Title: Mutliplicative Bell inequalities

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Abstract: Bell inequalities are important tools in contrasting classical and quantum behaviors. To date, most Bell inequalities are linear combinations of statistical correlations between remote parties. Nevertheless, finding the classical and quantum mechanical (Tsirelson) bounds for a given Bell inequality in a general scenario is a difficult task which rarely leads to closed-form solutions. Here we introduce a new class of Bell inequalities based on products of correlators that alleviate these issues. Each such Bell inequality is associated with a non-cooperative coordination game. In the simplest case, Alice and Bob, each having two random variables, attempt to maximize the area of a rectangle and the rectangleâ€™s area is represented by a certain parameter. This parameter, which is a function of the correlations between their random variables, is shown to be a Bell parameter, i.e. the achievable bound using only classical correlations is strictly smaller than the achievable bound using non-local quantum correlations We continue by generalizing to the case in which Alice and Bob, each having now n random variables, wish to maximize a certain volume in n-dimensional space. We term this parameter a multiplicative Bell parameter and prove its Tsirelson bound. Finally, we investigate the case of local hidden variables and show that for any deterministic strategy of one of the players the Bell parameter is a harmonic function whose maximum approaches the Tsirelson bound as the number of measurement devices increases. Some implications of these results are discussed.

- In existing Bell-type inequalities, the Bell parameter is the sum of correlations between measurements made by spacelike-separated parties
- We present a new class of Bell inequalities, which are based on a product of correlations
- What are the classical (Bell) and quantum (Tsirelson) limits?

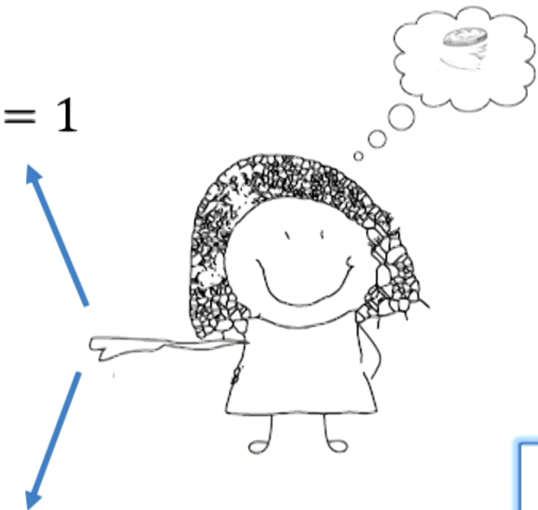
The main findings in this work have appeared in:
A. Te'eni, B. Y. Peled, E. Cohen and A. Carmi, "Multiplicative Bell inequalities," in Physical Review A, vol. 99, no. 4, p. 040102, 2019 (as a Rapid Communication).

Alice & Bob – the CHSH game

Quantum entanglement enables Alice & Bob to win the game more often than any classical strategy

$i \in \{1,2\}$

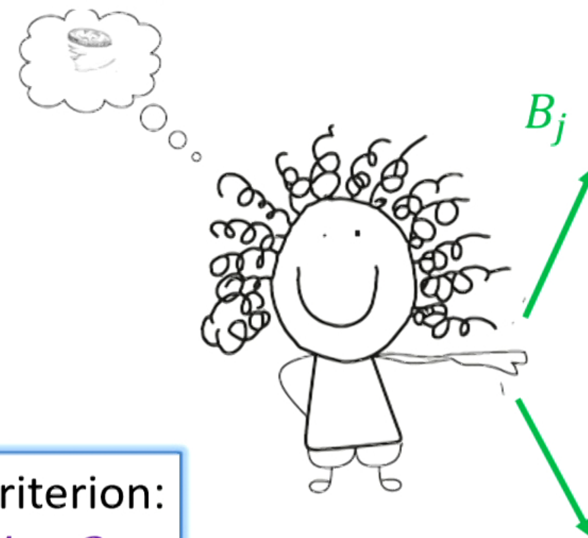
$A_i = 1$



$A_i = -1$

$j \in \{1,2\}$

$B_j = 1$



$B_j = -1$

Winning criterion:
$A_i \neq B_j \; for \; i = j = 2$
$A_i = B_j \; else$

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Bell-CHSH parameter

The chances of winning the game are represented by

the Bell-CHSH parameter:

$$\mathfrak{B}_{CHSH} = |c_{12} + c_{21} + c_{11} - c_{22}|$$

$$\Pr(win) = \frac{2 + \mathfrak{B}/2}{4}$$

Correlation: $c_{ij} = E[A \cdot B | i,j]$

A,B – random variables

$$\max \mathfrak{B}_{CHSH} \leq \begin{cases} 2 & \textit{local realism (Bell limit)} \\ 2\sqrt{2} & \textit{Quantum (Tsirelson limit)} \end{cases}$$

$2 < \mathfrak{B}_{CHSH} \leq 2\sqrt{2}$

Needs quantum entanglement

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Tsirelson's bound and the quantum correlation matrix

- **Theorem:** The second moment matrix for the vector of operators is positive semi-definite (A. Carmi and E. Cohen, "Relativistic independence bounds nonlocality," Science advances, 2019):

$$\begin{bmatrix} B_j \\ A_1 \\ \vdots \\ A_n \end{bmatrix} \forall j \in \{1, \dots, n\}$$

$$\begin{bmatrix} \boxed{\langle B_j B_j \rangle} & \langle A_1 \otimes B_j \rangle & \cdots & \langle A_n \otimes B_j \rangle \\ \langle A_1 \otimes B_j \rangle & \langle A_1 A_1 \rangle & \cdots & \langle A_1 A_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle A_n \otimes B_j \rangle & \langle A_n A_1 \rangle & \cdots & \langle A_n A_n \rangle \end{bmatrix} \succcurlyeq 0$$

From Schur's Complement:

$$\underbrace{\begin{bmatrix} \langle A_1 A_1 \rangle & \cdots & \langle A_1 A_n \rangle \\ \vdots & \ddots & \vdots \\ \langle A_n A_1 \rangle & \cdots & \langle A_n A_n \rangle \end{bmatrix}}_{R_A} \succcurlyeq \underbrace{\begin{bmatrix} \langle A_1 \otimes B_j \rangle \\ \vdots \\ \langle A_n \otimes B_j \rangle \end{bmatrix}}_{\vec{c}_j} \cancelto{1}{\langle B_j B_j \rangle}^{-1} \underbrace{\left[\langle A_1 \otimes B_j \rangle \quad \cdots \quad \langle A_n \otimes B_j \rangle \right]}_{\vec{c}_j^{\,T}}$$

Tsirelson's bound and the quantum correlation matrix (cont'd)

- Bell-CHSH Tsirelson's bound follows from PSD of the second moment matrix – specifically, from $R_A \succcurlyeq \vec{c}_j \vec{c}_j^T$
- On both sides, take the quadratic forms with vectors $[1 \quad \pm 1]^T$:

$$[1 \quad \pm 1] R_A \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} \geq [1 \quad \pm 1] \vec{c}_j \vec{c}_j^T \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} \Rightarrow \sqrt{2 \pm \langle \{A_1, A_2\} \rangle} \geq \left| c_{1j} \pm c_{2j} \right|$$

- Substitue $j = 1$ for $+$ sign and $j = 2$ for $-$ sign, add the two inequalities and use the triangle inequality:

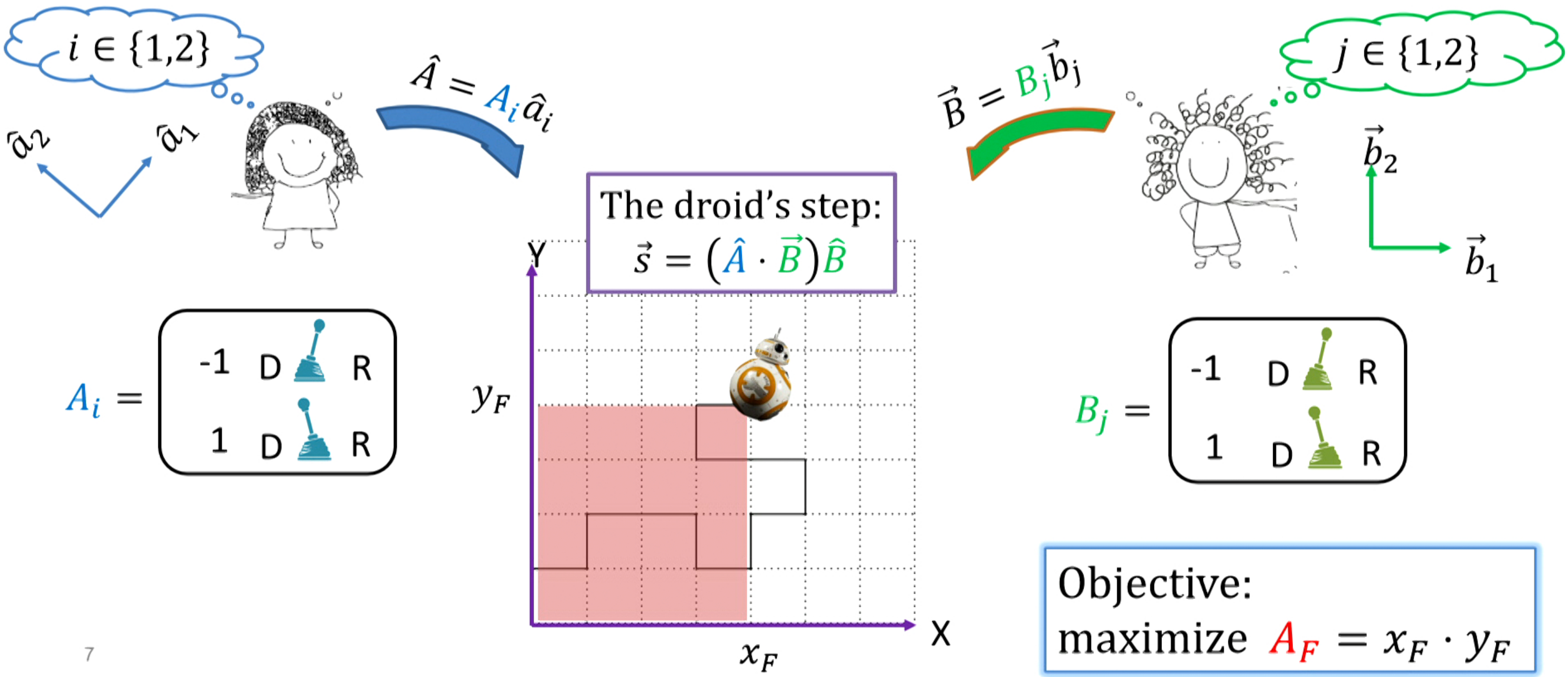
$$\mathfrak{B}_{CHSH} =$$

$$|c_{11} + c_{21} + c_{12} - c_{22}| \leq |c_{11} + c_{21}| + |c_{12} - c_{22}| \leq \sqrt{2 + \langle \{A_1, A_2\} \rangle} + \sqrt{2 - \langle \{A_1, A_2\} \rangle} \leq 2\sqrt{2}$$

- What if instead of adding, you multiply?

$$\underbrace{|c_{11} + c_{21}||c_{12} - c_{22}|}_{\text{New multiplicative Bell parameter}} \leq \sqrt{4 - \langle \{A_1, A_2\} \rangle^2} \leq \underbrace{2}_{\text{Tsirelson bound}}$$

Alice & Bob – “multiplicative” 2-device game



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Multiplicative 2-device Bell parameter

The expected area, $E[A_F]$, for the droid, is proportional

to the multiplicative 2-device Bell parameter:

$$\mathfrak{B}_2 = |(c_{12} + c_{22})(c_{11} - c_{21})|$$

Reminder:

$$\mathfrak{B}_{CHSH} = |c_{12} + c_{22} + c_{11} - c_{21}|$$

Correlation: $c_{ij} = E[A \cdot B | i, j]$

A, B – random variables

Theorem:

$$\max \mathfrak{B}_2 \leq \begin{cases} 1 & \textit{Bell limit} \\ 2 & \textit{Tsirelson limit} \end{cases}$$

$$1 < \mathfrak{B}_2 \leq 2$$

Needs quantum entanglement

Multiplicative n-device Bell parameter

We based our parameter on the orthogonal vectors:

$$\boxed{\mathfrak{B}_n \triangleq \prod_{j=1}^{n} \left| \vec{v}_j \cdot \vec{c}_j \right|} \qquad \vec{c}_j \triangleq \begin{bmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccccc} & \vec{v}_{j=2} & & & \end{array} \\ \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ & -2 & \ddots & \vdots & 1 \\ & & \ddots & 1 & \vdots \\ & & & -(n-1) & 1 \end{bmatrix} \end{array}$$

$$\mathfrak{B}_n = |c_{1n} + \cdots + c_{nn}| \prod_{j=1}^{n-1} \left| c_{1j} + \cdots + c_{jj} - j c_{j+1,j} \right|$$

Correlation: $c_{ij} = E[A \cdot B | i, j]$

A, B – random variables

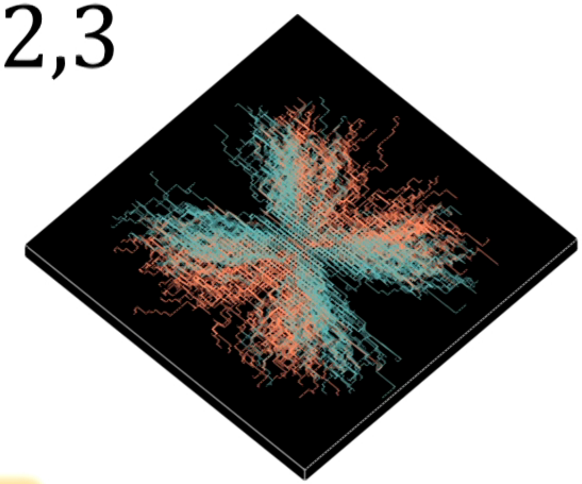
$n = 2$:

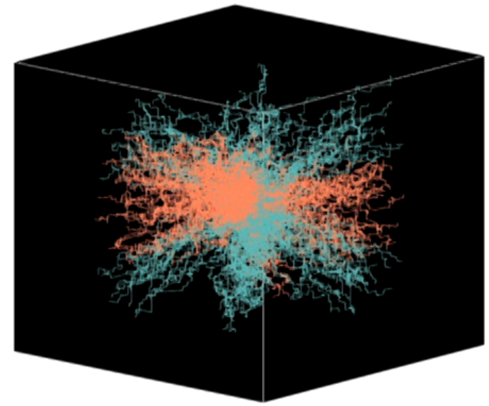
$\mathfrak{B}_2 = |(c_{12} + c_{22})(c_{11} - c_{21})|$

$n = 3$:

$\mathfrak{B}_3 = |(c_{13} + c_{23} + c_{33})(c_{11} - c_{21})(c_{12} + c_{22} - 2c_{32})|$

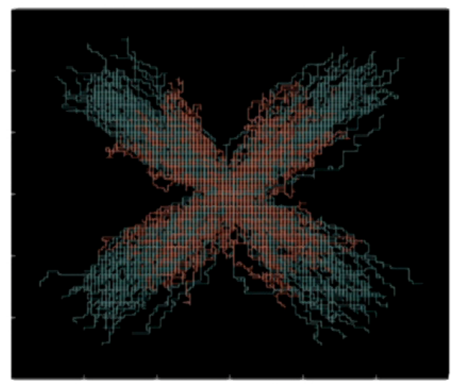
Simulation for $n = 2,3$





Orange – local correlations

Blue – nonlocal correlations



Tsirelson bound

Theorem:

$$\boxed{\mathfrak{B}_n \leq n!}$$

$\boxed{n!\ \text{is the Tsirelson bound}}$

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Tsirelson bound – proof (1)

$$\begin{array}{c} \qquad\quad \vec{v}_{j=2} \\ \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ & -2 & \ddots & \vdots & 1 \\ & & \ddots & 1 & \vdots \\ & & & -(n-1) & 1 \end{bmatrix} \end{array}$$

$\boxed{\text{First part – proof that } \mathfrak{B}_n \leq n!}$

<u>Outline:</u>

$$\boxed{\mathfrak{B}_n \triangleq \prod_{j=1}^{n} \left| \vec{v}_j \cdot \vec{c}_j \right|}$$

1. Eigenvectors' norms' product is $n!$

$$\|\vec{v}_n\|^2 \prod_{k=1}^{n-1} \|\vec{v}_k\|^2 = n \prod_{k=1}^{n-1} (k + k^2) = (n!)^2$$

2. From PSD of the covariance matrix:

$$\forall j \in \{1,2, \dots, n\}, \vec{v}_j^T \vec{c}_j \vec{c}_j^T \vec{v}_j \leq \vec{v}_j^T R_A \vec{v}_j$$

3. AM-GM inequality:

$$\prod_{j=1}^{n} \hat{v}_j^T R_A \hat{v}_j \leq \left(\frac{1}{n} \underbrace{\sum_{j=1}^{n} \hat{v}_j^T R_A \hat{v}_j}_{= n} \right)^n = 1$$

$$\underbrace{\begin{bmatrix} \langle A_1 A_1 \rangle & \cdots & \langle A_1 A_n \rangle \\ \vdots & \ddots & \vdots \\ \langle A_n A_1 \rangle & \cdots & \langle A_n A_n \rangle \end{bmatrix}}_{R_A}$$

Tsirelson bound – proof (2)

Second part – proof that $\mathfrak{B}_n = n!$ can be reached with QM

Suppose Alice and Bob share the two-qubit state $|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

- Alice can saturate the inequality:

$$\prod_{j=1}^{n} \hat{v}_j^T R_A \hat{v}_j \leq 1$$

- Her measurement operators $A_i = \hat{a}_i \cdot \vec{\sigma}$:
  - Choose $\hat{a}_1$ arbitrarily
  - For each $i \in \{2,3,\dots,n\}$, choose $\hat{a}_i$ which is orthogonal to the sum of all previously chosen vectors:

$$\hat{a}_i \cdot \sum_{j=1}^{i-1} \hat{a}_j = 0$$

- Bob can saturate the inequalities:

$$\forall j \in \{1,2,\dots,n\}, \vec{v}_j^T \vec{c}_j \vec{c}_j^T \vec{v}_j \leq \vec{v}_j^T R_A \vec{v}_j$$

- His measurement operators $B_j = \hat{b}_j \cdot \vec{\sigma}$:
  - $\vec{b}_j = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \vdots & & \vdots \\ \hat{a}_1 & \cdots & \hat{a}_n \\ \vdots & & \vdots \end{bmatrix}_{3\times n} \vec{v}_j$
  - And then normalize: $\hat{b}_j = \frac{\vec{b}_j}{\|\vec{b}_j\|}$

Reaching the quantum limit – example for $n = 3$

1. Choose $\hat{a}_1$ arbitrarily

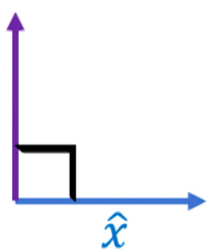
$$A_i = \hat{a}_i \cdot \vec{\sigma}$$

$$\hat{a}_1 = \hat{x}$$

2. For each $i \in \{2,3,\dots,n\}$, choose $\hat{a}_i$:

$$\hat{a}_2 \cdot \hat{a}_1 = 0$$

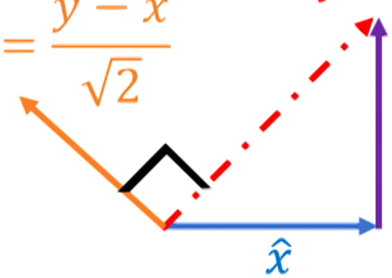
$$\hat{a}_2 = \hat{y}$$



$$\hat{a}_3 \cdot (\hat{a}_1 + \hat{a}_2) = 0$$

$$\hat{y} + \hat{x}$$

$$\hat{a}_3 = \frac{\hat{y} - \hat{x}}{\sqrt{2}}$$

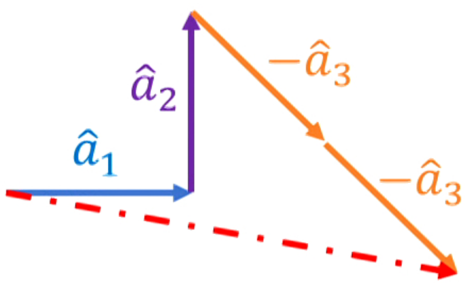


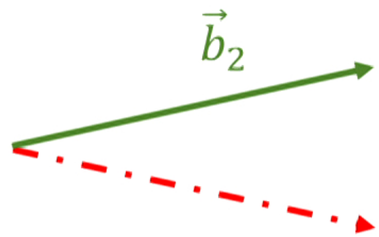
Reaching the quantum limit – example for $n = 3$

1. Construct $\vec{b}_j$ by: $\quad B_j = \hat{b}_j \cdot \vec{\sigma}$

   Demonstration for $j = 2$:

$$\vec{b}_2 = \Lambda \begin{bmatrix} 1 & 0 & -1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}_{\vec{v}_2} = \begin{bmatrix} 1+\sqrt{2} \\ -1+\sqrt{2} \\ 0 \end{bmatrix}$$





Λ flips y component

2. Normalize $\vec{b}_j$

Bell limit – Bob's strategy is deterministic

- Assuming local hidden variables, $\Pr(A_1, \ldots, A_n, B_1, \ldots, B_n)$ exists
- Using linearity of expectations, we can write the Bell parameter as follows:

$$\mathfrak{B}_n = |E[B_n(A_1 + \cdots + A_n)]| \prod_{j=1}^{n-1} |E[B_j(A_1 + \cdots + A_j - jA_{j+1})]|$$

- If Bob's strategy is deterministic $(\forall j, \Pr(B_j = 1) \in \{0,1\})$:

$$\boxed{\mathfrak{B}_n = |P_n(\vec{\mu})|, \mu_i = E[A_i]}$$

- Where $P_n(\vec{\mu})$ is the following function:

$$P_n(\vec{\mu}) = \left(\sum_{i=1}^{n} \mu_i\right) \prod_{k=1}^{n-1} \left(\sum_{j=1}^{k} \mu_j - k \cdot \mu_{k+1}\right)$$

- **Theorem:** $P_n(\vec{\mu})$ is a harmonic function

$$\boxed{\nabla^2 P_n(\vec{\mu}) = 0}$$

What happens when we increase n?

- $\max P_n(\vec{\mu})$ is also hard to find
- We found a "fully-deterministic" strategy, which achieves a special case of $P_n(\vec{\mu})$, denoted by FD_n
- Thus we conclude that: $\boxed{FD_n \leq \max P_n(\vec{\mu}) \leq Bell\ limit \leq n!}$

Theorem:

$$\boxed{\lim_{n\to\infty} \frac{FD_n}{n!} = \sqrt{\frac{\pi}{2e}} \approx 0.76}$$

Which would imply that in the limit of infinitely many possible measurement devices, the Bell and Tsirelson bounds are proportional

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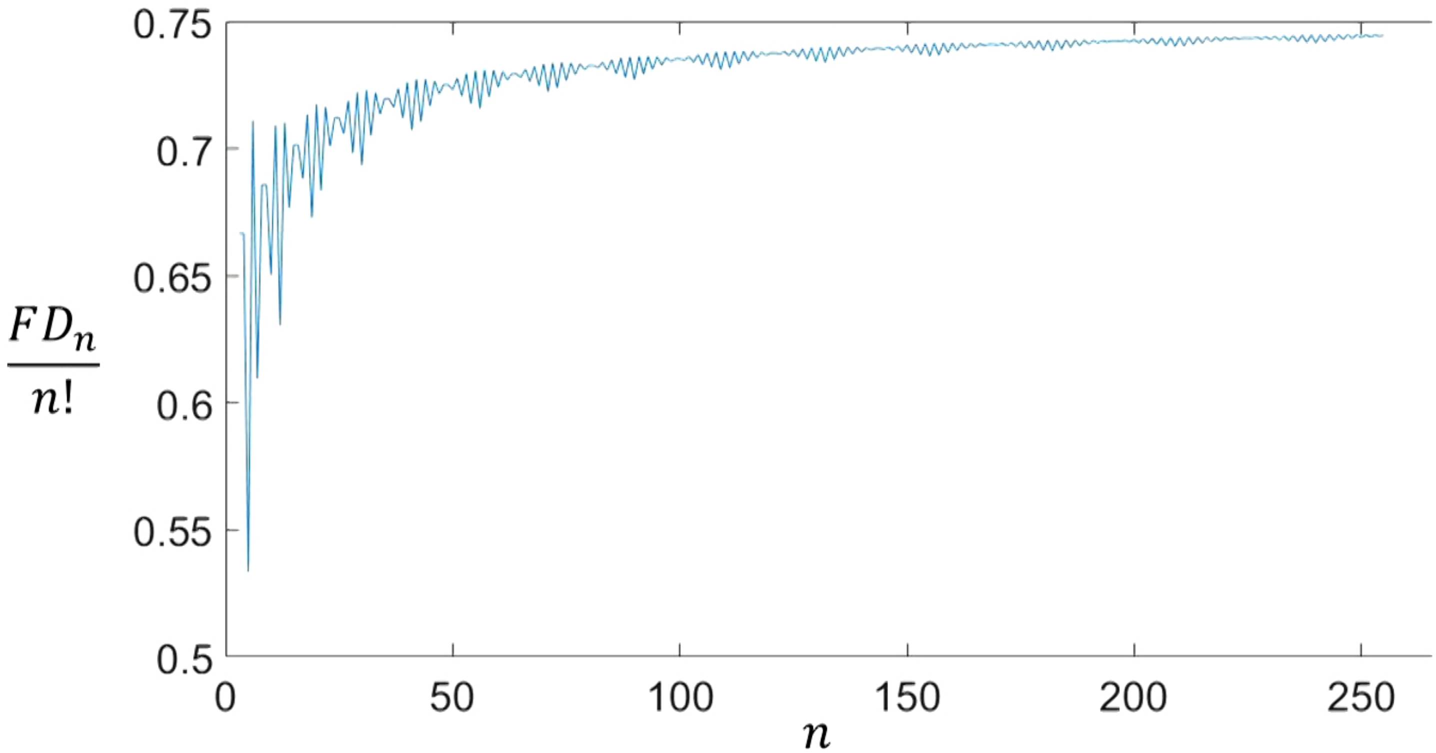
Simulation & numerical results

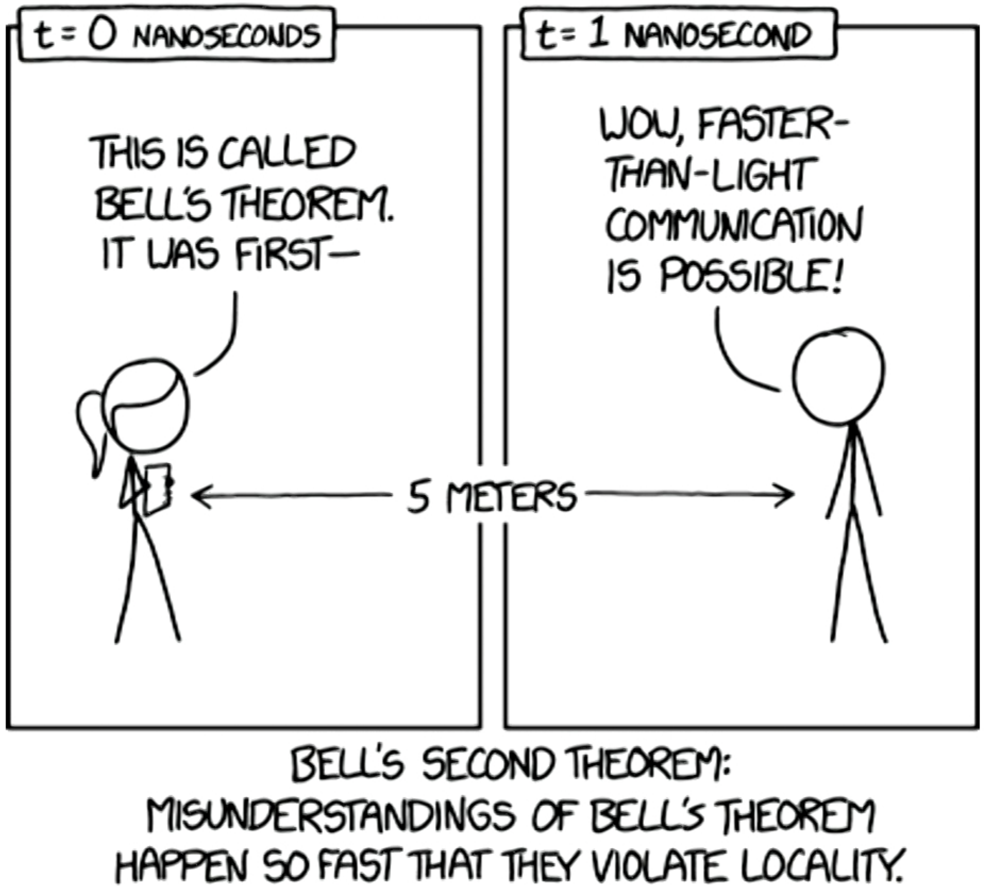


n	FD_n	Tsirelson / quantum limit ($n!$)
2	1	2
3	4	6
4	16	24
5	64	120
6	512	720
7	3072	5040
8	27648	40320
9	248832	362880
10	2359296	3628800

Summary

- New class of Bell inequalities
- General expression for Tsirelson's bound for an arbitrary number of devices
- Lower bound for Bell (easy to compute) which is proportional to Tsirelson's bound for a large number of devices

Thank you for listening!



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