Title: Fun with path integrals I

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Collection: Simplicity III

Date: September 12, 2019 - 11:00 AM

URL: http://pirsa.org/19090089

Fun With Path Integrals

Neil Turok

collaborators: J. Feldbrugge, A. Fertig, J-L Lehners, L. Sberna, U-L. Pen

interference

basic to quantum physics

universal

all known physics

$$\Psi = \int e^{\frac{i}{\hbar}\int\left(\frac{R}{16\pi G} - \frac{1}{4}F^2 + \bar{\psi} i \not{D} \psi - \lambda H \bar{\psi}\psi + |DH|^2 - V(H)\right)}$$

Schrödinger, Feynman, Euler, Planck, Einstein, Newton, Maxwell-Yang-Mills, Dirac, Kobayashi-Maskawa, Yukawa, Lagrange, Higgs

dark energy

$$\psi = (u_L, d_L, u_R, d_R, u_L, d_L, u_R, d_R, u_L, d_L, u_R, d_R, e_L, \nu_L, e_R, \nu_R) \times 3$$

$$M \nu_R \nu_R$$

dark matter?

Highly oscillatory integrals $\Psi = \left(\frac{\nu}{\pi}\right)^{\frac{D}{2}} \int d^D\vec{x}\, e^{i\nu\varphi(\vec{x})}$

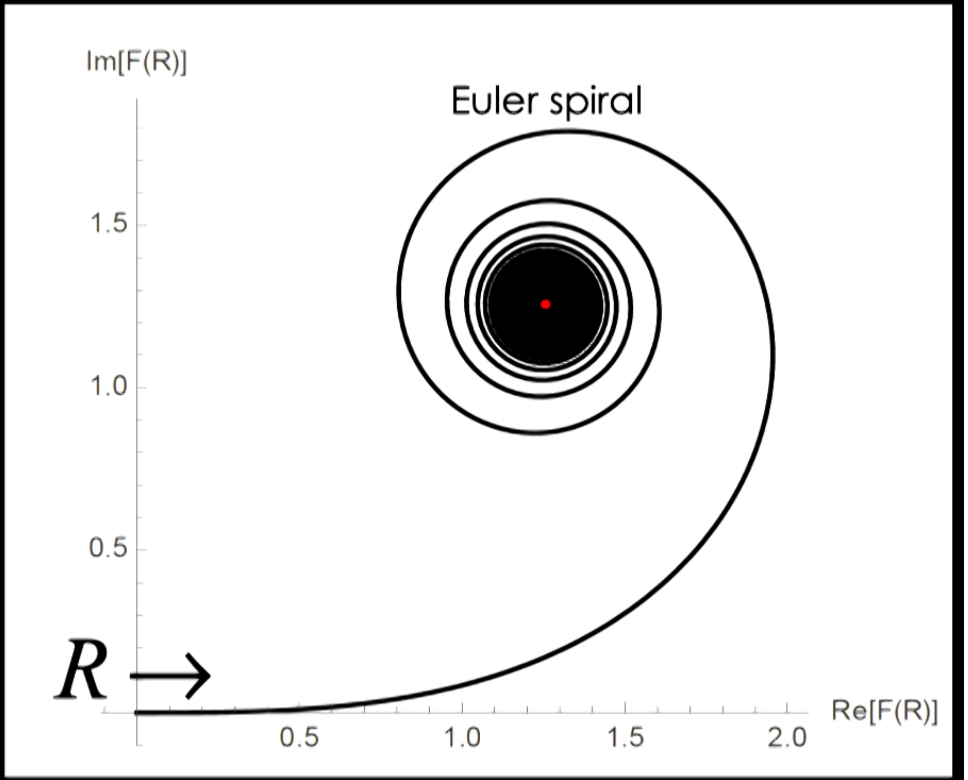
e.g. Fresnel integral

$$F(R) = \int_{-R}^{+R} e^{ix^2}\, dx$$

$$I = \lim_{R\to\infty} F(R) = e^{i\frac{\pi}{4}}\sqrt{\pi}$$

Conditionally, not absolutely convergent

Higher dimensional case?



2d: square cutoff

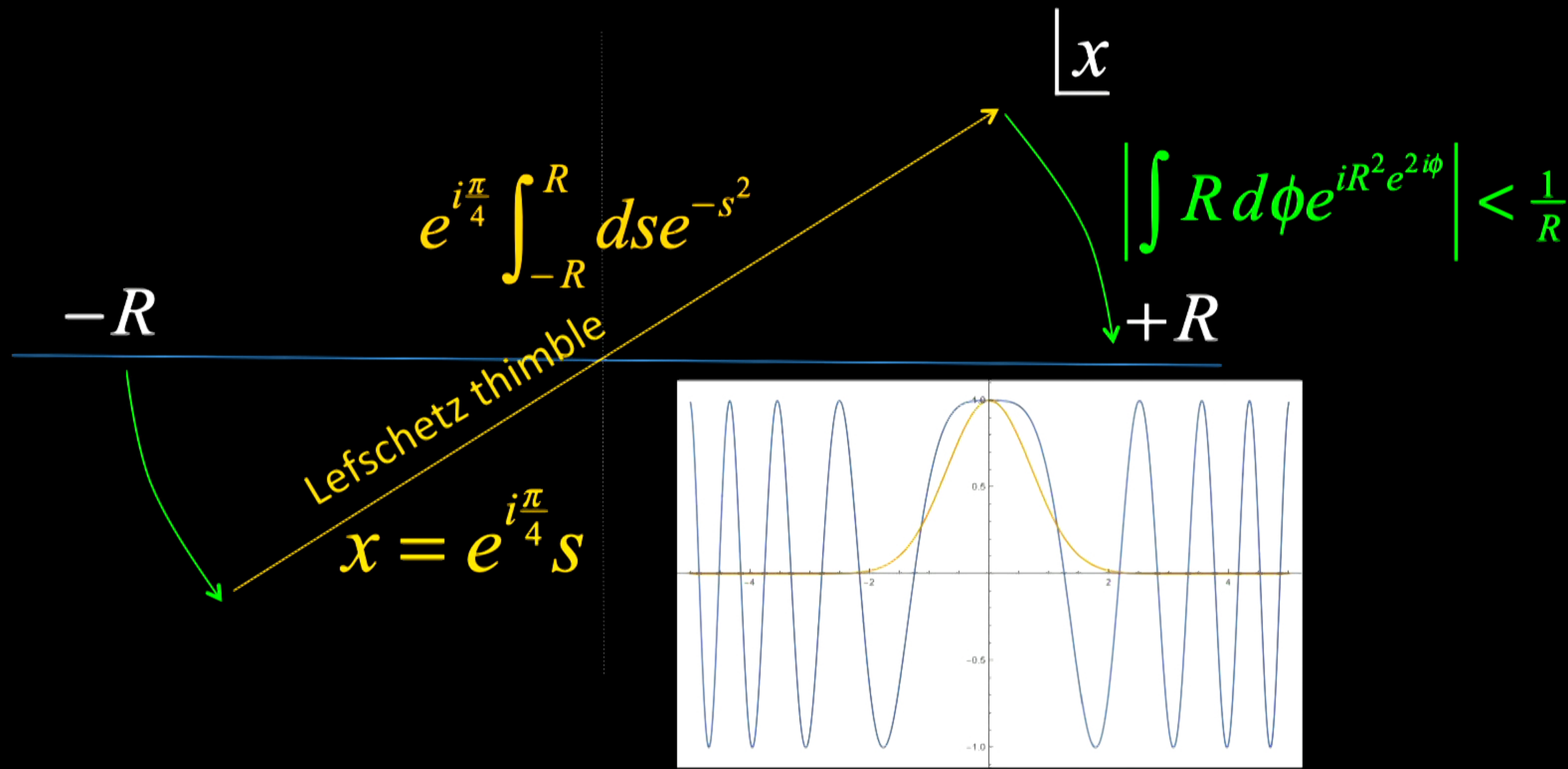
$$\iint dx\, dy e^{i(x^2+y^2)} = I^2 = i\pi$$

2d: round cutoff

$$\lim_{R\to\infty} 2\pi \int_0^R R\, dR e^{iR^2} = \lim_{R\to\infty} \frac{\pi}{i}\left(e^{iR^2} - 1\right)$$

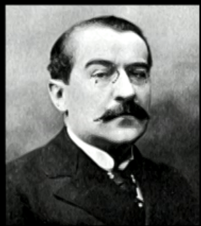
???

Resolution: use complex analyticity and Cauchy's theorem



Deforming the integration contour renders the integral *absolutely* convergent: arcs at infinity vanish as $R \longrightarrow \infty$

higher dimensions:
Picard-Lefschetz

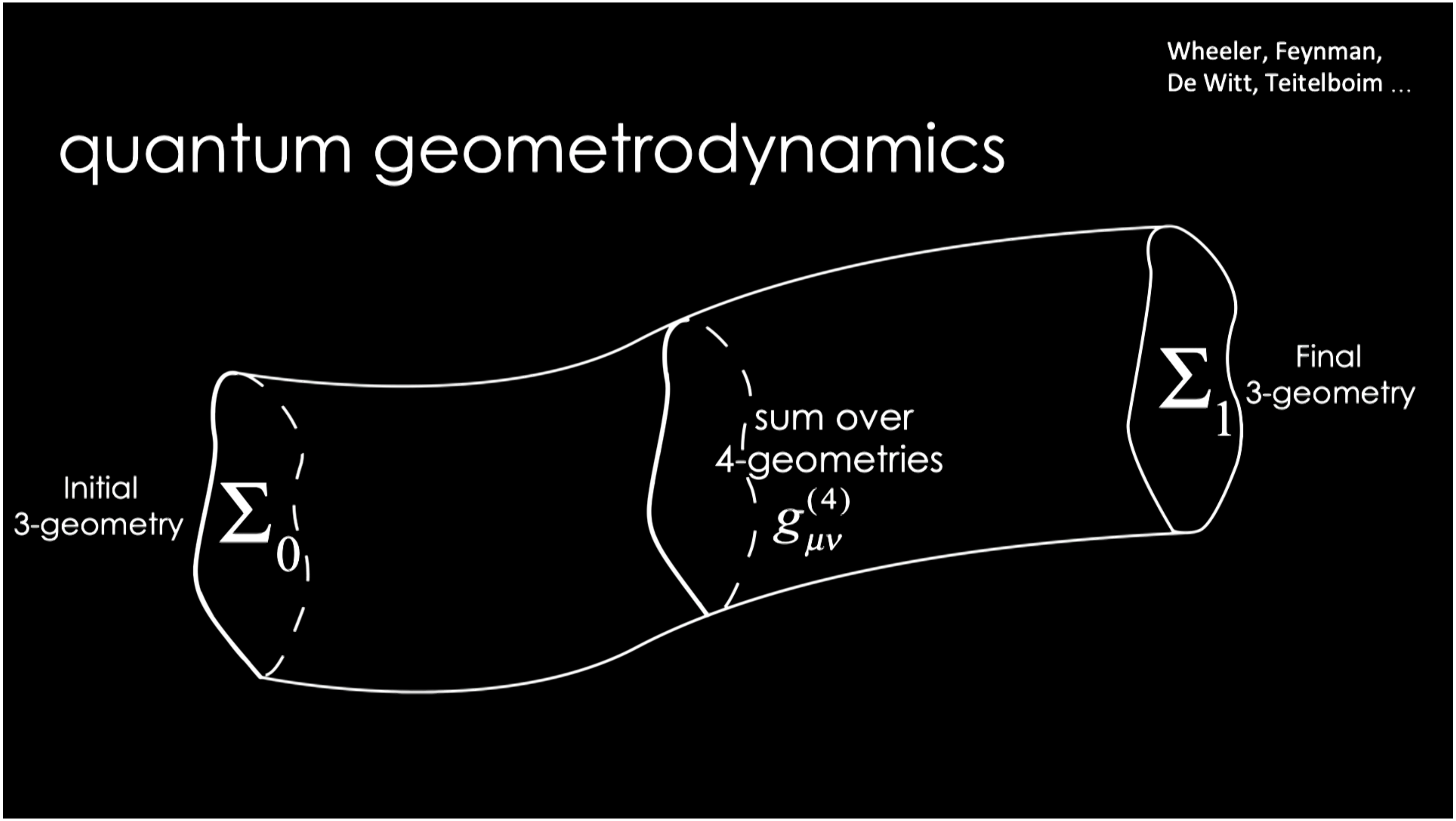
 

general method for analysing and performing highly oscillatory integrals:

``flow" the original contour onto a series of relevant "Lefschetz thimbles"

generalizes to arbitrary finite dimension

new approach to "real" path integrals (Feldbrugge and NT, in prep)


Wheeler, Feynman,
De Witt, Teitelboim …
quantum geometrodynamics
Initial
3-geometry
Σ_0
sum over
4-geometries
$g^{(4)}_{\mu\nu}$
Σ_1
Final
3-geometry

particles without quantum fields
e.g. pair creation in an electric field

(see Feynman's Dirac Memorial Lecture)

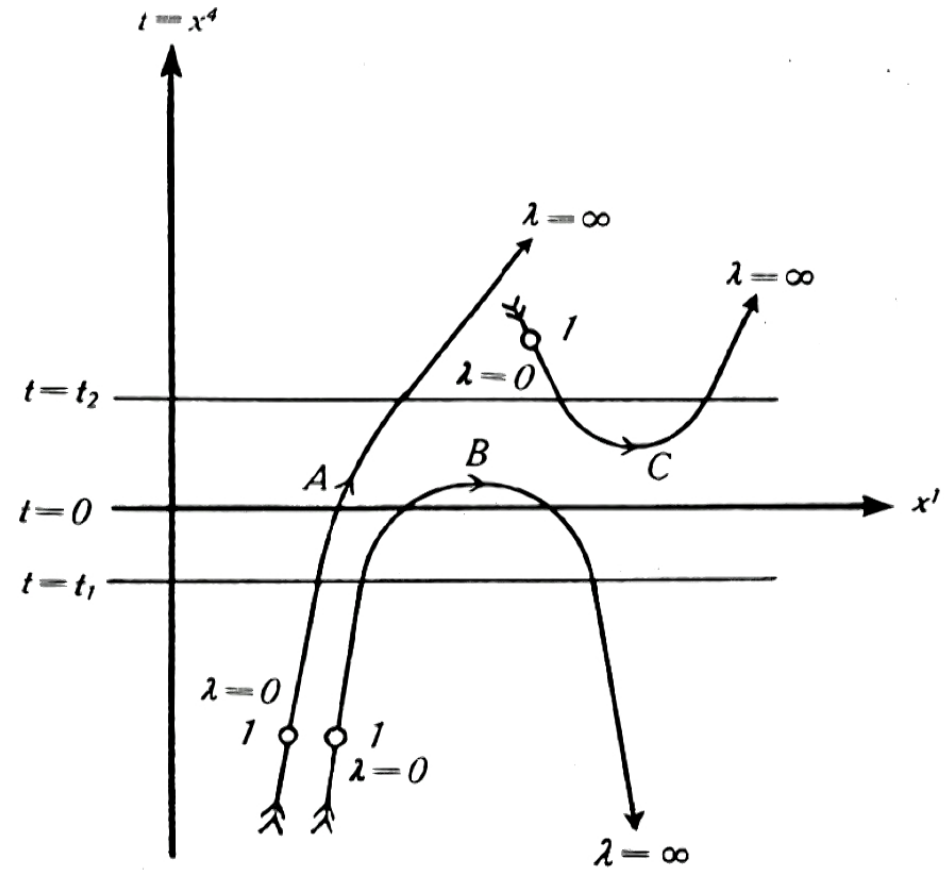
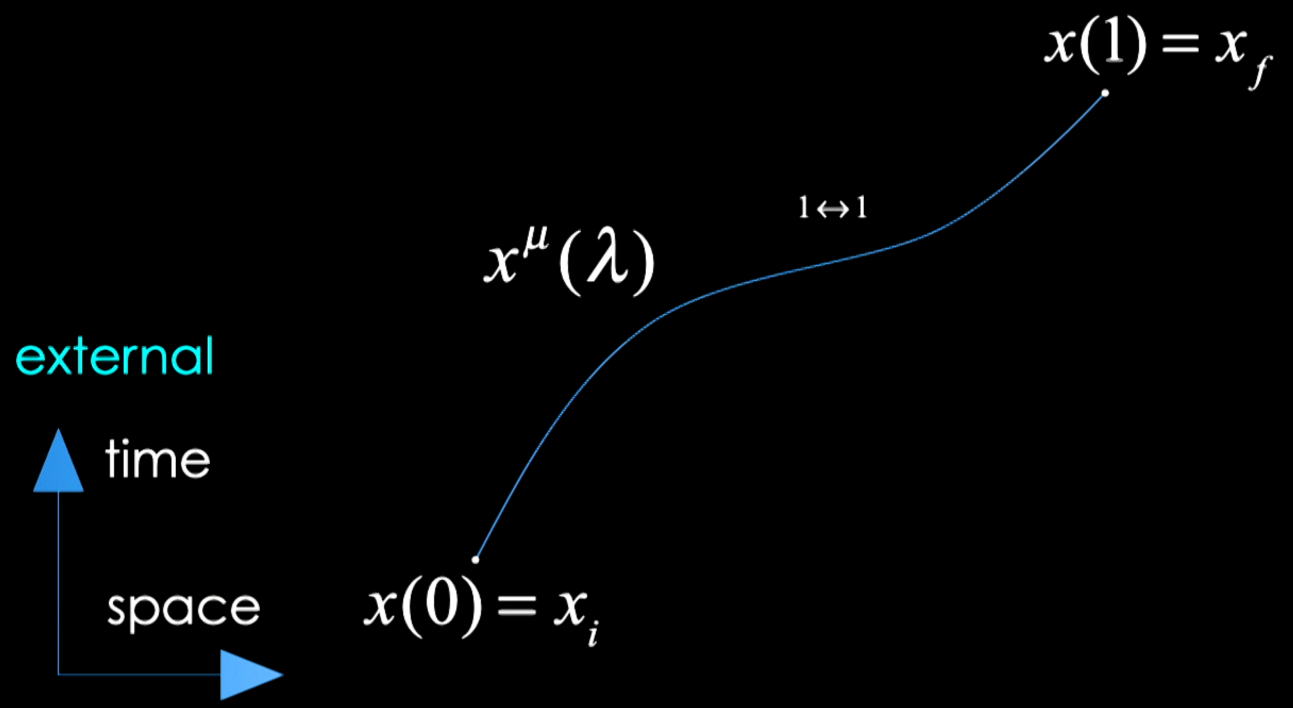


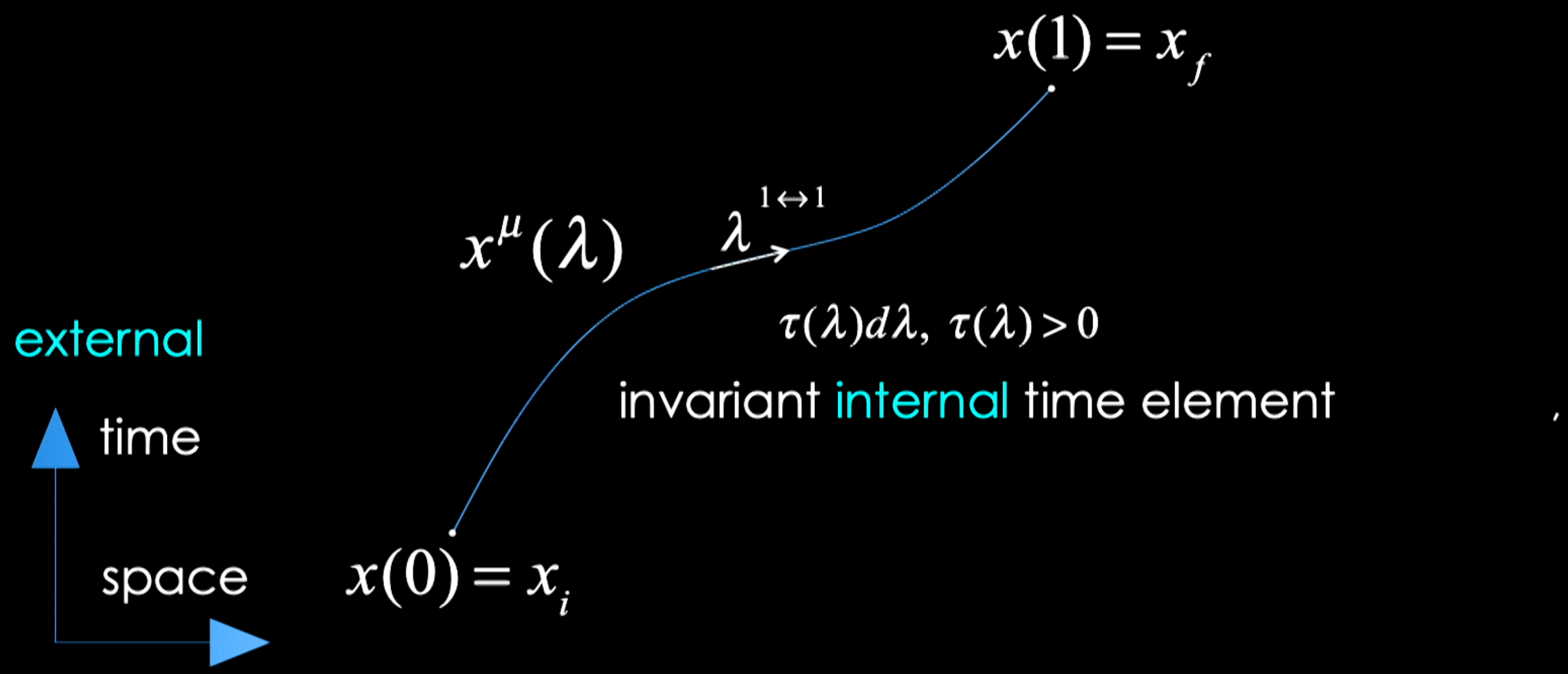
Fig. 1.

Ernst Stueckelberg 1941

relativistic particle



relativistic particle



Spacetime amplitude approach to relativistic quantum mechanics

w/ J. Feldbrugge, A. Fertig, J-L Lehners, L. Sberna
to appear shortly

Kinematics

$\Psi(x)$ - amplitude for particle to be at spacetime point x

Probability $\int d^4x\ \Psi^*(x)\Psi(x) = 1$ (Schrodinger)

$$\left[\hat{x}^\mu, \hat{p}_\nu\right] = i\hbar\delta^\mu_\nu \qquad \left\langle x^\mu \right\rangle = \int d^4x\ \Psi^*(x)\ x^\mu\ \Psi(x),$$

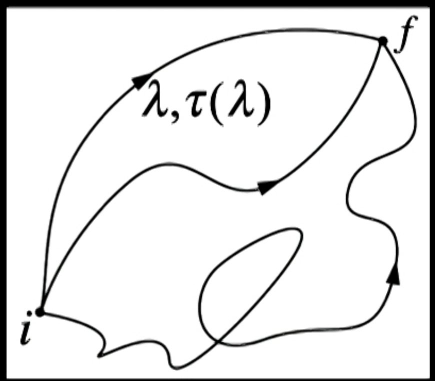
$$\left\langle p^\mu \right\rangle = \int d^4x\ \Psi^*(x)(-i\hbar\partial^\mu)\Psi(x), \qquad \text{etc}$$

energy-time uncertainty relation built in

Dynamics

e.g. in Minkowski

$$H = \eta^{\mu\nu} p_\mu p_\nu + m^2 \Leftrightarrow S = \int_0^1 d\lambda \; (\tfrac{1}{4\tau}\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - m^2\tau);$$



amplitude for a particle at x_i^μ to be observed at x_f^μ

$$\int_{0^+}^{\infty} d\tau \left\langle x_f \middle| e^{-i\hat{H}\tau/\hbar} \middle| x_i \right\rangle = \int_{0^+}^{\infty} d\tau \int_{x_i}^{x_f} Dx e^{iS/\hbar} = G_F(x_f, x_i)$$

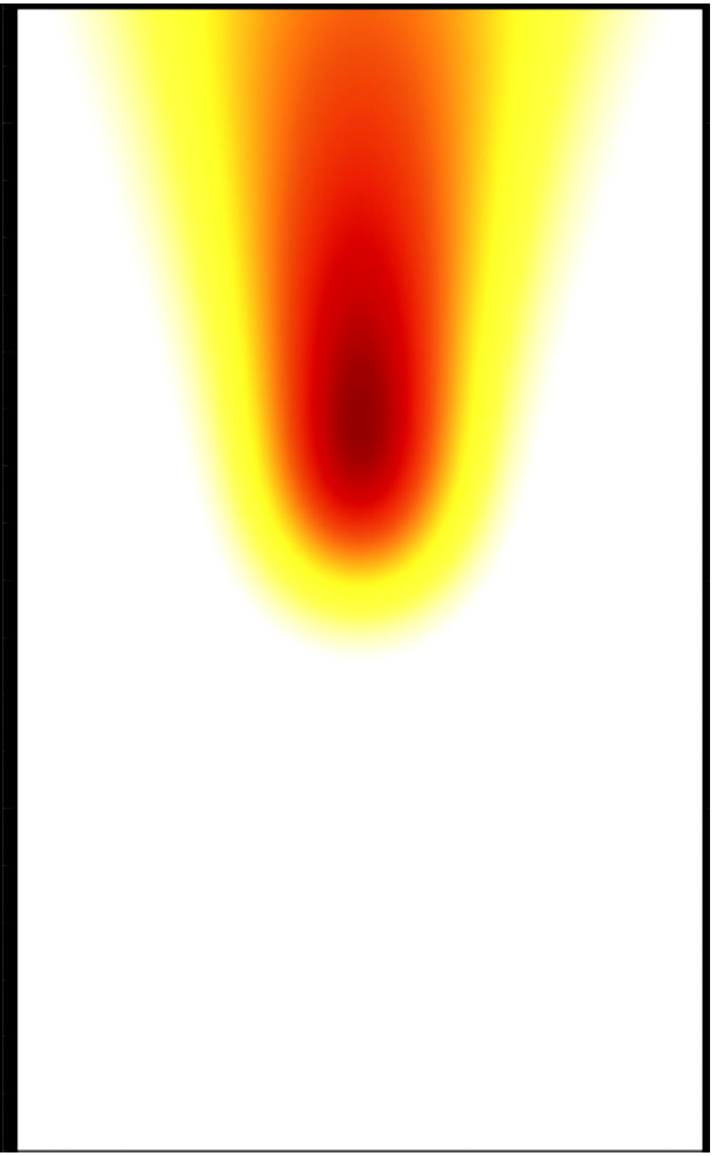
(in gauge $\tau = \int_0^1 \tau(\lambda)\, d\lambda = const$) $\hat{H} G_F(x,x') = -i\hbar\delta(x-x')$

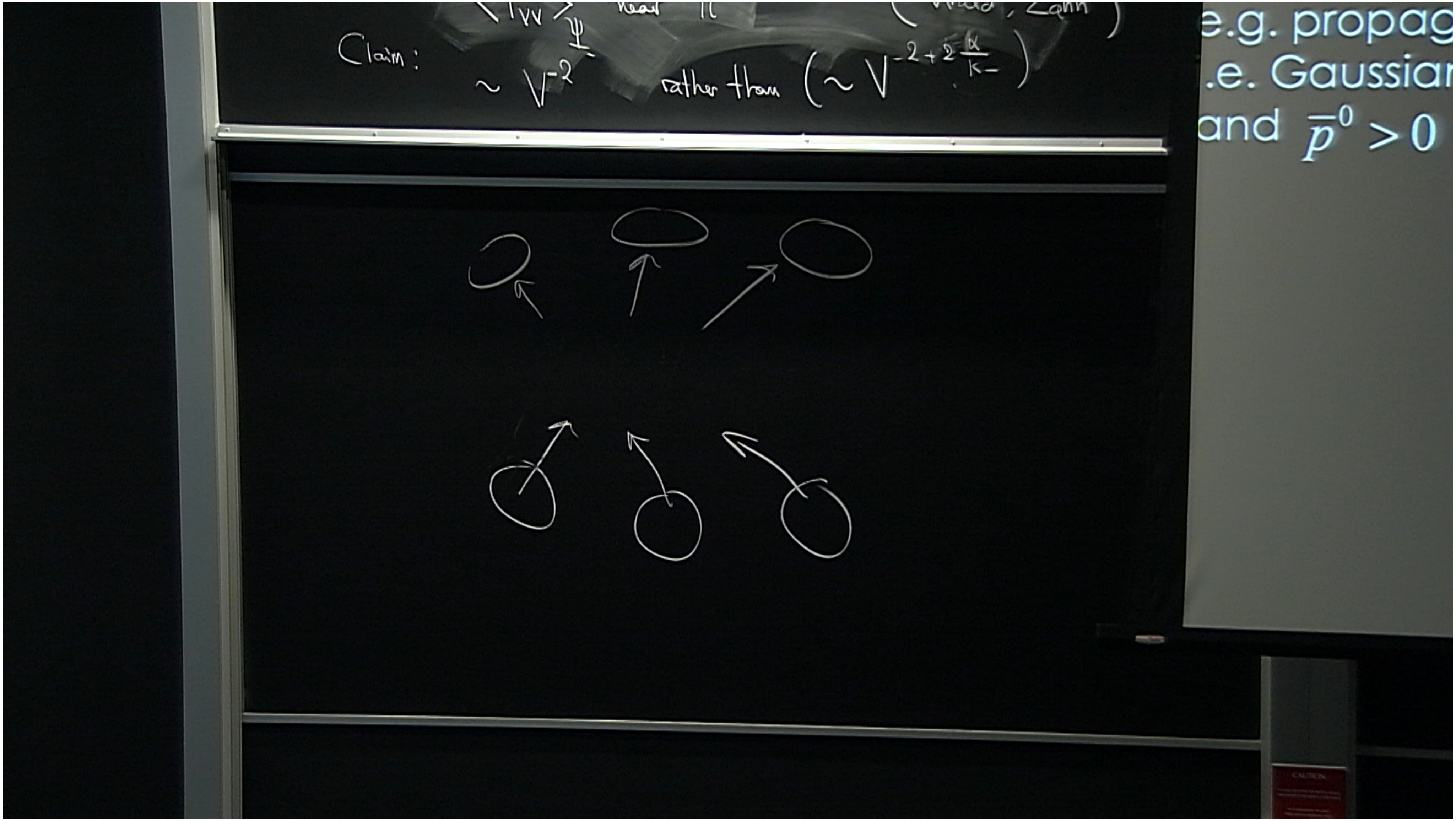
Feynman propagator is a consequence of geometry and QM

unambiguous, no need for $i\varepsilon$ (gives simplified passage to NR QM)
will be interesting to calculate in other spacetimes

e.g. propagate a localized particle
i.e. Gaussian with spacetime spread
and $\bar{p}^0 > 0$

$$\left| \int d^4x' G_F(x,x') \Psi(x') \right|^2$$

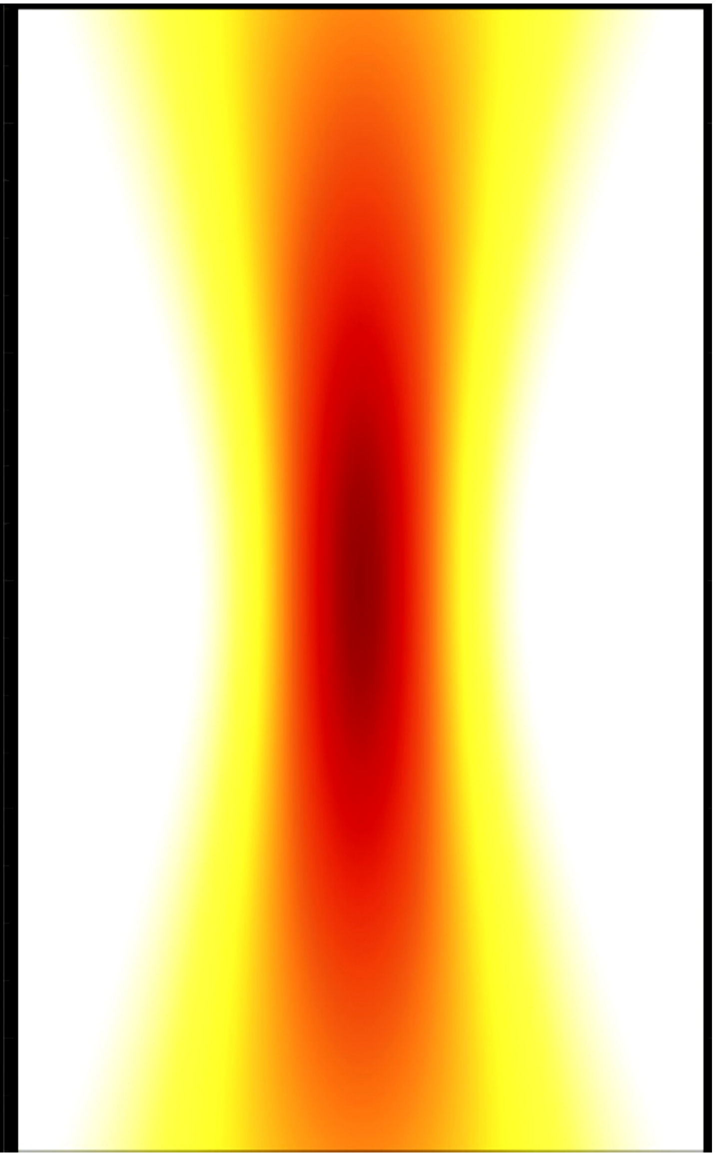



Claim:
~ V^-2 rather than (~ V^{-2+2 α/κ_-})
e.g. propag
.e. Gaussiar
and p̄^0 > 0

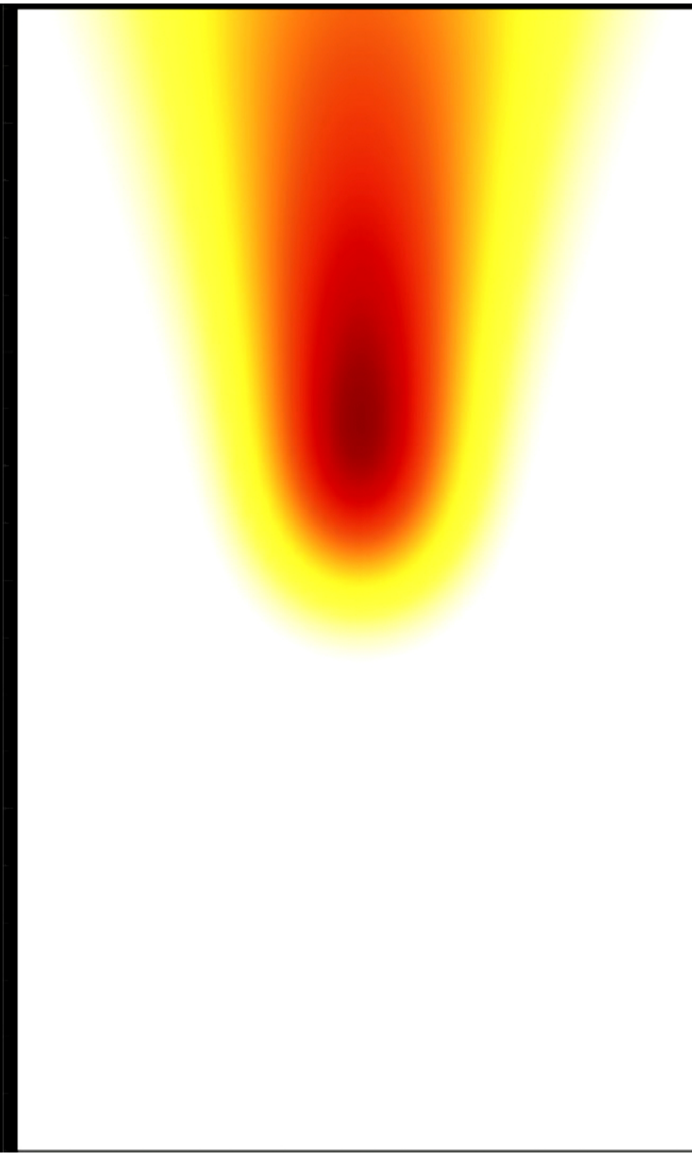
connection with on-shell ("physical") states

$$\int_{-\infty}^{\infty} d\tau \left\langle x_f \middle| e^{-i\hat{H}\tau/\hbar} \middle| x_i \right\rangle = \hbar 2\pi \delta(\hat{H}) = \mathrm{Re}[G_F(x_f, x_i)]$$

$$\left|\int d^4x' \,\mathrm{Re}[G_F(x,x')]\Psi(x')\right|^2$$



$$\left| \int d^4x' G_F(x,x') \Psi(x') \right|^2$$

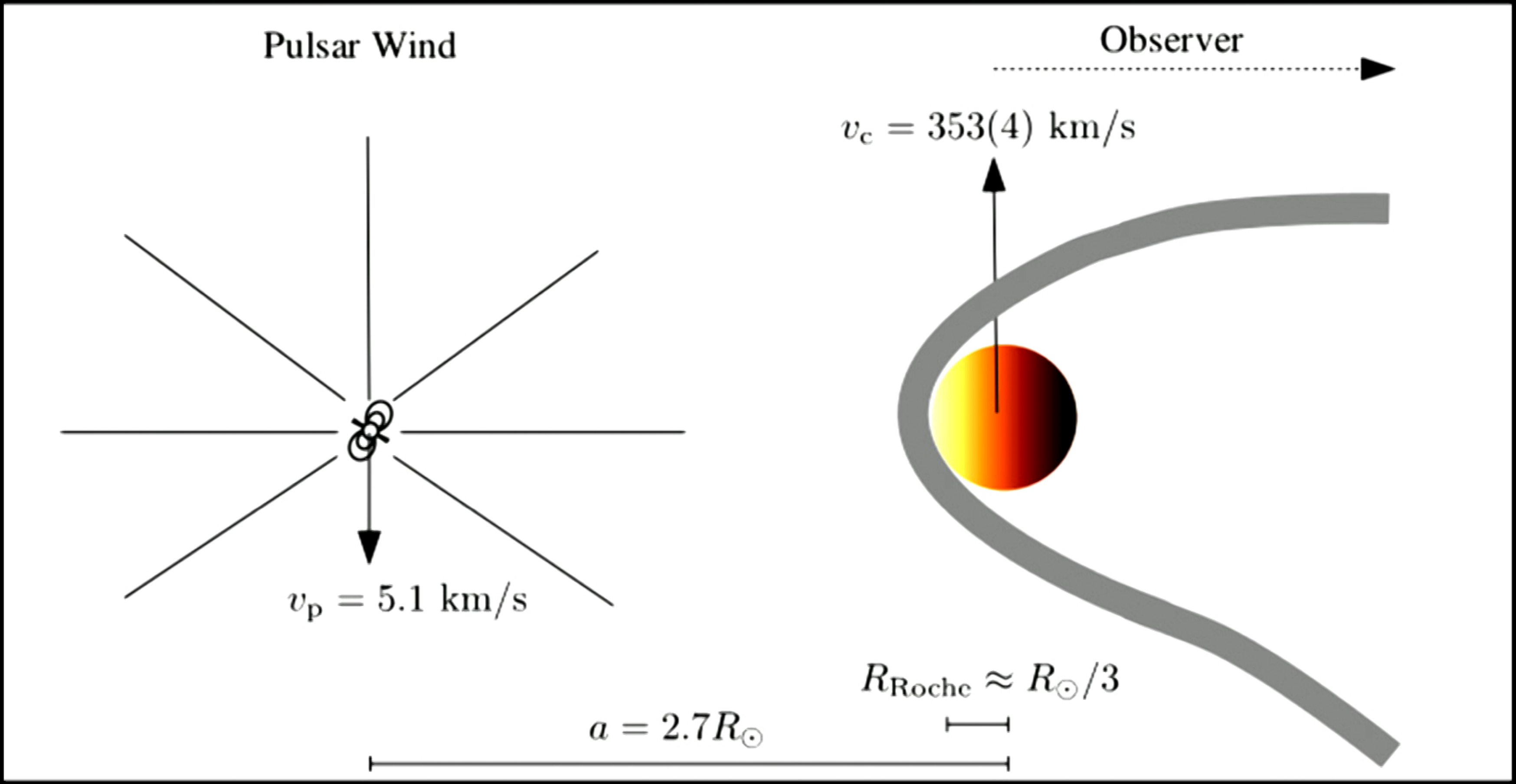


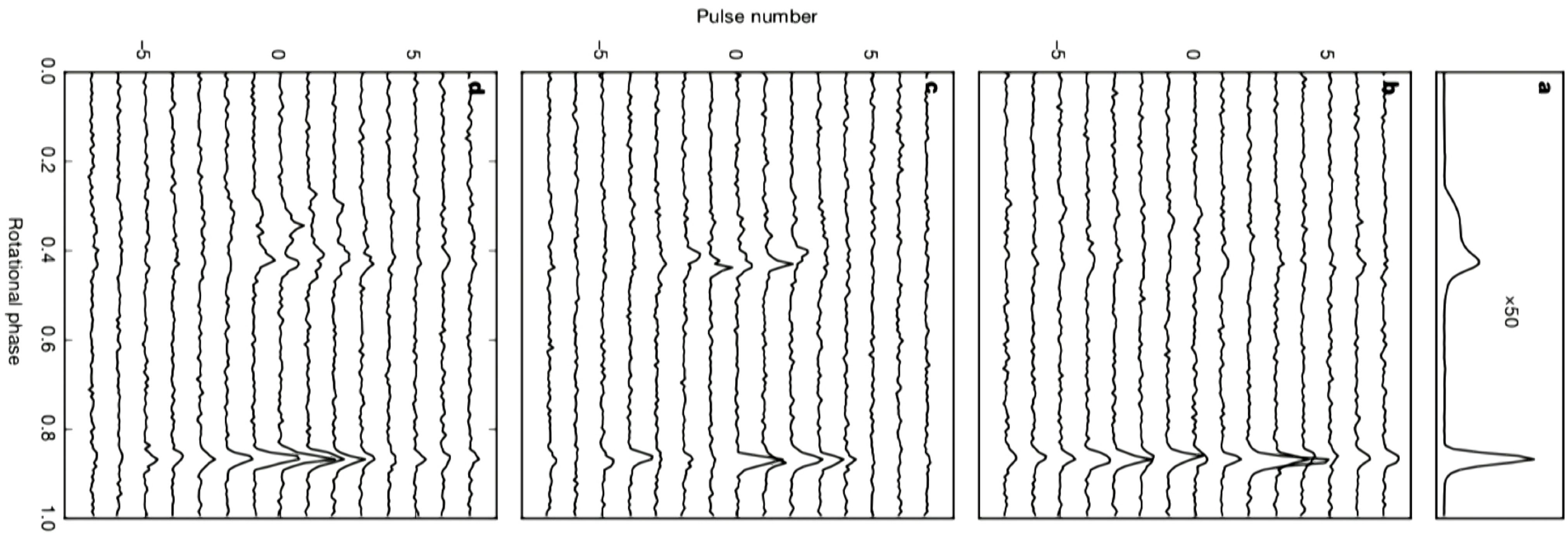
application to radio astronomy

Feldbrugge, Pen, NT 1909.04632

Plasma lensing of the "black widow" pulsar PL1957+20

R. Main et al 2018





Pulse number
Rotational phase
a
b
c
d
×50

Feynman path integral $\Longrightarrow$ Fermat's principle of least time

start from dispersion relation in plasma: $\omega^2 = k^2 c^2 + \omega_p^2(\vec{x})$

phase speed $\frac{\omega}{k} = c\sqrt{1+\frac{\omega_p^2(\vec{x})}{k^2c^2}}$; group speed $\left|\frac{d\omega}{d\vec{k}}\right| = \frac{c}{\sqrt{1+\frac{\omega_p^2(\vec{x})}{k^2c^2}}}$;

Hamiltonian $H = -p_0^2 c^2 + p^2 c^2 + \hbar^2 \omega^2(\vec{x})$

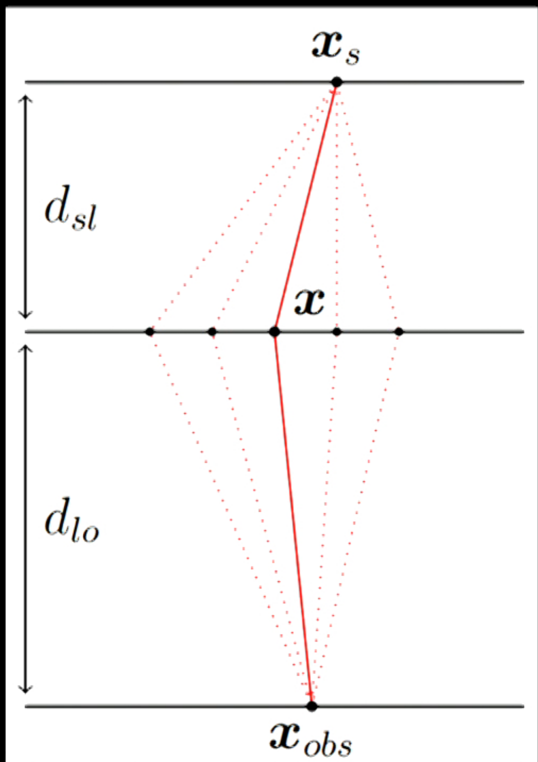
generates reparameterisations: $H \approx 0$ for classical solutions

action $S = \int_0^1 d\lambda (p_0 \dot{x}^0 + \vec{p}\cdot\dot{\vec{x}} - \tau(\lambda) H)$

BUT if we fix initial and final x^0 how can Fermat's principle arise?

Answer: fix initial *energy* (monochromatic beam) and final time

thin lens



$$\int d\vec{x}_\perp e^{i\omega \int |d\vec{x}| \frac{n(\vec{x})}{c}}$$

Pythagoras

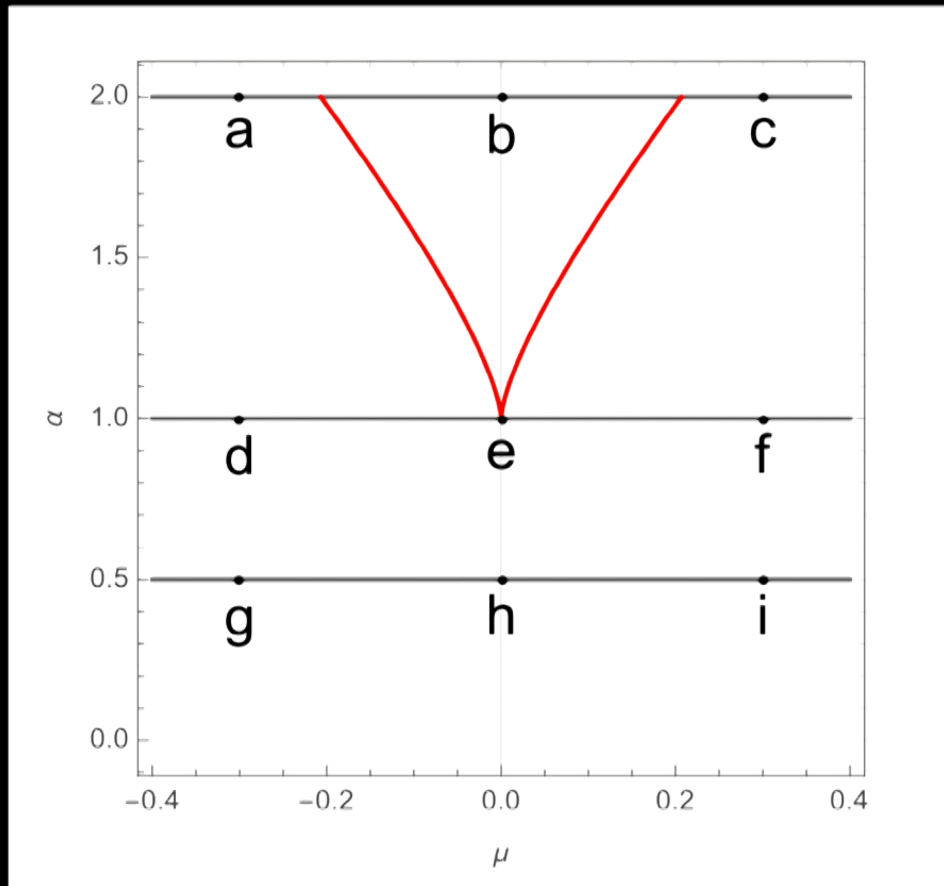
Refraction

$$\approx \int d\vec{x}_\perp e^{i\frac{\omega}{2c}\left[\frac{(\vec{x}_\perp - \vec{\mu})^2}{\bar{d}} - \int dz \frac{\omega_p^2(\vec{x}_\perp, z)}{\omega^2}\right]}$$

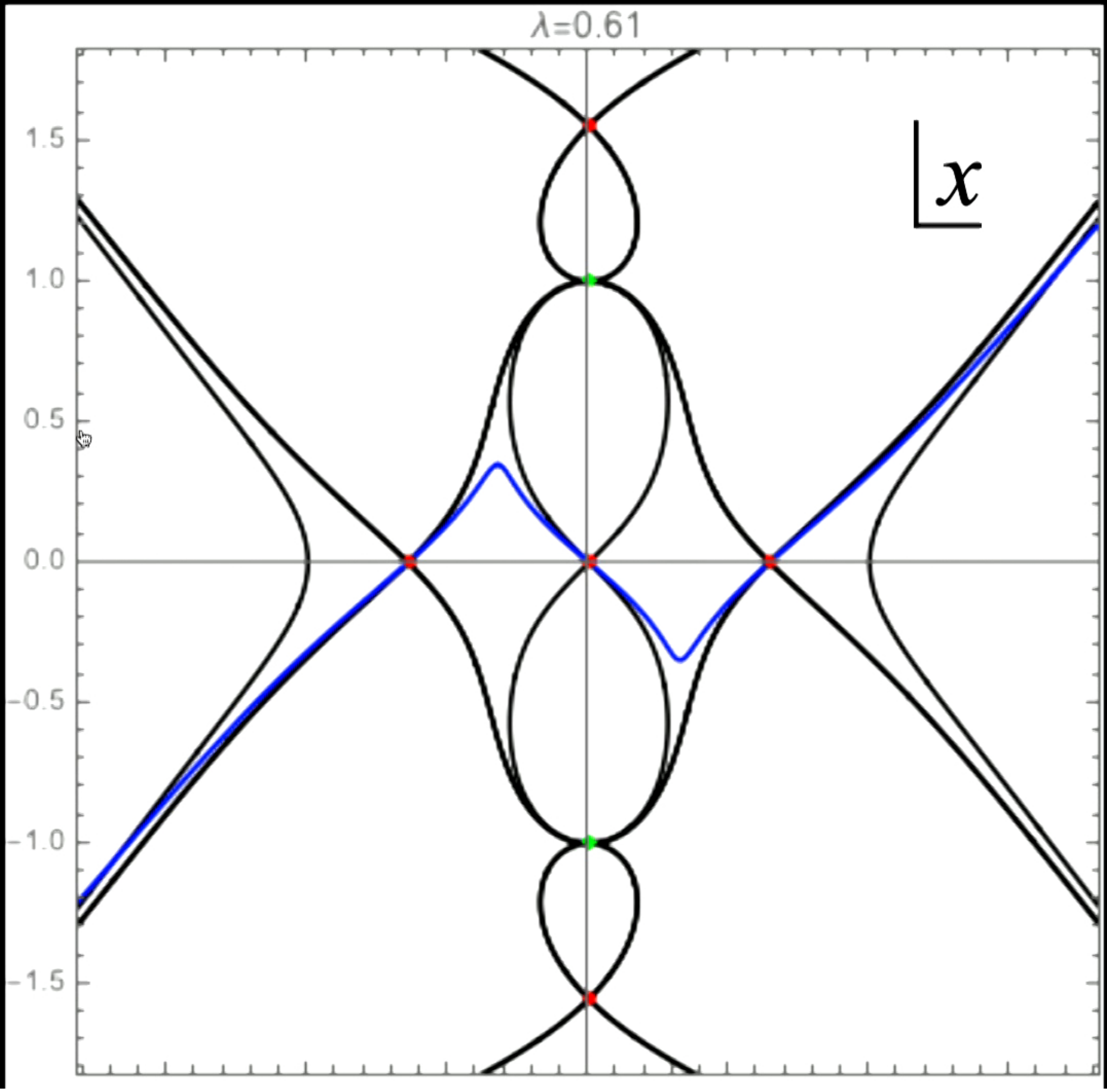
lensing strongest at low frequencies

$$\frac{1}{\bar{d}} \equiv \frac{1}{d_{sl}} + \frac{1}{d_{lo}}; \quad n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}\ ; \quad \omega_p^2 = \frac{n_e(\vec{x})e^2}{\varepsilon_0 m_e}$$

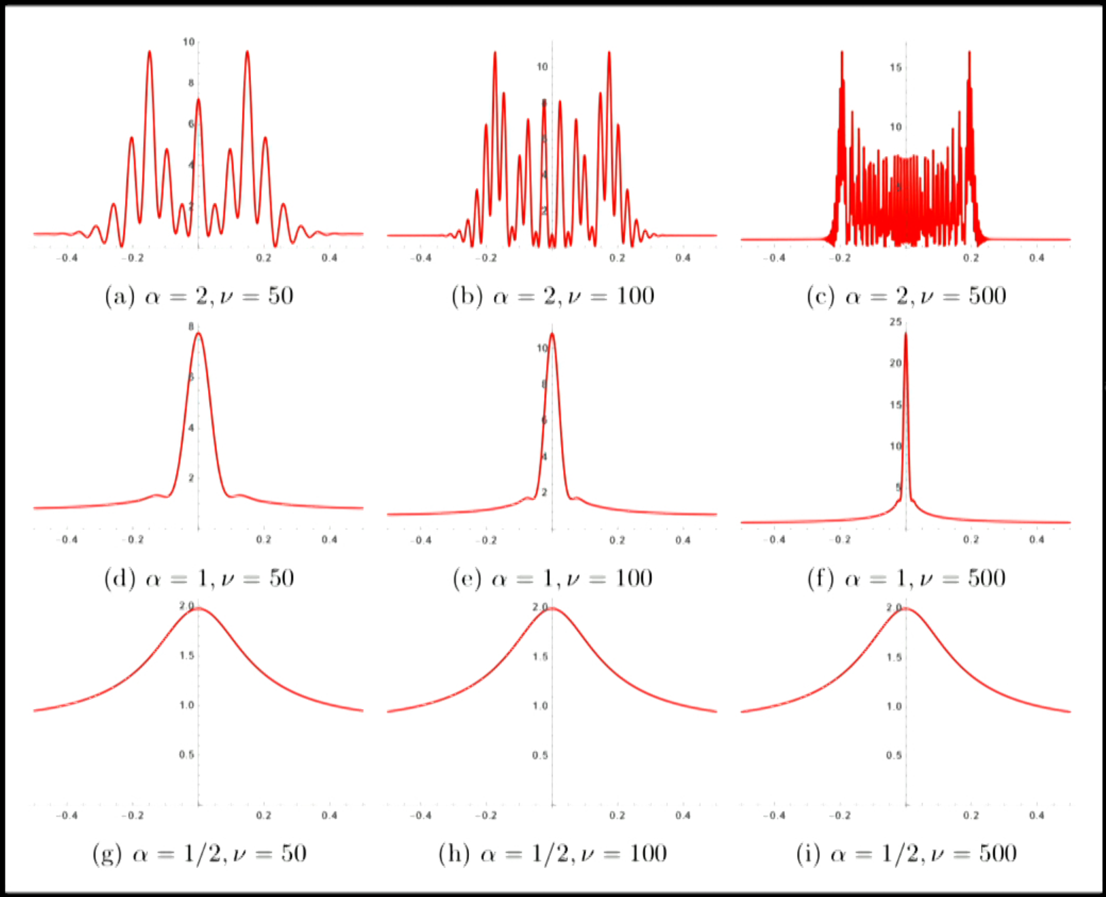
e.g. 1d localised lens: $\Phi(x) = (x-\mu)^2 + \frac{\alpha}{(1+x^2)}$

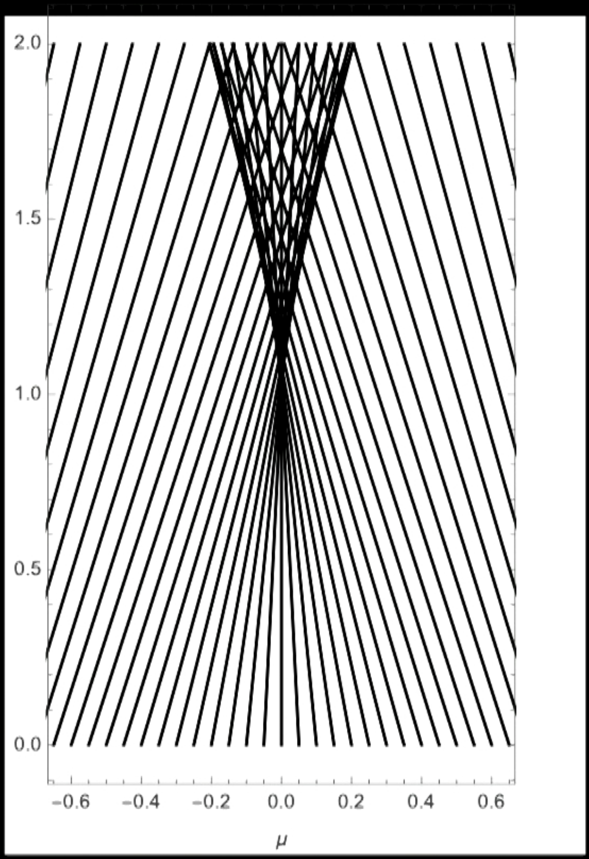


Flowing the contour (case b)

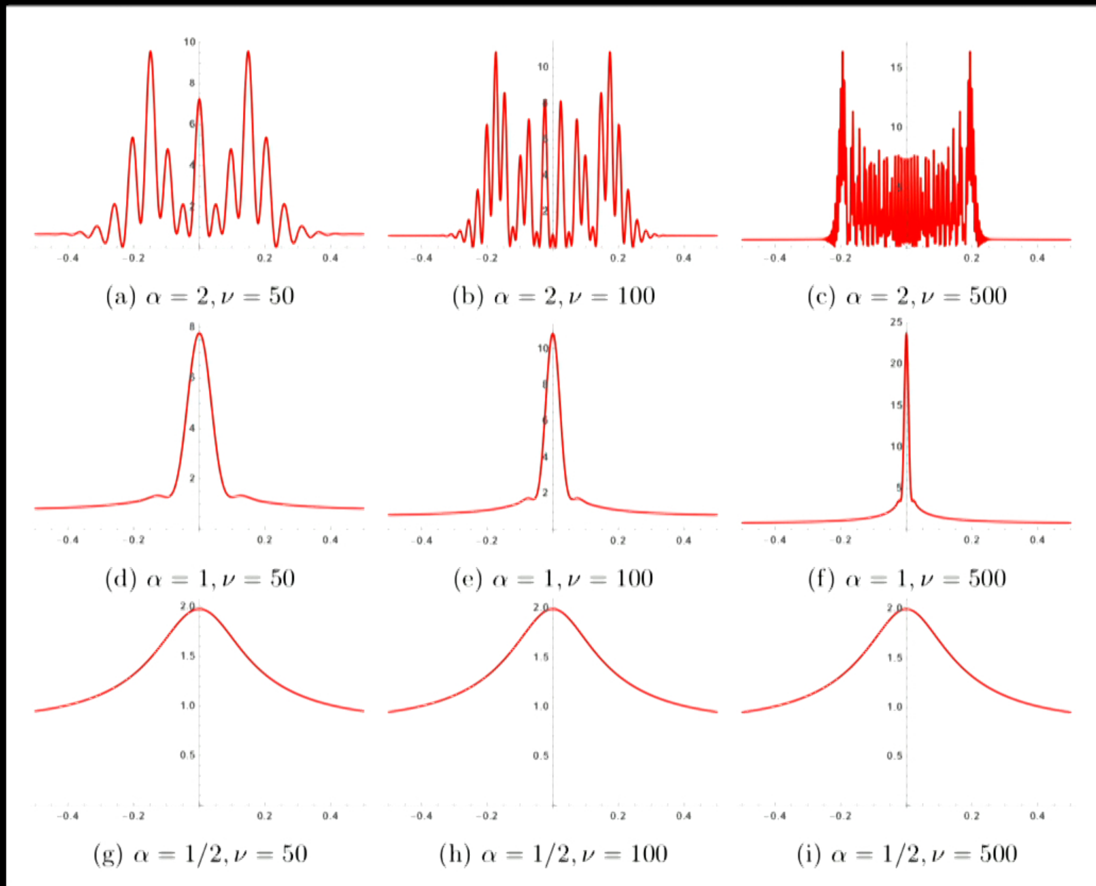


Interference patterns at increasing ν geometric optics limit

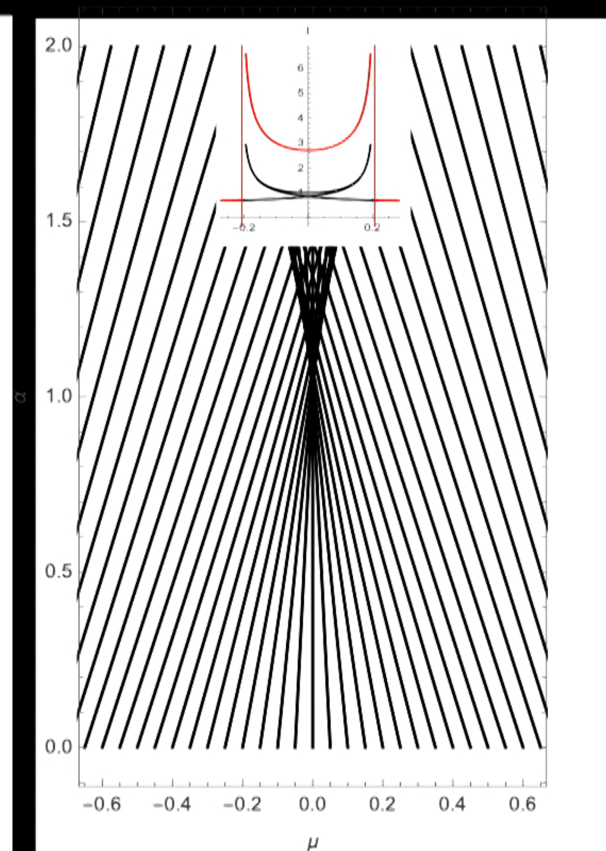


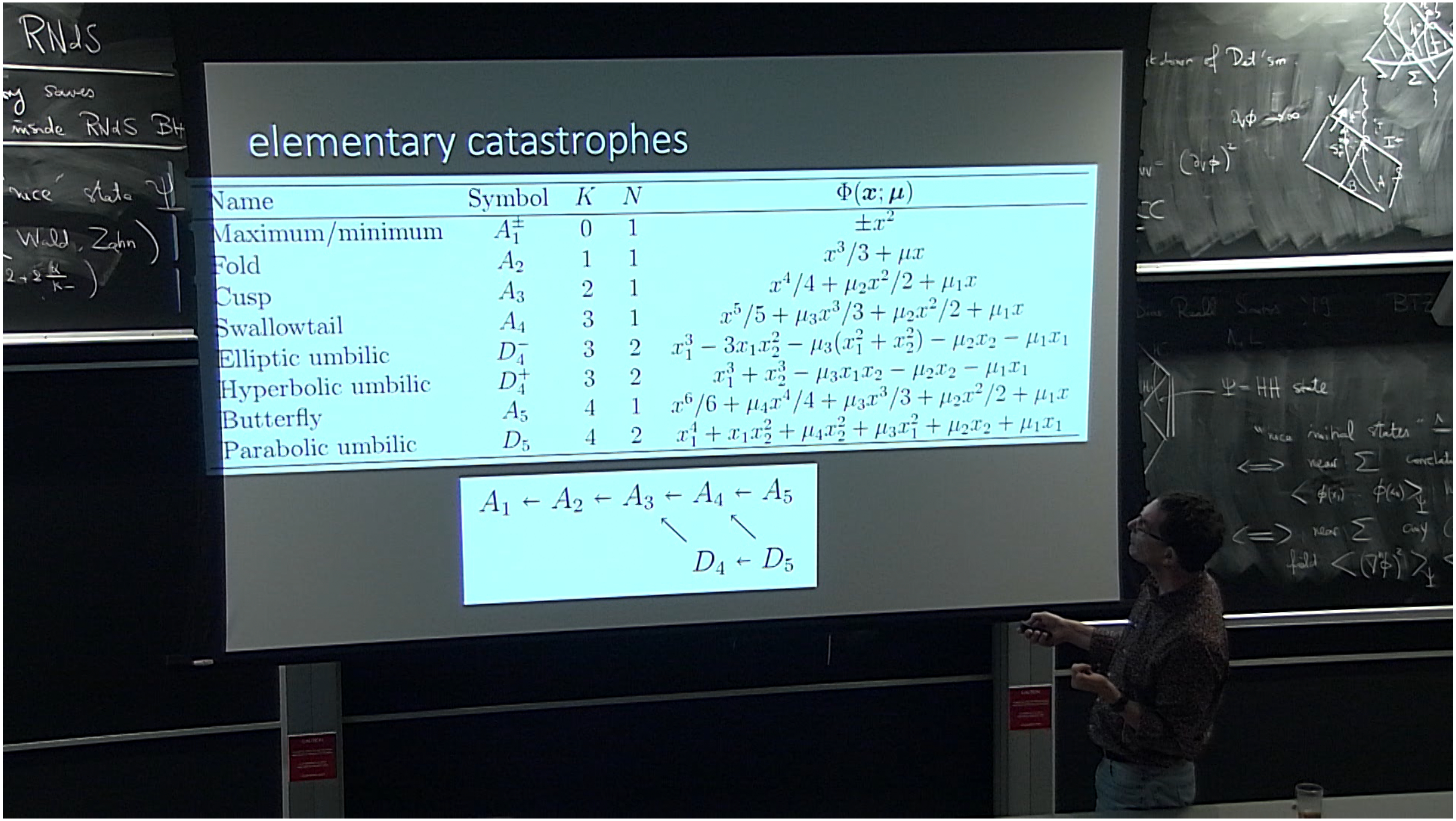


Interference patterns at increasing ν



geometric optics limit



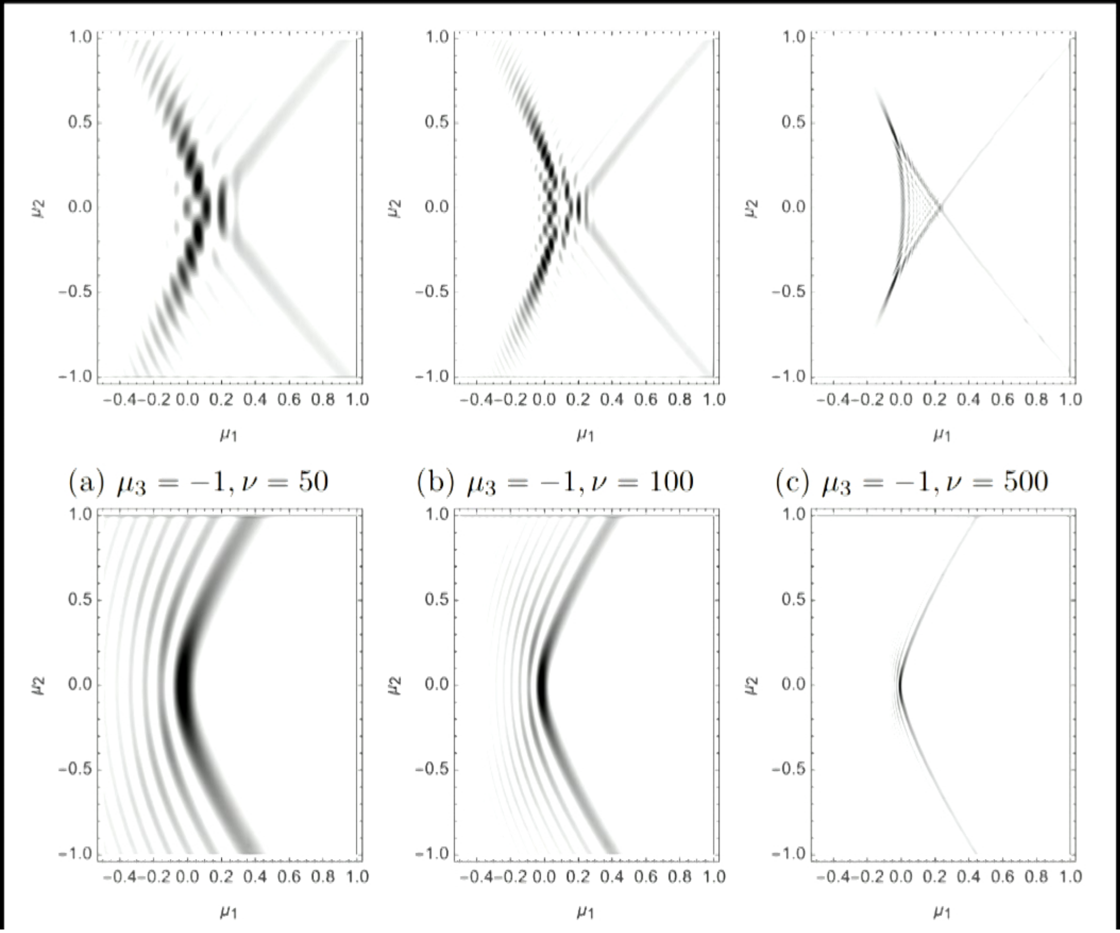

elementary catastrophes
Name | Symbol | K | N | $\Phi(\boldsymbol{x};\boldsymbol{\mu})$
Maximum/minimum | $A_1^{\pm}$ | 0 | 1 | $\pm x^2$
Fold | A_2 | 1 | 1 | $x^3/3 + \mu x$
Cusp | A_3 | 2 | 1 | $x^4/4 + \mu_2 x^2/2 + \mu_1 x$
Swallowtail | A_4 | 3 | 1 | $x^5/5 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x$
Elliptic umbilic | D_4^- | 3 | 2 | $x_1^3 - 3x_1 x_2^2 - \mu_3(x_1^2 + x_2^2) - \mu_2 x_2 - \mu_1 x_1$
Hyperbolic umbilic | D_4^+ | 3 | 2 | $x_1^3 + x_2^3 - \mu_3 x_1 x_2 - \mu_2 x_2 - \mu_1 x_1$
Butterfly | A_5 | 4 | 1 | $x^6/6 + \mu_4 x^4/4 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x$
Parabolic umbilic | D_5 | 4 | 2 | $x_1^4 + x_1 x_2^2 + \mu_4 x_2^2 + \mu_3 x_1^2 + \mu_2 x_2 + \mu_1 x_1$
$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow A_4 \leftarrow A_5$
$D_4 \leftarrow D_5$
RNdS
Ψ = HH state

elementary catastrophes

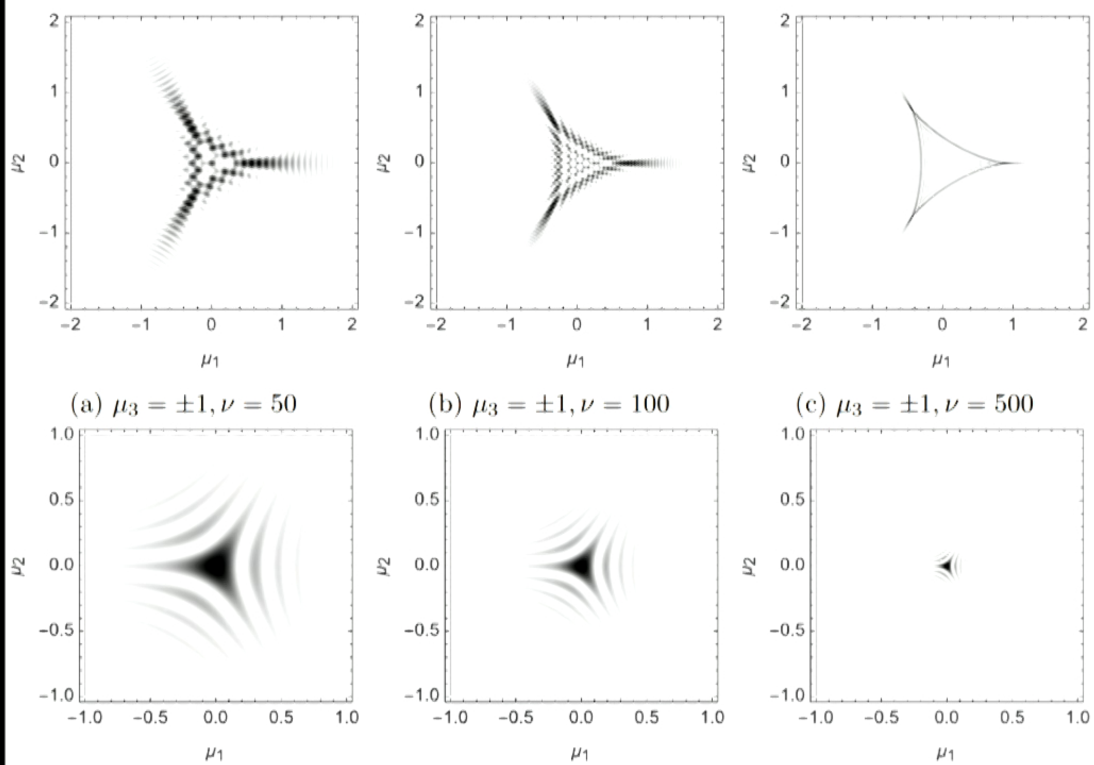
Name	Symbol	K	N	$\Phi(\boldsymbol{x};\boldsymbol{\mu})$
Maximum/minimum	$A_1^{\pm}$	0	1	$\pm x^2$
Fold	A_2	1	1	$x^3/3 + \mu x$
Cusp	A_3	2	1	$x^4/4 + \mu_2 x^2/2 + \mu_1 x$
Swallowtail	A_4	3	1	$x^5/5 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x$
Elliptic umbilic	D_4^-	3	2	$x_1^3 - 3x_1 x_2^2 - \mu_3(x_1^2 + x_2^2) - \mu_2 x_2 - \mu_1 x_1$
Hyperbolic umbilic	D_4^+	3	2	$x_1^3 + x_2^3 - \mu_3 x_1 x_2 - \mu_2 x_2 - \mu_1 x_1$
Butterfly	A_5	4	1	$x^6/6 + \mu_4 x^4/4 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x$
Parabolic umbilic	D_5	4	2	$x_1^4 + x_1 x_2^2 + \mu_4 x_2^2 + \mu_3 x_1^2 + \mu_2 x_2 + \mu_1 x_1$

$$
\begin{array}{ccccccccc}
A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & A_4 & \leftarrow & A_5 \\
 & & & & & \nwarrow & & \nwarrow & \\
 & & & & & & D_4 & \leftarrow & D_5
\end{array}
$$

swallowtail



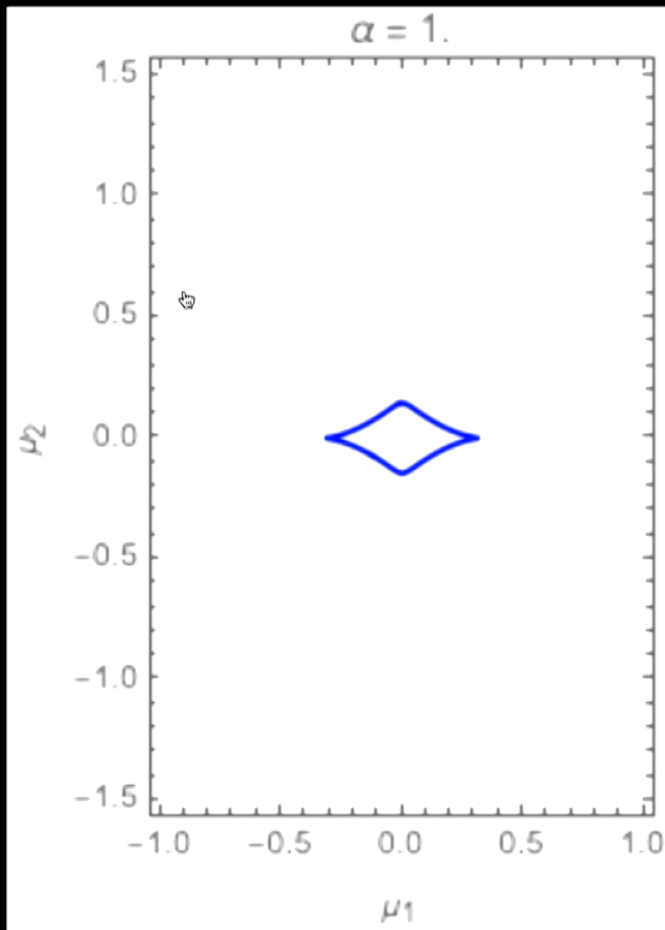
elliptic umbilic



(a) $\mu_3 = \pm 1, \nu = 50$ (b) $\mu_3 = \pm 1, \nu = 100$ (c) $\mu_3 = \pm 1, \nu = 500$

(d) $\mu_3 = 0, \nu = 50$ (e) $\mu_3 = 0, \nu = 100$ (f) $\mu_3 = 0, \nu = 500$

back to localised lenses

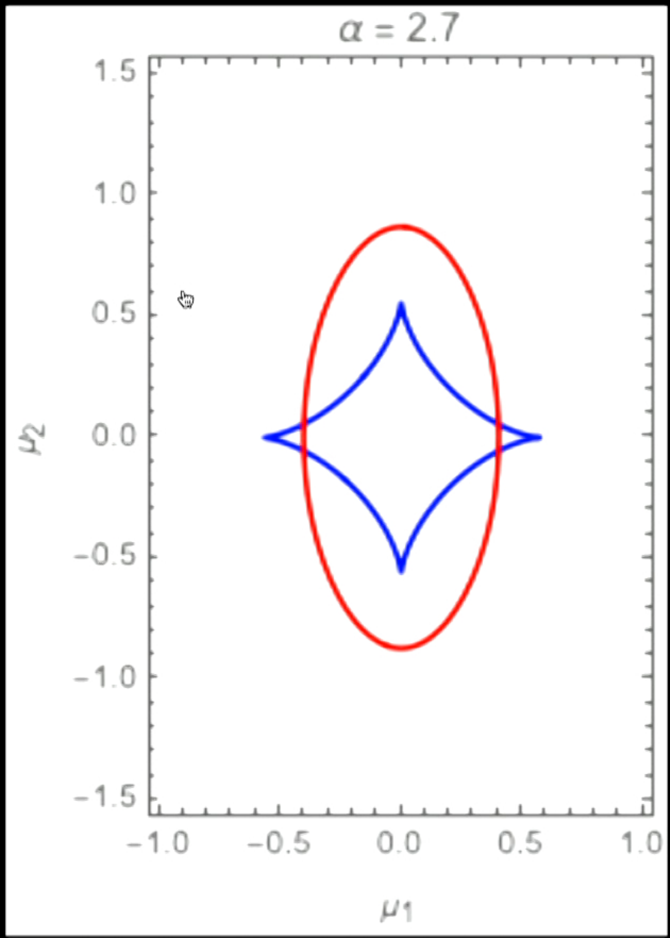


$2d:$

$$\Phi(\vec{x}) = (\vec{x} - \vec{\mu})^2 + \frac{\alpha}{(1 + x_1^2 + 2x_2^2)}$$

As α increases, evolves from pair of twin cusps with folds, through hyperbolic umbilic to create an elliptical fold with no cusps

back to localised lenses



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$$\Phi(\vec{x}) = (\vec{x} - \vec{\mu})^2 + \frac{\alpha}{(1+x_1^2+2x_2^2)}$$

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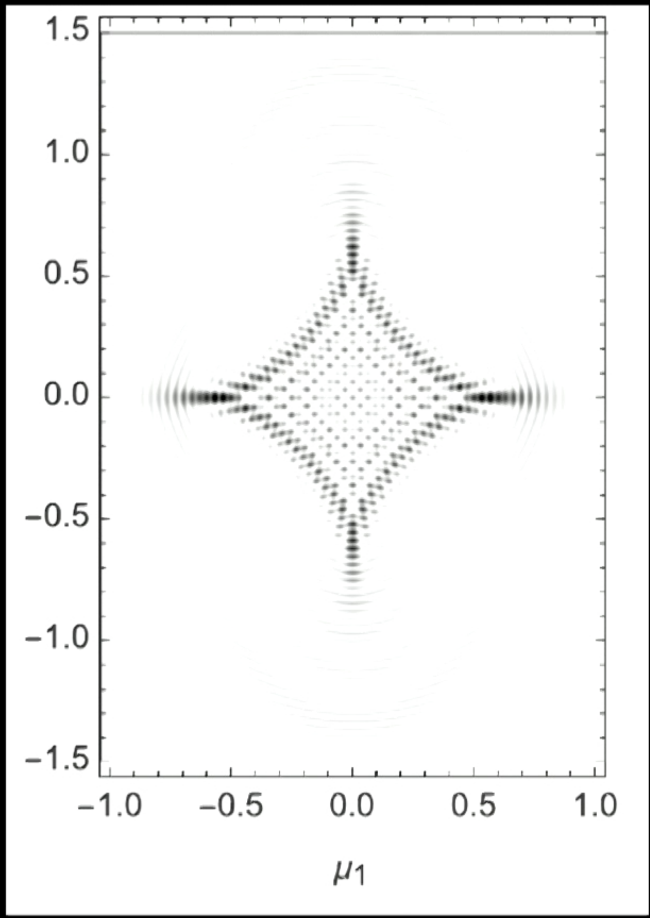
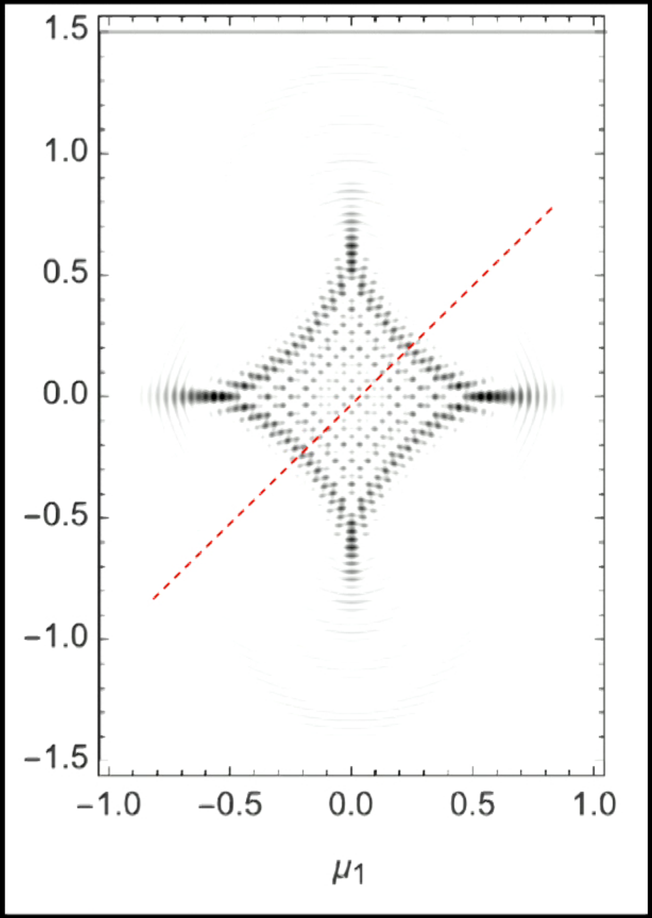
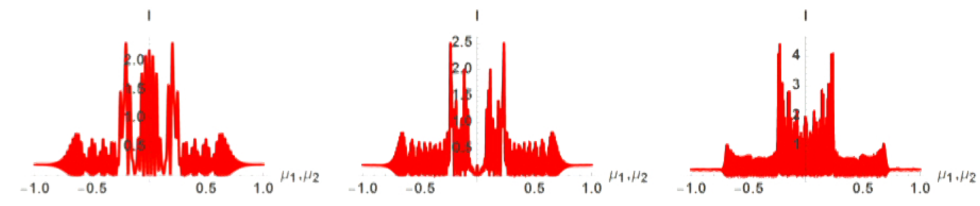
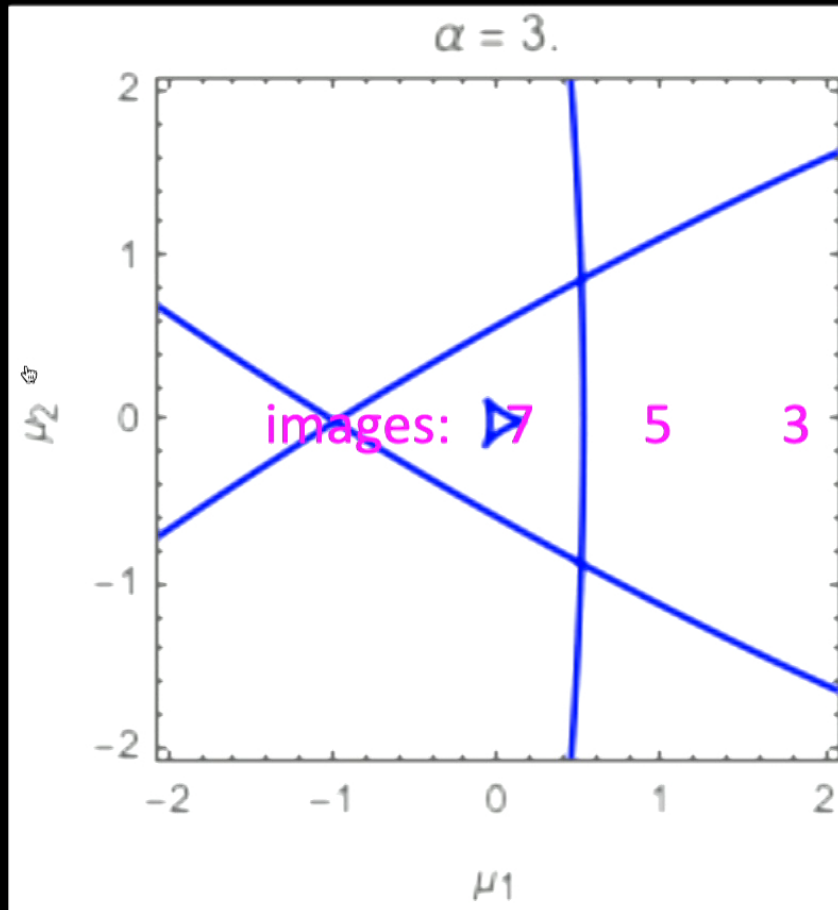

1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
μ_1





(a) $\nu = 50$ (b) $\nu = 100$ (c) $\nu = 500$



$2d:$

$$\Phi(\vec{x}) = (\vec{x} - \vec{\mu})^2 + \frac{\alpha(x_1^3 - 3x_1x_2^2)}{(1 + x_1^2 + x_2^2)}$$

Formation of elliptic umbilic catastrophe (when inner triangle shrinks to zero) via merger of three cusps

Summary

New approach to the *Lorentzian* path integral

QM is all about interference

Illustration: path integrals in the sky!

Goals: existence of QM path integrals without Wick rotation

QFT without fields

Geometrodynamics, emergence of time and the cosmos