\nabla_\mu V^\nu \ldots$ (1,1) TENSOR WE REQUIRE

$$\boxed{\Gamma'^\alpha{}_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\kappa}\frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial x^\lambda}{\partial x'^\gamma}\Gamma^\kappa{}_{\delta\lambda} - \frac{\partial^2 x'^\alpha}{\partial x^\kappa \partial x^\delta}\frac{\partial x^\kappa}{\partial x'^\beta}\frac{\partial x^\delta}{\partial x'^\gamma}}$$

CONNECTION

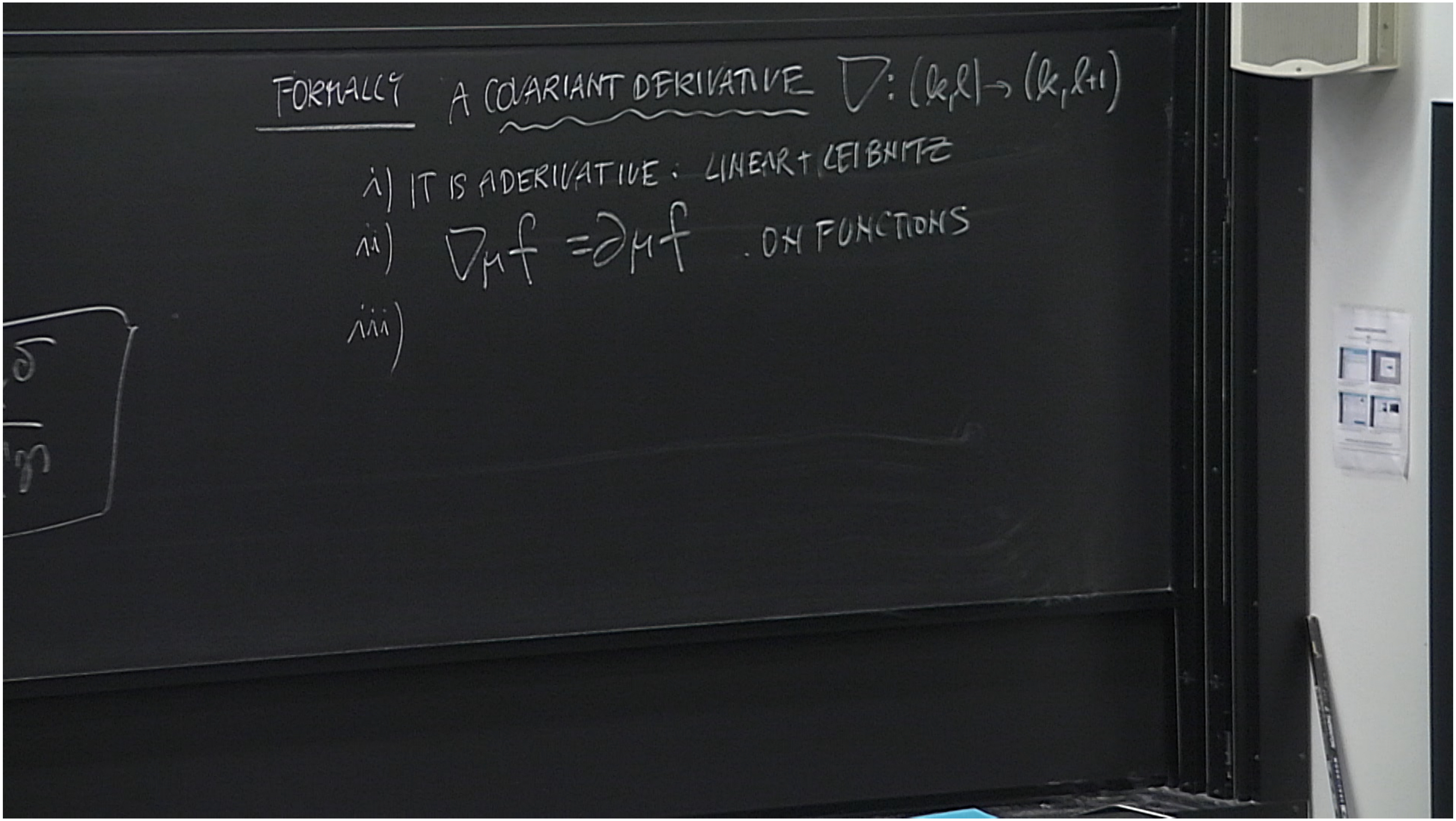
RIEMANN (1830)

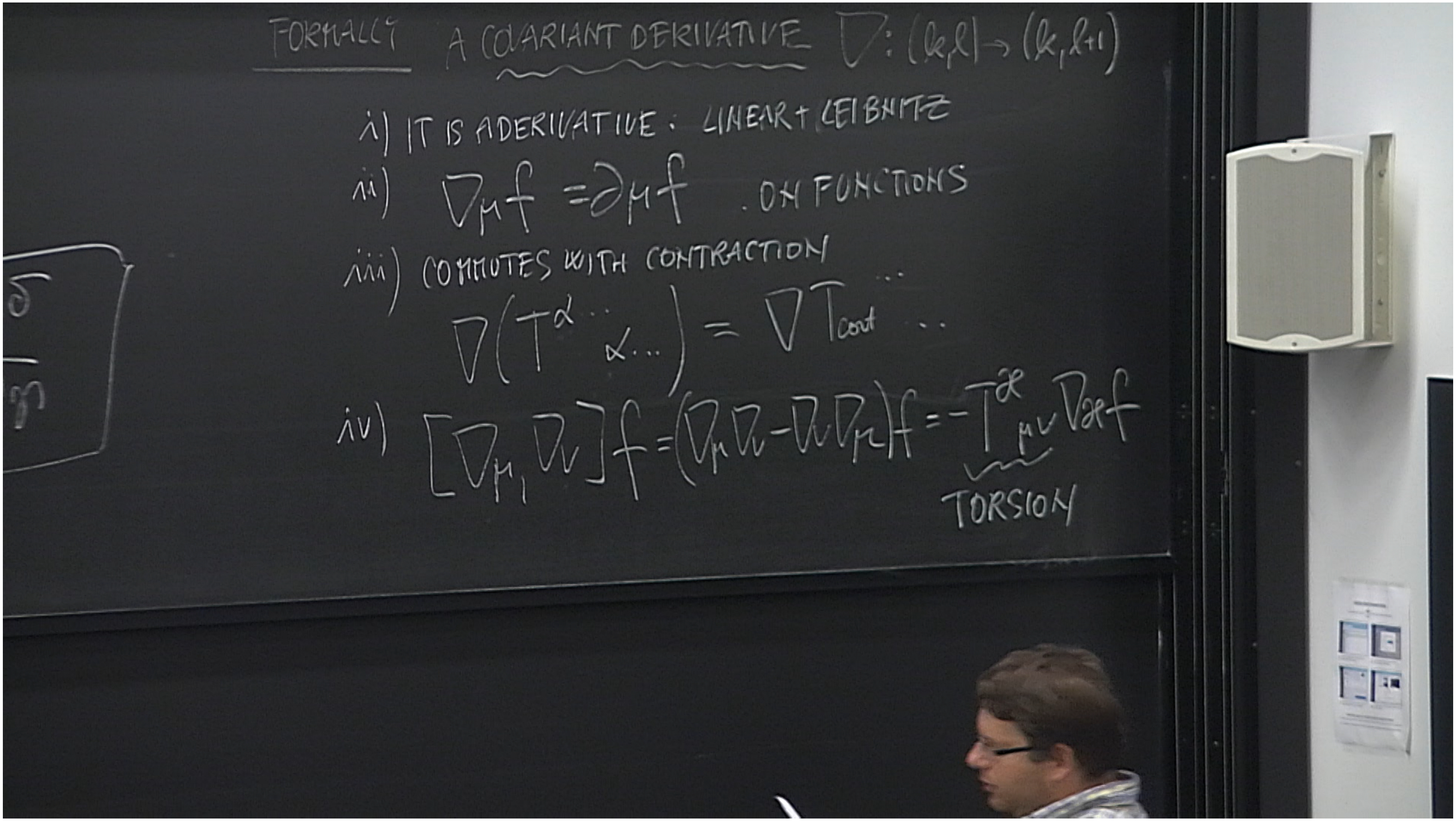
$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\alpha\mu} V^\alpha}$$

IN ORDER $\nabla_\mu V^\nu$ ... (1,1) TENSOR WE REQUIRE

$$\boxed{\Gamma'^\alpha{}_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\kappa}\frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial x^2}{\partial x'^\gamma}\Gamma^\kappa{}_{\delta 2} - \underbrace{\frac{\partial^2 x'^\alpha}{\partial x^\kappa \partial x^\delta}\frac{\partial x^\kappa}{\partial x'^\beta}\frac{\partial x^\delta}{\partial x'^\gamma}}_{\text{UGLY TERM}}}$$

CONNECTION


FORMALLY A COVARIANT DERIVATIVE $\nabla : (k,l) \to (k,l+1)$
i) IT IS A DERIVATIVE : LINEAR + LEIBNITZ
ii) $\nabla_\mu f = \partial_\mu f$. ON FUNCTIONS
iii)


FORMALLY A COVARIANT DERIVATIVE $\nabla : (k,l) \to (k,l+1)$
i) IT IS A DERIVATIVE: LINEAR + LEIBNITZ
ii) $\nabla_\mu f = \partial_\mu f$ ON FUNCTIONS
iii) COMMUTES WITH CONTRACTION
$\nabla \left(T^{\alpha \cdots}{}_{\alpha \cdots} \right) = \nabla T_{cont} \cdots$
iv) $[\nabla_\mu, \nabla_\nu] f = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = -T^\alpha{}_{\mu\nu} \nabla_\alpha f$
TORSION

FORMALLY A COVARIANT DERIVATIVE $\nabla : (k,l) \to (k,l+1)$

i) IT IS A DERIVATIVE: LINEAR + LEIBNITZ

ii) $\nabla_\mu f = \partial_\mu f$. ON FUNCTIONS

iii) COMMUTES WITH CONTRACTION

$$\nabla\left(T^{\alpha\dots}{}_{\alpha\dots}\right) = \nabla T_{cont} \dots$$

iv) $$[\nabla_\mu, \nabla_\nu] f = (\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu) f = -\underbrace{T^\alpha{}_{\mu\nu}}_{\text{TORSION}}\nabla_\alpha f$$

$$\boxed{T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta} - \Gamma^\alpha{}_{\beta\gamma}}$$ TENSOR.

IT FOLLOWS:

$$\boxed{\nabla_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^{\gamma}_{\beta\alpha}\,\omega_\gamma}$$

IT FOLLOWS:

$$\boxed{\nabla_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^{\gamma}{}_{\beta\alpha}\, \omega_\gamma}$$

IDEA OF PROOF:

$$\nabla(V\cdot\omega) \overset{i)}{=} (\nabla V)\cdot\omega + V\cdot(\nabla\omega) \overset{ii)}{=} \partial(V\cdot\omega)$$

$$= (\partial V + \Gamma V)\cdot\omega + V\cdot(\partial\omega - \Gamma\omega)$$

$$= \partial V\cdot\omega + V\cdot\partial\omega = \partial(V\cdot\omega)$$

IN GENERAL:

$$\nabla_\alpha T_{\beta\ldots}{}^{\gamma\ldots} = \partial_\alpha T_{\beta\ldots}{}^{\gamma\ldots} + \Gamma^{\gamma}{}_{\delta\alpha} T_{\beta\ldots}{}^{\delta\ldots} + \ldots$$
$$- \Gamma^{\rho}{}_{\beta\alpha} T_{\rho\ldots}{}^{\gamma\ldots} - \ldots$$

IN GENERAL:

$$\nabla_\alpha T_{\beta\ldots}{}^{\gamma\ldots} = \partial_\alpha T_{\beta\ldots}{}^{\gamma\ldots} + \Gamma^{\gamma}{}_{\delta\alpha} T_{\beta\ldots}{}^{\delta\ldots} + \ldots - \Gamma^{\rho}{}_{\beta\alpha} T_{\rho}{}^{\gamma\ldots} - \ldots$$

$\cdot\,\omega)$

d) METRIC = SYMMETRIC, NON-DEGENERATE, (0,2) TENSOR

+ PSEUDO-RIEM. SIGNATURE (1 EIGENVALUE $-$, 3 —"— $+$)

IN GENERAL:

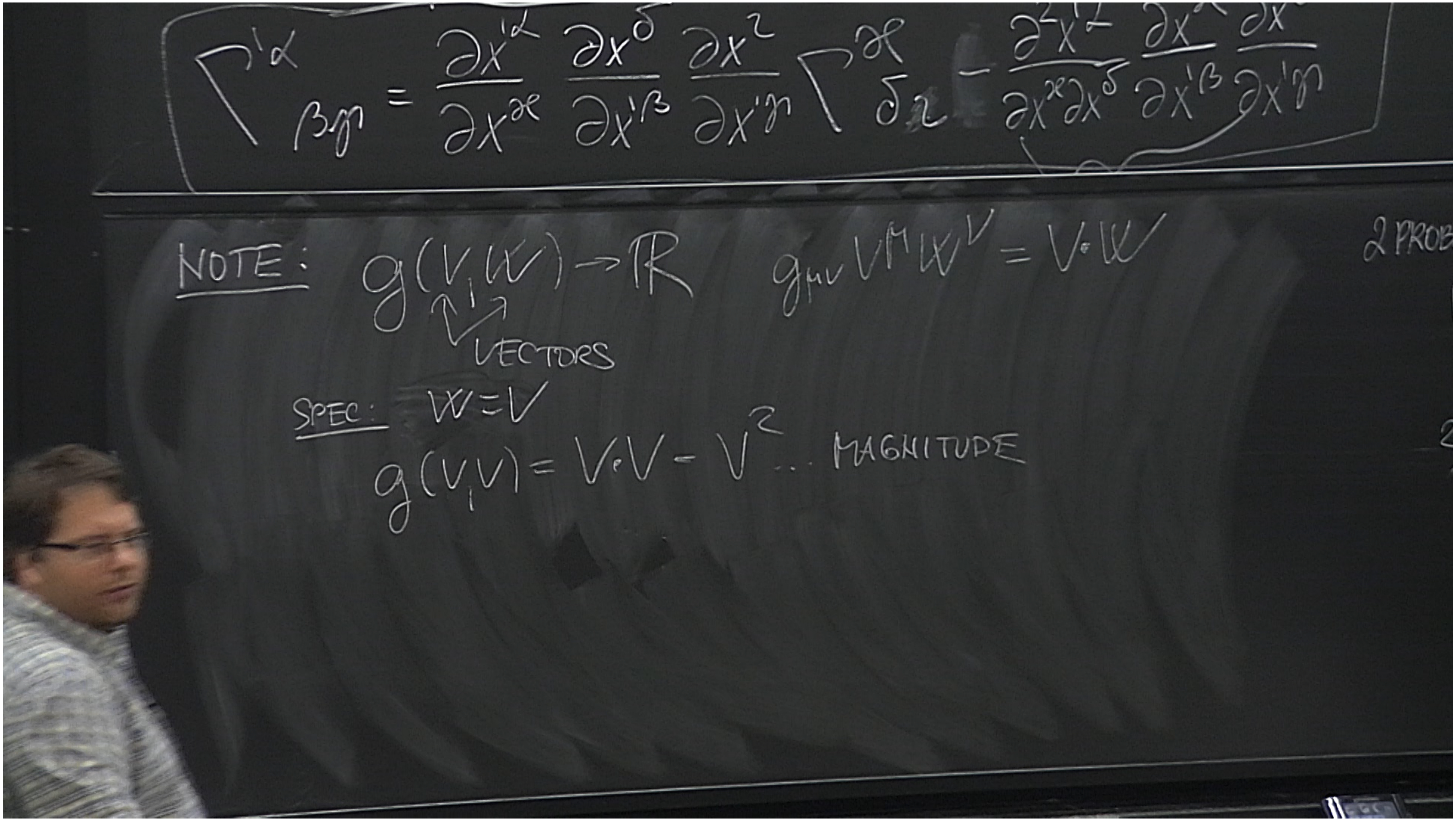
$$\nabla_\alpha T_{\beta\ldots}{}^{\gamma\ldots} = \partial_\alpha T_{\beta\ldots}{}^{\gamma\ldots} + \Gamma^\gamma{}_{\delta\alpha} T_{\beta\ldots}{}^{\delta\ldots} + \ldots - \Gamma^\rho{}_{\beta\alpha} T_{\rho\ldots}{}^{\gamma\ldots} - \ldots$$

d) METRIC = SYMMETRIC, NON-DEGENERATE, (0,2) TENSOR

PSEUDO-RIEM. SIGNATURE (1 EIGENVALUE ⊖, 3 —11— S ⊕)

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = g_{\mu\nu}\, dx^\mu dx^\nu = ds^2$$

$\nabla \cdot \omega$)



$\Gamma'^{\alpha}_{\beta\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\kappa}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} \frac{\partial x^{\lambda}}{\partial x'^{\gamma}} \Gamma^{\kappa}_{\delta\lambda} - \frac{\partial^2 x'^{\alpha}}{\partial x^{\kappa} \partial x^{\delta}} \frac{\partial x^{\kappa}}{\partial x'^{\beta}} \frac{\partial x^{\delta}}{\partial x'^{\gamma}}$
NOTE: $g(V, W) \to \mathbb{R}$ $g_{\mu\nu} V^{\mu} W^{\nu} = V \cdot W$
VECTORS
SPEC: $W = V$
$g(V, V) = V \cdot V = V^2$ ... MAGNITUDE
2 PROB

$$\Gamma'^{\alpha}{}_{\beta\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\kappa}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} \frac{\partial x^{\lambda}}{\partial x'^{\gamma}} \Gamma^{\kappa}{}_{\delta\lambda} - \frac{\partial^2 x'^{\alpha}}{\partial x^{\kappa}\partial x^{\delta}} \frac{\partial x^{\kappa}}{\partial x'^{\beta}} \frac{\partial x^{\delta}}{\partial x'^{\gamma}}$$

2 PROB

NOTE: $g(V, W) \to \mathbb{R}$ $\quad g_{\mu\nu} V^{\mu} W^{\nu} = V \cdot W$

VECTORS

SPEC: $W = V$

$g(V, V) = V \cdot V = V^2$ ... MAGNITUDE

$$p \xrightarrow{dx^{\mu}} p + \delta p$$

$$ds^2 = g(dx, dx) = g_{\mu\nu}\, dx^{\mu} dx^{\nu}$$

2 VECTORS CONNECTING NEARBY POINTS

• CHRISTOFFEL SYMBOLS

THEOREM: IF $T^{\alpha}{}_{\beta\gamma}=0$ & $\underbrace{M_{\alpha\beta\gamma}}_{\text{METRICITY TENSOR}} = \nabla_\alpha g_{\beta\gamma} = 0$

=> UNIQUE CONNECTION... GIVE BY CHRISTOFFEL SYMBOLS

$$\boxed{\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}\left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}\right)}$$

• CHRISTOFFEL SYMBOLS

THEOREM: IF $T^{\alpha}{}_{\beta\gamma}=0$ & $\underbrace{M_{\alpha\beta\gamma}}_{\text{METRICITY TENSOR}} = \nabla_{\alpha} g_{\beta\gamma} = 0$

$\Rightarrow$ UNIQUE CONNECTION... GIVE BY CHRISTOFFEL SYMBOLS

$$\boxed{\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}\left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}\right)}$$

IN GR: THE ONLY DYNAMICAL IS $g_{\mu\nu}$

(BEYOND GR BOTH $T^{\alpha}{}_{\beta\gamma}$, $M_{\alpha\beta\gamma}$.. CAN BE DYN.)

PRINCIPLE OF EQUIVALENCE . AT EVERY SPACETIME POINT p WE CAN INTRODUCE LOCAL INERTIAL (FREELY FALLING) FRAME:

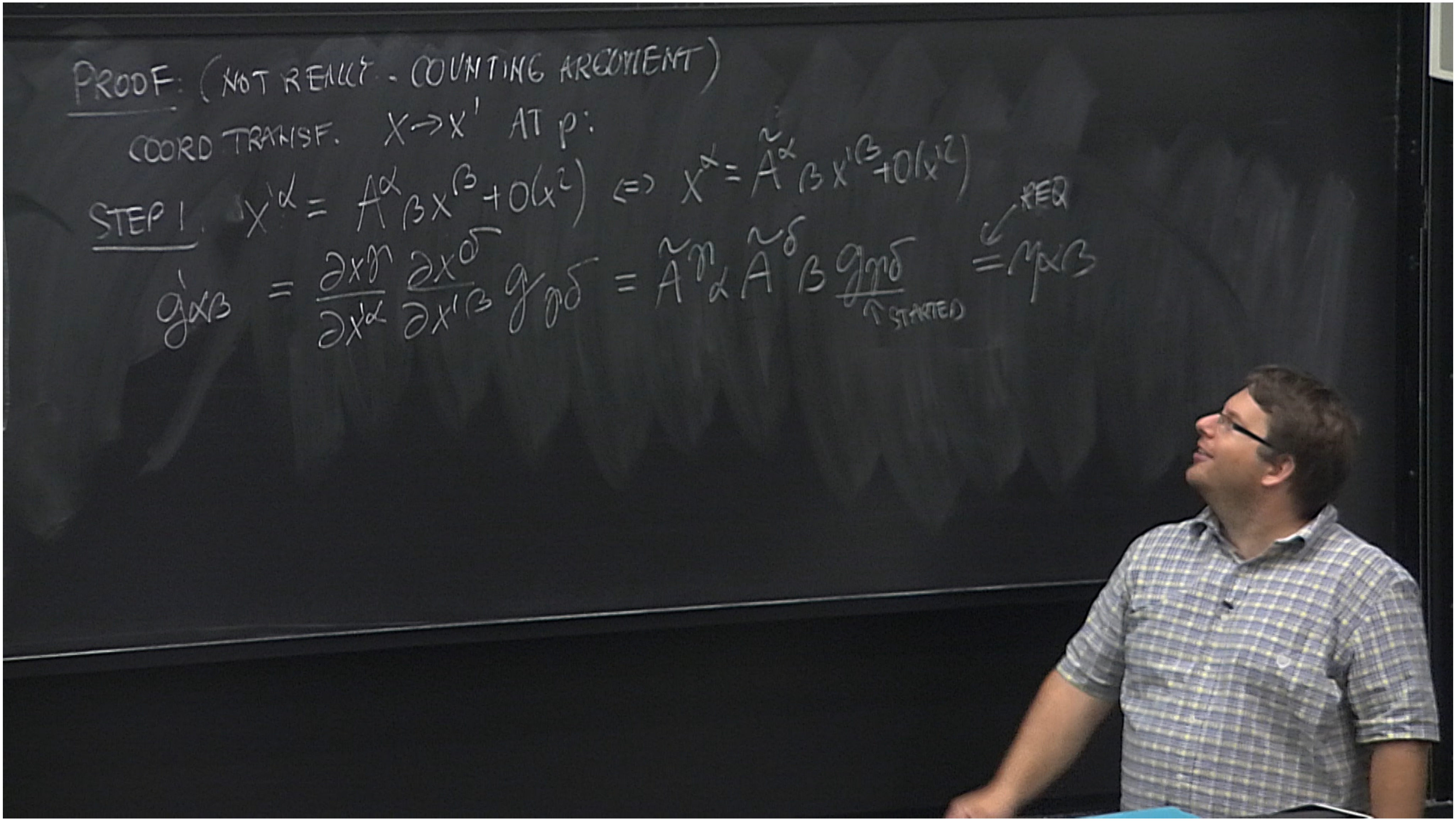
$$g_{\mu\nu} = \eta_{\mu\nu}$$

PRINCIPLE OF EQUIVALENCE. AT EVERY SPACETIME POINT p WE CAN INTRODUCE LOCAL INERTIAL (FREELY FALLING) FRAME:

$$g_{\mu\nu} = \eta_{\mu\nu}, \quad g_{\mu\nu,\rho} = 0 \quad \text{AT } p$$

$$\Updownarrow$$

$$\Gamma^{\mu}{}_{\nu\rho} = 0$$



PROOF: (NOT REALLY - COUNTING ARGUMENT)
COORD TRANSF. $x \to x'$ AT p:
STEP 1: $x'^\alpha = A^\alpha{}_\beta x^\beta + O(x^2) \Leftrightarrow x^\alpha = \tilde{A}^\alpha{}_\beta x'^\beta + O(x'^2)$
$g'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} g_{\gamma\delta} = \tilde{A}^\gamma{}_\alpha \tilde{A}^\delta{}_\beta g_{\gamma\delta} \overset{?}{=} \eta_{\alpha\beta}$
REQ
STARTED

PROOF: (NOT REALLY - COUNTING ARGUMENT)

COORD TRANSF. $X \to X'$ AT p:

STEP 1: $x'^{\alpha} = A^{\alpha}{}_{\beta}\, x^{\beta} + O(x^2) \iff x^{\alpha} = \tilde{A}^{\alpha}{}_{\beta}\, x'^{\beta} + O(x'^2)$

$$g'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} g_{\gamma\delta} = \tilde{A}^{\gamma}{}_{\alpha} \tilde{A}^{\delta}{}_{\beta} \underbrace{g_{\gamma\delta}}_{\text{STARTED}} \overset{\text{REQ}}{=} \eta_{\alpha\beta}$$

16 $\tilde{A}^{\alpha}{}_{\beta}$ AT MY DISPOSE

PROOF (NOT REALLY - COUNTING ARGUMENT)

COORD TRANSF. $x \to x'$ AT p:

STEP 1. $x'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + O(x^2) \Leftrightarrow x^{\alpha} = \tilde{A}^{\alpha}{}_{\beta} x'^{\beta} + O(x'^2)$

$$g'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} g_{\gamma\delta} = \tilde{A}^{\gamma}{}_{\alpha} \tilde{A}^{\delta}{}_{\beta} \underbrace{g_{\gamma\delta}}_{\text{STARTED}} \overset{\text{REQ}}{=} \eta_{\alpha\beta}$$

16 $\tilde{A}^{\alpha}{}_{\beta}$ AT MY DISPOSE FOR 10 EQUATIONS

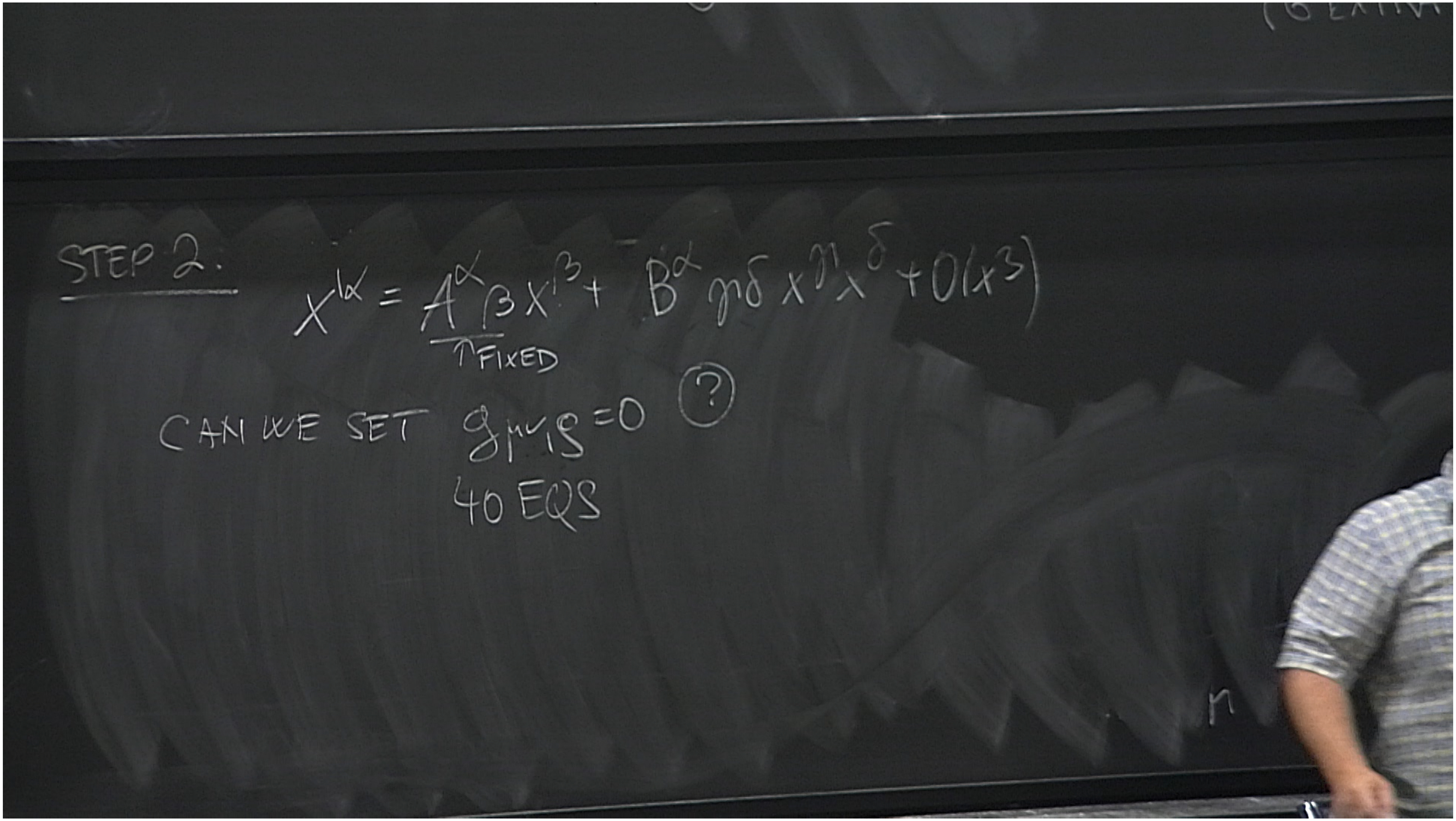
IN PRINCIPLE CAN SOLVE.

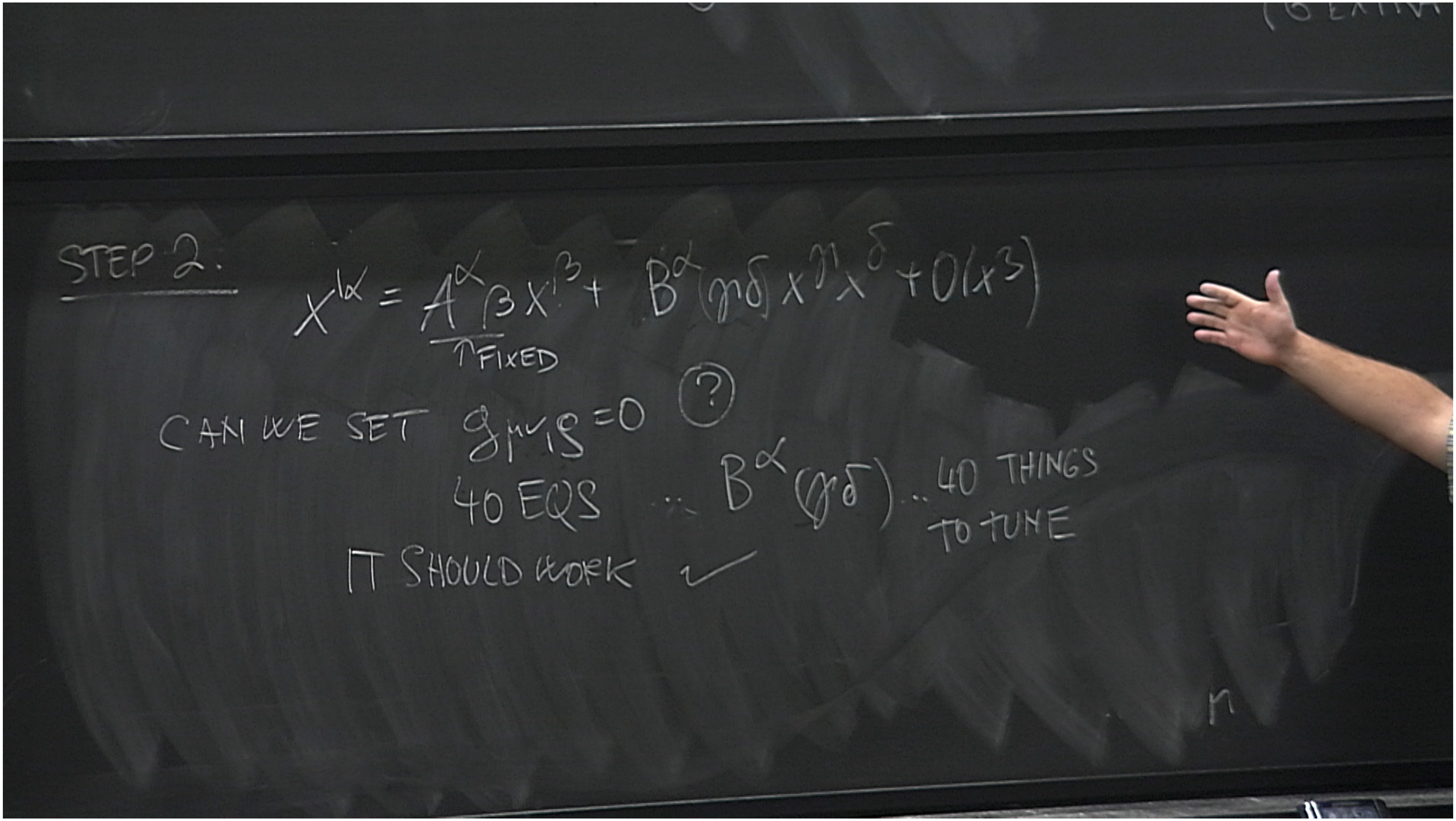
(6 EXTRA $\simeq$ LORENTZ TRANSF FREEDOM)

STEP 2:

$$x'^{\alpha} = \underbrace{A^{\alpha}{}_{\beta}}_{\uparrow\,\text{FIXED}} x^{\beta} + B^{\alpha}{}_{\gamma\delta}\, x^{\gamma} x^{\delta} + O(x^3)$$

CAN WE SET $g_{\mu\nu,\rho} = 0$?


STEP 2:
$x'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + B^{\alpha}{}_{\gamma\delta} x^{\gamma} x^{\delta} + O(x^3)$
FIXED
CAN WE SET $g_{\mu\nu,\rho} = 0$?
40 EQS


STEP 2:
$x'^{\alpha} = A^{\alpha}{}_{\beta} x'^{\beta} + B^{\alpha}{}_{\gamma\delta} x'^{\gamma} x'^{\delta} + O(x^3)$
FIXED
CAN WE SET $g_{\mu\nu,\rho} = 0$?
40 EQS ... $B^{\alpha}{}_{(\gamma\delta)}$ ... 40 THINGS TO TUNE
IT SHOULD WORK

STEP 3: WHAT ABOUT $g_{\mu\nu,\rho\sigma\lambda} = 0$?

$$x'^{\alpha} = \underbrace{A^{\alpha}{}_{\beta}}_{\text{FIXED}} x^{\beta} + \underbrace{B^{\alpha}{}_{\gamma\delta}}_{\text{FIXED}} x^{\gamma} x^{\delta} + C^{\alpha}{}_{(\gamma\delta\epsilon)} x^{\gamma} x^{\delta} x^{\epsilon} + O(x^4)$$

| 4 | 5 | 6 |

$$\frac{4\cdot5\cdot6}{1\cdot2\cdot3} = 20$$

100 EQUATIONS

4×20 .. COMPONENTS OF $C^{\alpha}(\gamma\delta\epsilon)$ $= \underline{80}$

$\boxed{\text{20 LESS COMPTS}}$. AT LEAST 20 $g_{\mu\nu,\alpha\beta\gamma} \neq 0$

STEP 3: WHAT ABOUT $g_{\mu\nu,\beta\gamma\lambda} = 0$?

$$x'^{\alpha} = \underbrace{A^{\alpha}{}_{\beta}}_{\text{FIXED}} x^{\beta} + \underbrace{B^{\alpha}{}_{\gamma\delta}}_{\text{FIXED}} x^{\gamma} x^{\delta} + C^{\alpha}{}_{(\gamma\delta z)} x^{\gamma} x^{\delta} x^{z} + O(x^{4})$$

$$\boxed{4 \,|\, 5 \,|\, 6} \qquad \frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3} = 20$$

100 EQUATIONS

4×20.. COMPONENTS OF $C^{\alpha}{}_{(\gamma\delta z)}$ $= \underline{80}$

$\boxed{\text{20 LESS COMPTS}}$. AT LEAST 20 $g_{\mu\nu,\alpha\beta\gamma} \neq 0$

HIDING IN RIEMANN TENSOR

$$\underbrace{A^\alpha{}_\beta}_{\text{FIXED}} x^\beta + B^\alpha{}_{(\gamma\delta)} x^\gamma x^\delta + O(x^3)$$

$g_{\mu\nu,\rho} = 0$?

40 EQS ∴ $B^\alpha{}_{(\gamma\delta)}$ .. 40 THINGS TO TUNE

SHOULD WORK ✓

$R^\alpha{}_{\beta\gamma\delta}$ .. LOT'S SYMMETRIES

STEP 3: WHAT ABOUT $g_{\mu\nu,\rho\sigma} = 0$

$$x'^\alpha = \underbrace{A^\alpha{}_\beta}_{\text{FIXED}} x^\beta + \underbrace{B^\alpha{}_{\gamma\delta}}_{\text{FIXED}} x^\gamma x^\delta$$

100 EQUATIONS

$\boxed{\text{20 LESS COMPTS}}$. AT LE

HIDING IN RIEMANN TENSOR