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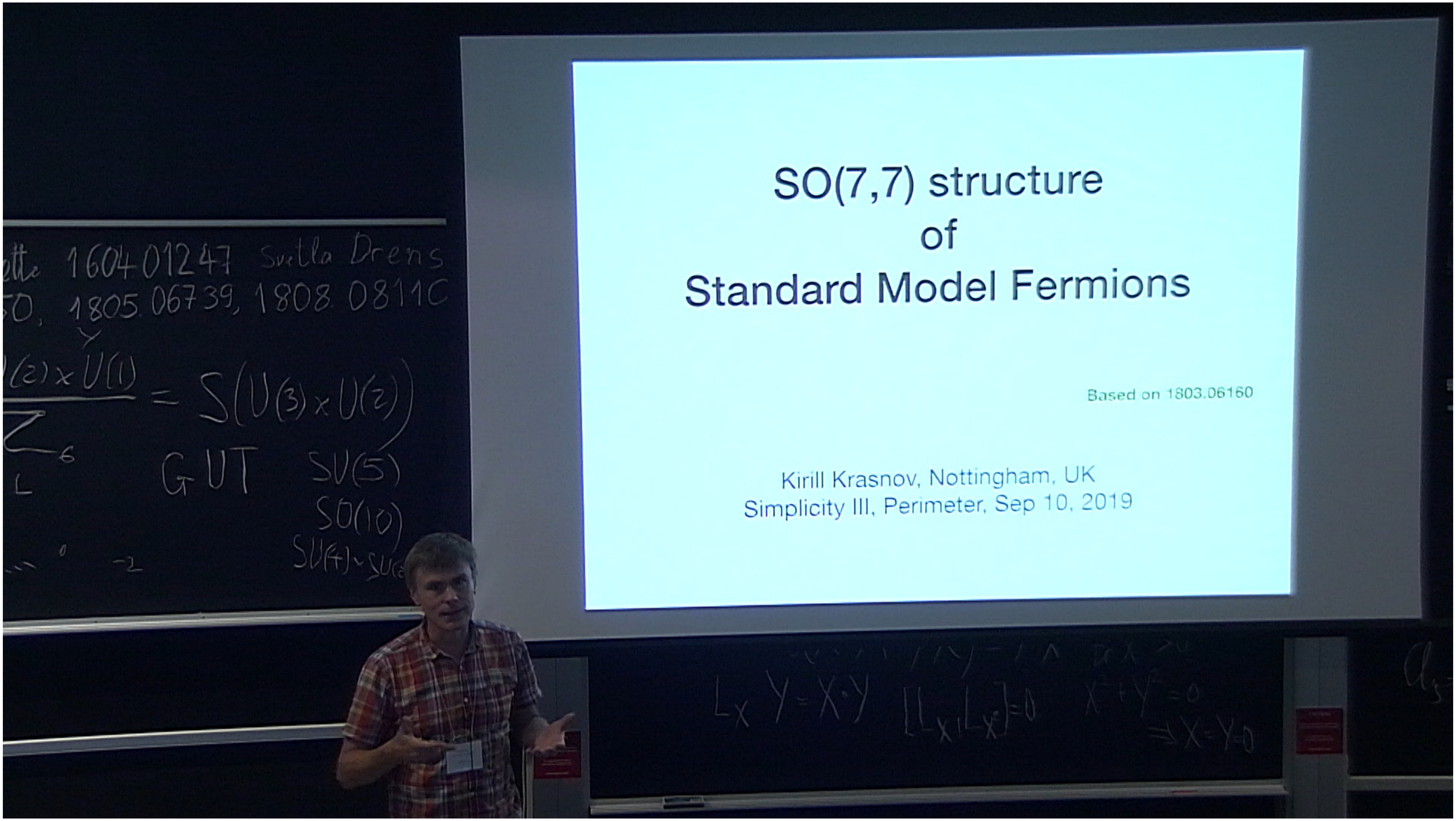
Dynamical Constraints on RG Flows and Cosmology

Based on: 1906.10226 with Baumann and Green

Tom Hartman
Cornell University

Perimeter ✦ September 2019


SO(7,7) structure
of
Standard Model Fermions
Based on 1803.06160
Kirill Krasnov, Nottingham, UK
Simplicity III, Perimeter, Sep 10, 2019

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Motivations

GR is mostly likely an effective field theory describing collective low energy excitations of some unknown to us microscopic DOF.

How can we say anything about the mysterious UV in the absence of any hints from experimental or observational data?

It may be that we will have no further hints apart from the structure of all known to us particles and interactions.

So it may be sensible to try to take this structure seriously and see where it can lead. This talk is an attempt in this direction.

I will present a suggestive rewriting of the SM free fermionic Lagrangian and make some speculations as to what can be next

Suggestive analogy

$e+\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0$	(1)	Gauss' Law
$\mu\alpha=\frac{dH}{dy}-\frac{dG}{dz}$ $\mu\beta=\frac{dF}{dz}-\frac{dH}{dx}$ $\mu\gamma=\frac{dG}{dx}-\frac{dF}{dy}$	(2)	Equivalent to Gauss' Law for magnetism
$P=\mu\left(\gamma\frac{dy}{dt}-\beta\frac{dz}{dt}\right)-\frac{dF}{dt}-\frac{d\Psi}{dx}$ $Q=\mu\left(\alpha\frac{dz}{dt}-\gamma\frac{dx}{dt}\right)-\frac{dG}{dt}-\frac{d\Psi}{dy}$ $R=\mu\left(\beta\frac{dx}{dt}-\alpha\frac{dy}{dt}\right)-\frac{dH}{dt}-\frac{d\Psi}{dz}$	(3)	Faraday's Law (with the Lorentz Force and Poisson's Law)
$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=4\pi p'$ $\quad p'=p+\frac{df}{dt}$ $\frac{d\alpha}{dz}-\frac{d\gamma}{dx}=4\pi q'$ $\quad q'=q+\frac{dg}{dt}$ $\frac{d\beta}{dx}-\frac{d\alpha}{dy}=4\pi r'$ $\quad r'=r+\frac{dh}{dt}$	(4)	Ampère-Maxwell Law
$P=-\zeta p \quad Q=-\zeta q \quad R=-\zeta r$		Ohm's Law
$P=kf \quad Q=kg \quad R=kh$		The electric elasticity equation (**E** = **D**/ε)
$\frac{de}{dt}+\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}=0$		Continuity of charge

Fig. 5. *Maxwell's equations in his original notation in "A dynamical theory of the electromagnetic field." The modern and original variables correspond as follows:* $\mathbf{E}\leftrightarrow(P,Q,R)$; $\mathbf{D}\leftrightarrow(f,g,h)$; $\mathbf{H}\leftrightarrow(\alpha,\beta,\gamma)$; $\mathbf{B}\leftrightarrow\mu(\alpha,\beta,\gamma)$; $\mathbf{J}\leftrightarrow(p,q,r)$; $\rho\leftrightarrow e$; *Ψ is the electric potential; (F,G,H) is the magnetic potential. Note that the original set of equations includes Ohm's law, the Lorentz force, and the continuity equation for charge.*

$$dF=0$$

$$d^*F=0$$

This talk

I will describe representation theory that leads to the realisation that all spinor fields of one generation of the SM arise as components of a single real Weyl (Majorana) irreducible representation of a group whose complexification is Spin(14,C)

I will describe an elegant way to obtain the correct kinetic terms for all the spinor fields (or free fermion Lagrangian) using dimensional reduction from 14D to 4D

I will describe the beautiful geometry of spinors in 14D that is potentially related to the issue of symmetry breaking required to go from Spin(14) to Lorentz times the SM gauge group

I want to start by explaining you the construction of spinor representations of the orthogonal groups that in particular leads to the following important statement

A Weyl spinor representation of SO(2n), when restricted to SO(2k) x SO(2(n-k)) embedded into SO(2n) in the standard way, will split as a Weyl spinor of both SO(2k) and SO(2(n-k)), plus another Weyl spinor of both SO(2k) and SO(2(n-k)), of opposite chiralities

The construction I will explain is standard in the maths literature, but very few physicists know it.

It turns out that spinors are differential forms in disguise!

Spinor representations of orthogonal groups

Spinors of SO(n,n) admit a beautiful explicit description in terms of differential forms. Complexifying, everything works also for arbitrary signature.

Clifford algebra in n+n dimensions can be realised by operators acting on differential forms in n dimensions

$$(a^i)^\dagger := dx^i \qquad\qquad a_i := i_{\partial/\partial x^i}$$

They satisfy the following anti-commutation relations

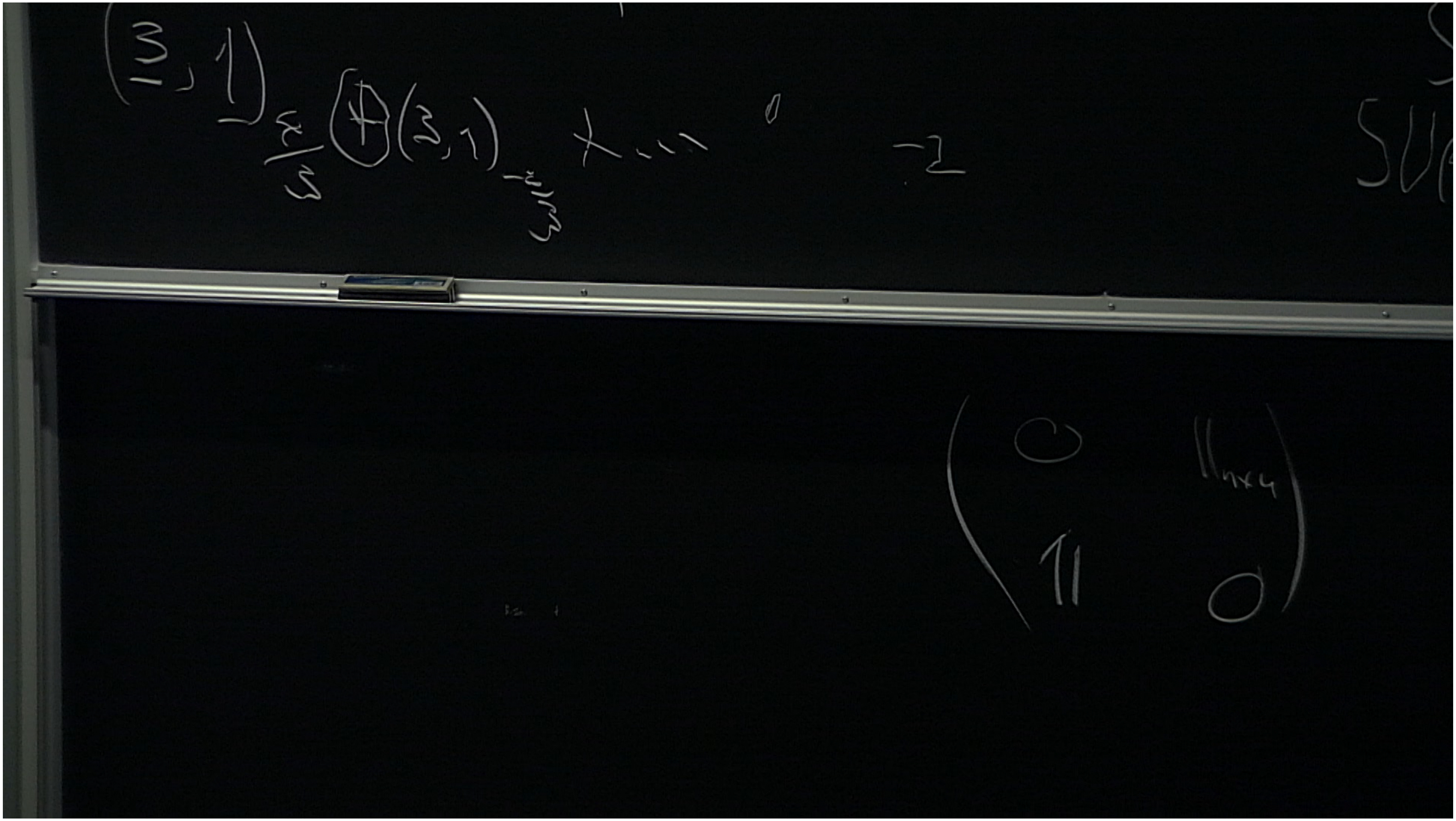
$$(a^i)^\dagger a_j + a_j (a^i)^\dagger = \delta^i_j$$

Gives split signature metric

All others anti-commute

This gives a realisation of Cliff(n,n)

Weyl representations are those in fixed parity (even or odd) differential forms


(3, 1)
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Lie algebra of Spin(n,n)

Most general quadratic operators constructed from $a_i, a_i^\dagger$

Concretely, let $T := \mathbb{R}^n$

Consider matrices of block form

$$M = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix}, \qquad A \in \mathrm{End}(T), \beta \in T \otimes T, B \in T^* \otimes T^*$$

With matrices B, β anti-symmetric

$$\mathrm{spin}(n,n) = \mathrm{so}(n) \oplus \mathrm{so}(n) \oplus \mathrm{gl}(n)$$

The action of Lie algebra element M on a differential form ϕ

$$c(M)\phi = B \wedge \phi - i_\beta \phi - A^T \phi + \frac{1}{2}\mathrm{Tr}(A)\phi.$$

Example of SO(2,2)

$$(a^1)^\dagger, (a^2)^\dagger, a_1, a_2$$

SO(2,2) Lie algebra is realised by all quadratic operators

Get two commuting copies of SL(2) Lie algebra

$$H = a_1 a_1^\dagger - a_2 a_2^\dagger, \qquad E_+ = a_1 a_2^\dagger, \qquad E_- = a_2 a_1^\dagger.$$

$$[E_+, E_-] = H, \qquad [H, E_\pm] = \pm 2E_\pm.$$

$$\bar{H} = a_1 a_1^\dagger + a_2 a_2^\dagger - 1 \equiv a_1 a_1^\dagger - a_2^\dagger a_2, \qquad \bar{E}_+ = a_1 a_2, \qquad \bar{E}_- = a_2^\dagger a_1^\dagger.$$

$$[\bar{E}_+, \bar{E}_-] = \bar{H}, \qquad [\bar{H}, \bar{E}_\pm] = \pm 2\bar{E}_\pm.$$

The action on odd forms

$$H dx^2 = (a_1 a_1^\dagger - a_2 a_2^\dagger) dx^2 = dx^2, \qquad H dx^1 = (a_1 a_1^\dagger - a_2 a_2^\dagger) dx^1 = -dx^1,$$

$$E_- dx^2 = a_2 a_1^\dagger dx^2 = -dx^1, \qquad E_+ dx^1 = a_1 a_2^\dagger dx^1 = -dx^2$$

The action on even forms

$$\bar{H}\,1 = (a_1 a_1^\dagger - a_2^\dagger a_2)\,1 = 1, \qquad \bar{H} dx^1 dx^2 = (a_1 a_1^\dagger - a_2^\dagger a_2) dx^1 dx^2 = -dx^1 dx^2,$$
$$\bar{E}_-\,1 = a_2^\dagger a_1^\dagger\,1 = -dx^1 dx^2, \qquad \bar{E}_+ dx^1 dx^2 = a_1 a_2 dx^1 dx^2 = -1.$$

Overall, get SO(2,2) = SL(2) x SL(2)

Weyl spinors transforming non-trivially with respect to the first SL(2) are odd forms, and non-trivially with respect to the second SL(2) are even forms

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\alpha + \beta dx^1 dx^2 \qquad \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} = -\bar{\alpha} dx^2 + \bar{\beta} dx^1.$$

Two types of 2-component spinors of SO(2,2)

Decomposition under $\mathrm{Spin}(2k) \times \mathrm{Spin}(2(n-k)) \subset \mathrm{Spin}(2n)$

We realise a Weyl representation of Spin(2n) as e.g. even degree differential forms in n dimensions. We then split coordinates

$$n = (n-k) + k$$

We have

$$\Lambda_{even}(\mathbb{R}^n) = \Lambda_{odd}(\mathbb{R}^{n-k}) \otimes \Lambda_{odd}(\mathbb{R}^k) \oplus \Lambda_{even}(\mathbb{R}^{n-k}) \otimes \Lambda_{even}(\mathbb{R}^k)$$

Weyl spinor with respect to both Spin(2(n-k)) and Spin(2k)

Weyl spinor of opposite chirality (with respect to both groups)

This proves the decomposition rule

Dirac operator

To describe Dirac operator in $\mathbb{R}^{n,n}$
will describe spinors as differential forms in $\mathbb{R}^n$
with coefficient functions depending on both $x^i, \tilde{x}_i$

The Dirac operator is

$$D\psi = c(dx^i)\frac{\partial}{\partial x^i}\psi + c(d\tilde{x}_i)\frac{\partial}{\partial \tilde{x}_i}\psi$$

where c is Clifford multiplication

Explicitly $c(dx^i) = dx^i = (a^i)^\dagger$

$$c(d\tilde{x}_i) = i_{\partial/\partial x^i} = a_i$$

Dirac operator as a version of the exterior derivative operator

Dirac operator in $\mathbb{R}^{2,2}$

$$ds^2 = dx^1 d\tilde{x}_1 + dx^2 d\tilde{x}_2$$

Off-diagonal form of the metric

$$x^{1,2} = u^{1,2} + \tilde{u}^{1,2}, \qquad \tilde{x}_{1,2} = u^{1,2} - \tilde{u}^{1,2}$$

$$ds^2 = (du^1)^2 + (du^2)^2 - (d\tilde{u}^1)^2 - (d\tilde{u}^2)^2$$

Diagonal form of the metric

Two chiral Dirac operators $\quad \partial^T : S_- \to S_+, \qquad \partial : S_+ \to S_-$

$$\partial_A{}^{A'} \equiv \partial^T = \begin{pmatrix} \partial/\partial u^2 - \partial/\partial\tilde{u}^2 & -\partial/\partial u^1 + \partial/\partial\tilde{u}^1 \\ -\partial/\partial u^1 - \partial/\partial\tilde{u}^1 & -\partial/\partial u^2 - \partial/\partial\tilde{u}^2 \end{pmatrix} = 2\begin{pmatrix} \partial/\partial\tilde{x}_2 & -\partial/\partial\tilde{x}_1 \\ -\partial/\partial x^1 & -\partial/\partial x^2 \end{pmatrix}$$

$$\partial_{A'}{}^{A} \equiv \partial = \begin{pmatrix} \partial/\partial u^2 + \partial/\partial\tilde{u}^2 & -\partial/\partial u^1 + \partial/\partial\tilde{u}^1 \\ -\partial/\partial u^1 - \partial/\partial\tilde{u}^1 & -\partial/\partial u^2 + \partial/\partial\tilde{u}^2 \end{pmatrix} = 2\begin{pmatrix} \partial/\partial x^2 & -\partial/\partial\tilde{x}_1 \\ -\partial/\partial x_1 & -\partial/\partial\tilde{x}_2 \end{pmatrix}$$

More compact notation $\quad \partial^T = 2\begin{pmatrix} \tilde{\partial}^2 & -\tilde{\partial}^1 \\ -\partial_1 & -\partial_2 \end{pmatrix}, \qquad \partial = 2\begin{pmatrix} \partial_2 & -\tilde{\partial}^1 \\ -\partial_1 & -\tilde{\partial}^2 \end{pmatrix}$

$$\partial\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 2\begin{pmatrix} \partial_2\alpha - \tilde{\partial}^1\beta \\ -\partial_1\alpha - \tilde{\partial}^2\beta \end{pmatrix} \qquad \partial^T\begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} = 2\begin{pmatrix} \tilde{\partial}^2\bar{\alpha} - \tilde{\partial}^1\bar{\beta} \\ -\partial_1\bar{\alpha} - \partial_2\bar{\beta} \end{pmatrix}$$

Dirac operator as exterior derivative

$$D(-\bar\alpha dx^2+\bar\beta dx^1)=-\partial_1\bar\alpha dx^1dx^2+\partial_2\bar\beta dx^2dx^1-\tilde\partial^2\bar\alpha d\tilde x_2dx^2+\tilde\partial^1\bar\beta d\tilde x_1dx^1= \\ (-\partial_1\bar\alpha-\partial_2\bar\beta)dx^1dx^2-(\tilde\partial^2\bar\alpha-\tilde\partial^1\bar\beta).$$

Same result more compactly

$$D\left(\begin{pmatrix} -dx^2 & dx^1\end{pmatrix}\begin{pmatrix}\bar\alpha\\ \bar\beta\end{pmatrix}\right)=\begin{pmatrix}-1 & dx^1dx^2\end{pmatrix}\frac{1}{2}\partial^T\begin{pmatrix}\bar\alpha\\ \bar\beta\end{pmatrix}$$

Same computation for the other chirality

$$D(-\alpha+\beta dx^1dx^2)=-\partial_1\alpha dx^1-\partial_2\alpha dx^2+\tilde\partial^1\beta d\tilde x_1dx^1dx^2+\tilde\partial^2\beta d\tilde x_2dx^1dx^2 \\ =-(\partial_2\alpha-\tilde\partial^1\beta)dx^2+(-\partial_1\alpha-\tilde\partial^2\beta)dx^1.$$

More compactly

$$D\left(\begin{pmatrix} -1 & dx^1dx^2\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right)=\begin{pmatrix}-dx^2 & dx^1\end{pmatrix}\frac{1}{2}\partial\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Do reproduce the correct Dirac operators!

Standard Model and GUT

The structure of the SM is most transparent in the 2-component spinor formalism

2-component spinors are of two types

$$\chi_A \qquad \chi^{\dagger}_{A'}$$

unprimed (undotted) spinor — **primed (dotted) spinor**

Both are irreducible representations of Lorentz

Complex (Hermitian) conjugates of each other

Weyl Lagrangian

$$L = i\,(\chi^{\dagger})^{A'}\partial_{A'}{}^{A}\chi_A \equiv i\,\chi^{\dagger}\partial\chi$$

Real (Hermitian) modulo a surface term

Fermions of the SM

Two-component fermion fields	SU(3)	$SU(2)_L$	Y	T_3	$Q = T_3 + Y$
$\boldsymbol{Q}_i \equiv \begin{pmatrix} u_i \\ d_i \end{pmatrix}$	triplet	doublet	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{2}{3}$
	triplet		$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{3}$
$\bar{u}^i$	anti-triplet	singlet	$-\frac{2}{3}$	0	$-\frac{2}{3}$
$\bar{d}^i$	anti-triplet	singlet	$\frac{1}{3}$	0	$\frac{1}{3}$
$\boldsymbol{L}_i \equiv \begin{pmatrix} \nu_i \\ \ell_i \end{pmatrix}$	singlet	doublet	$-\frac{1}{2}$	$\frac{1}{2}$	0
	singlet		$-\frac{1}{2}$	$-\frac{1}{2}$	-1
$\bar{\ell}^i$	singlet	singlet	1	0	1

All fields are 2-component spinors, transforming under SU(3) x SU(2) x U(1) as indicated

The generation indices i=1,2,3   Colour indices suppressed

Bar over a symbol is a part of the name, not to be confused with complex conjugation

SM Lagrangian

We describe it in words instead of writing a long expression

Every of the 2-component spinors in the table will have its Weyl kinetic term. Spinors are coupled to the SU(3) x SU(2) x U(1) gauge fields, and the Higgs field, which is a complex valued SU(2) doublet, of hypercharge Y=1/2. All terms of mass dimension four that are compatible with the gauge and Lorentz symmetry are written down, together with their Hermitian conjugates.

Plus there are kinetic terms for the gauge fields - usual F^2

Plus there is the kinetic plus potential term for the Higgs. Potential is quartic and makes Higgs acquire a non-trivial VEV.

Right-handed sterile neutrinos $\bar{\nu}_i$ can be added for free

If add Majorana mass terms for them, gets see-saw mechanism

SO(10) structure of SM fermions

To see all fermions of a single generation inside a single irreducible representation of SO(10) need to think of leptons as the fourth colour of quarks

$$\begin{pmatrix} \nu \\ l \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix}^{lepton}$$

Then have SU(4) mixing the four colours of quarks

$$\begin{pmatrix} u \\ d \end{pmatrix}^{red} \quad \begin{pmatrix} u \\ d \end{pmatrix}^{green} \quad \begin{pmatrix} u \\ d \end{pmatrix}^{blue} \quad \begin{pmatrix} u \\ d \end{pmatrix}^{lepton}$$

Also need to introduce a new gauge symmetry - $\mathrm{SU}(2)_R$ with respect to which the barred spinors form doublets

$$\bar{u}, \bar{d} \to \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}, \quad \bar{\nu}, \bar{l} \to \begin{pmatrix} \bar{\nu} \\ \bar{l} \end{pmatrix}$$

These spinors are also arranged as those of different SU(4) colour

$$\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}^{red} \quad \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}^{green} \quad \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}^{blue} \quad \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}^{lepton}$$

Overall, we have fields transforming under Pati-Salam group

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(4)$$

in the following representations

$$Q = \begin{pmatrix} u \\ d \end{pmatrix} \quad (\mathbf{2}, \mathbf{1}, \mathbf{4}) \qquad \bar{Q} = \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \quad (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}})$$

One then notes $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2 = \mathrm{SO}(4)$

$\mathrm{SU}(4)/\mathbb{Z}_2 = \mathrm{SO}(6)$ Cartan's isomorphisms

And $\mathrm{SO}(4) \times \mathrm{SO}(6) \subset \mathrm{SO}(10)$

This shows that all spinors of a single generation of SM arise as components of a single Weyl spinor of SO(10), with Pati-Salam group embedded into SO(10) in the standard way

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(4) \sim \mathrm{SO}(4) \times \mathrm{SO}(6) \subset \mathrm{SO}(10)$$

2-component spinors of single generation are components of $\mathbf{16}_{\mathbb{C}}$ irreducible Weyl representation of SO(10)

Fermions of the SM

Two-component fermion fields	SU(3)	$SU(2)_L$	Y	T_3	$Q = T_3 + Y$
$\boldsymbol{Q}_i \equiv \begin{pmatrix} u_i \\ d_i \end{pmatrix}$	triplet	doublet	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{2}{3}$
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Back to the main story:

In the description of SO(10) GUT Lorentz spinor indices played no role. The GUT fermion is an object

$$\Psi_{\alpha A}$$

plus complex conjugate

SO(10) spinor index

$$\alpha = 1, \ldots, 16$$

SO(1,3) 2-component spinor index

$$A = 1, 2$$

Overall, single generation of fermions is described by 16 x 2 complex functions or

64 real valued functions

Can “unify” the Lorentz and GUT spinor indices by repeating

$$\mathrm{SO}(2k) \times \mathrm{SO}(2(n-k)) \subset \mathrm{SO}(2n)$$

Should put the Lorentz SO(1,3) and GUT SO(10) groups together

Some real form of

$$\mathrm{SO}(4, \mathbb{C}) \times \mathrm{SO}(10, \mathbb{C}) \subset \mathrm{SO}(14, \mathbb{C})$$

Weyl spinor of SO(14,C) is 64 dimensional (complex), and splits

$$\mathbf{2}_{\mathbb{C}} \otimes \mathbf{16}_{\mathbb{C}} + \mathbf{2}_{\mathbb{C}} \otimes \mathbf{16}_{\mathbb{C}}$$

into a sum of two Weyl representations of opposite chiralities

This is as we want, should just select an appropriate real form

SO(14, C) real form

Standard representation theory of Clifford algebras shows that there are only two real forms that give a real 64-dimensional Weyl representation

Have $\mathrm{SO}(s,r) \qquad s+r=14$

To have Weyl representation being real need $s-r=0 \mod 8$

The two possibilities are

$$\mathrm{SO}(7,7) \qquad s-r=0$$
$$\mathrm{SO}(11,3) \qquad s-r=8$$

Both contain Lorentz SO(1,3) and Pati-Salam groups as subgroups

$$\mathrm{SO}(1,3)\times\mathrm{SO}(6,4)\subset\mathrm{SO}(7,7)$$

$$\mathrm{SO}(1,3)\times\mathrm{SO}(10)\subset\mathrm{SO}(11,3)$$

This talk is an advertisement of the first option

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Dimensional reduction: Weyl Lagrangian in $\mathbb{R}^{n,n}$

Weyl Lagrangian exists only for SO(n,n) with n odd

SO(n,n) invariant inner product

$$(\Psi_1, \Psi_2) = \sigma(\Psi_1)\Psi_2\Big|$$

canonical involution

$$v_1 \otimes \ldots \otimes v_k \to v_k \otimes \ldots \otimes v_1$$

restriction to top form

$$S[\Psi] = \int_{\mathbb{R}^{n,n}} (\Psi, D\Psi)$$

Vanishes by integration by parts for n=1 mod 4

Together always give an even form, but D changes degree by one. So n must be odd

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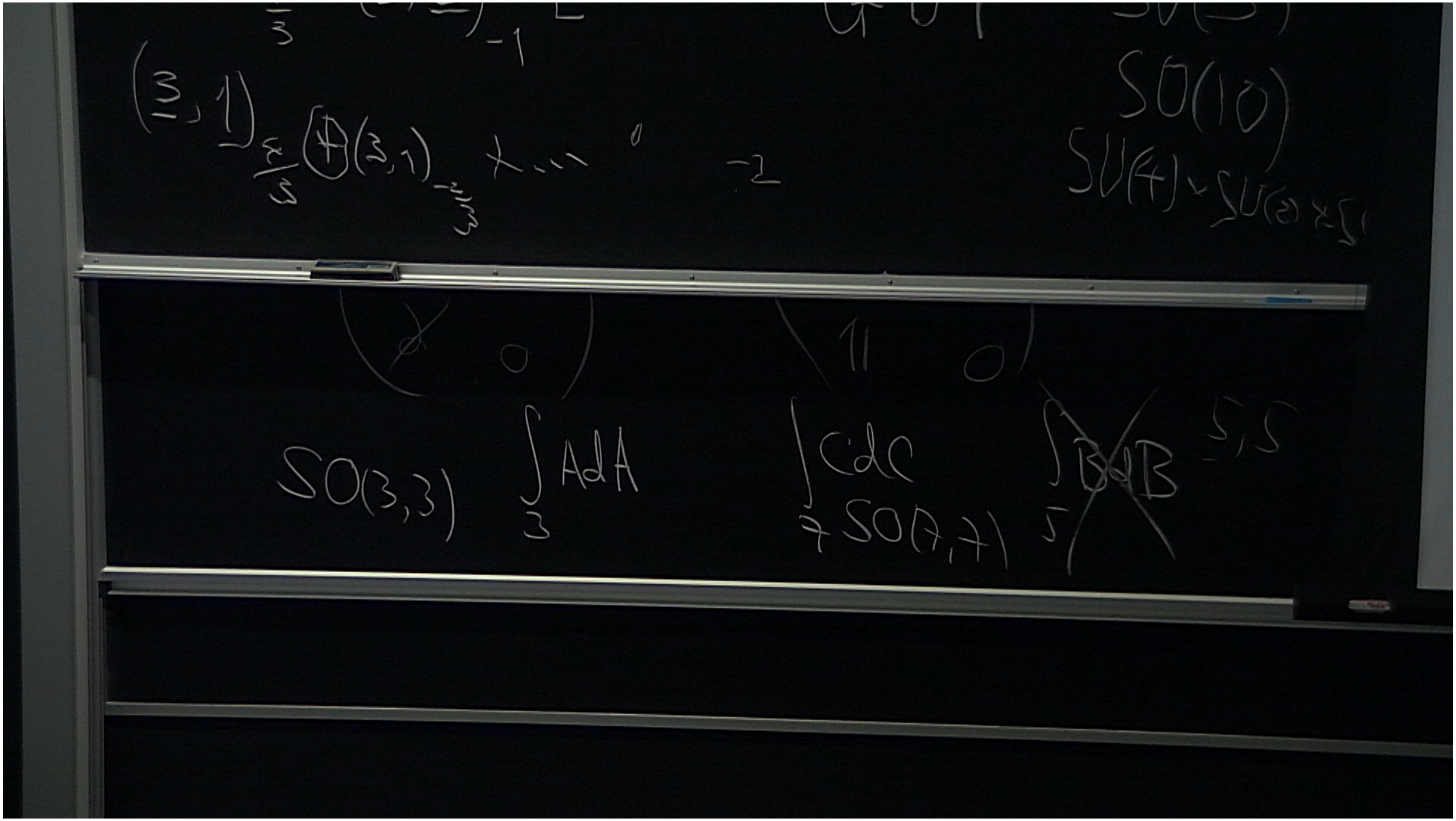
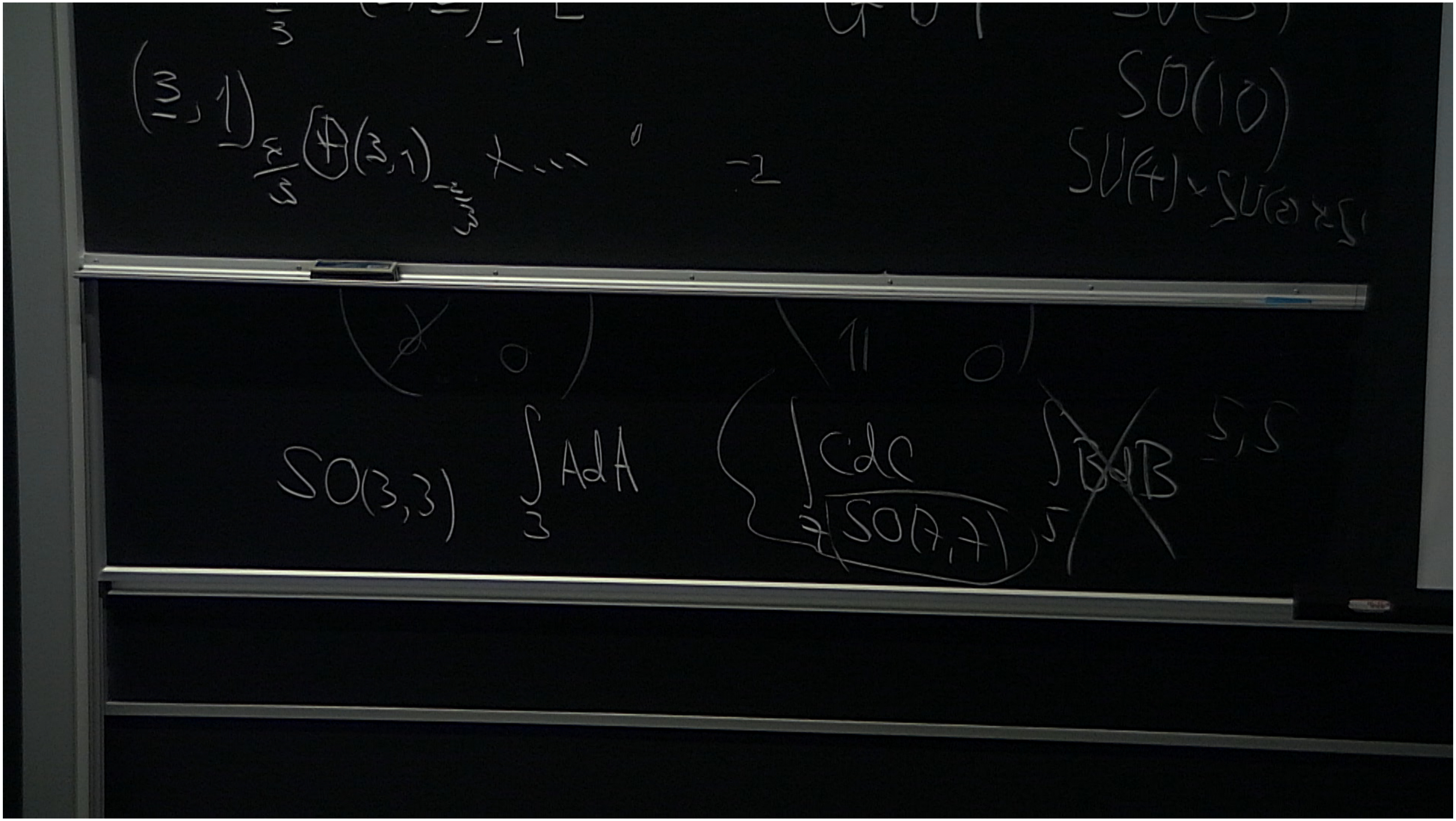

SO(10)
SO(3,3)
∫AdA
∫CdC
SO(7,7)
∫BdB


SO(10)
SU(4) × SU(2)
SO(3,3)
∫AdA
∫CdC
SO(7,7)
5,5

So, there is a non-trivial Weyl Lagrangian only for SO(4k-1,4k-1)

$k = 1, \ldots$

It is an exercise to check that dimensional reduction from 2n to 2k dimensions produces the correct Weyl Lagrangians in 2k dimensions. The signature in $\mathbb{R}^{2k}$ can be any desired one.

E.g. Weyl Lagrangian exists for SO(3,3)

$$\Lambda_{even}(\mathbb{R}^3) = \Lambda^0 \oplus \Lambda^2$$

4 real dimensional = 2 complex dimensional

Dimensional reduction to 3+1 gives

$$\mathrm{SO}(3,1) \times \mathrm{SO}(2) \subset \mathrm{SO}(3,3)$$

single electrically charged Weyl fermion in 3+1

Because we decided to reduce to 3+1 where spinors are complex-valued, the two Weyl spinors of opposite chirality are just a 2-component spinor and its complex conjugate

The next non-trivial case is for SO(7,7)

Dimensional reduction to 3+1 gives

$$\mathrm{SO}(3,1) \times \mathrm{SO}(4) \times \mathrm{SO}(6) \subset \mathrm{SO}(7,7)$$

the fermion content is that of the Pati-Salam version of the SM

Summary so far: We have re-written the SM fermion kinetic terms as dimensional reduction of

$$\int_{\mathbb{R}^{14}} \Psi \partial\!\!\!/ \, \Psi$$

Also explained the SM spinor content - The only simpler option is SO(3,3), which is too simple. But of course do not understand why need to reduce to 4D, and do not understand why need to break the symmetry further to that of the SM gauge group

I will now explain some further geometric (group theory) facts that select SO(7,7) as the group with certain unique properties

It is possible that this can lead to understanding of why need to reduce to 4D and why the SM gauge group arises

The idea is to assume that there is some mechanism that gives all of the SM spinor fields (or rather their bilinears) some non-zero expectation value. So that the quantum spinor fields that appear in the SM Lagrangian are perturbations around a non-trivial classical (spinor) background.

So, assume that there is a non-trivial Weyl spinor of SO(7,7)

A non-zero spinor generally breaks the Spin group to some stabiliser subgroup, and it is interesting to study these

Representation theory Fact #1

Consider the action of SO(2n) on its Weyl spinor representation

$$\dim(\mathrm{SO}(2n)) = \frac{2n(2n-1)}{2}$$

Dimension of the group

$$\dim(W_{2n}) = 2^{n-1}$$

Dimension of the Weyl representation

The dimension of the spinor representation grows with n much faster than dimension of the group

While for small n we have

$$\dim(\mathrm{SO}(2n)) > \dim(W_{2n})$$

The last n when this is true is n=7 giving SO(14)

This will not be true for sufficiently large n

Indeed, for n=7

$$\dim(\mathrm{SO}(14)) = 91$$

$$\dim(W_{14}) = 64$$

For n=8

$$\dim(\mathrm{SO}(16)) = 120$$

$$\dim(W_{16}) = 128$$

Last dimension when

$$\dim(\mathrm{SO}(2n)) > \dim(W_{2n})$$

Why is this interesting?

When dimension of the group is bigger than dimension of the space it acts on, generically, there is a non-trivial subgroup stabilising a point - symmetry breaking

This is very interesting for SO(7,7)!

Symmetry braking for SO(3,3)

Consider the action of so(3,3) on its Weyl spinor representation, now realised as odd degree forms in $\mathbb{R}^3$

General such form is $\phi = \phi_1 + \phi_3$

$$c(M)\phi = B \wedge \phi_1 - i_\beta \phi_3 + (\frac{1}{2}\mathrm{Tr}(A) - A^T)(\phi_1 + \phi_3)$$

Clear that can kill the ϕ_1 part using $\beta \in \mathrm{so}(3)$

The canonical form of the Weyl spinor of SO(3,3) $\phi = \phi_3$

Not surprising, because $SO(3,3) \sim \mathrm{SL}(4, \mathbb{R})$

Every spinor of SL(4) can be put into the form $\to$ $\begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Stabiliser $\mathrm{sl}(3) \oplus \mathrm{so}(3)$

Not particularly interesting

$$\begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \in \mathrm{SL}(4, \mathbb{R})$$

Symmetry breaking for SO(7,7)

The dimension of the generic orbit for SO(7,7) acting in its Weyl representation is 63 - the “scale” of the spinor can not be changed

$$\mathrm{dim}(\mathrm{SO}(7,7)) - \mathrm{dim}(\mathrm{orbit}) = 91 - 63 = 28$$

This suggests that the stabiliser is related to G_2 $\quad \mathrm{dim}(G_2) = 14$

(could also be SO(8) but this is not what happens)

There are three possible generic orbits, with stabilisers being

Cases 1,1’	$G_2 \times G_2$	Compact real form
	$G_2' \times G_2'$	Split real form
Case 2	$G_2^{\mathbb{C}}$	

Symmetry breaking for SO(7,7) - general case

Now general odd form $\phi = \phi_1 + \phi_3 + \phi_5 + \phi_7$

Dimensions	7	35	21	1	= 64

Action of so(7,7)

$$c(M)\phi = -i_\beta \phi_3$$ ← **1-forms**

$$+B \wedge \phi_1 - i_\beta \phi_5$$ ← **3-forms**

$$+B \wedge \phi_3 - i_\beta \phi_7$$ ← **5-forms**

$$+B \wedge \phi_5$$ ← **7-forms**

$$+(\frac{1}{2}\mathrm{Tr}(A) - A^T)\phi$$

Can use β to kill ϕ_5 part

Generic form can always be put into the form

$$\phi = \phi_1 + \phi_3 + \phi_7$$

Selects a special direction!

The appearance of SU(3)

The subgroup of G_2 arising at the stabiliser of ϕ_3 that fixes the special direction ϕ_1 is precisely SU(3)!

Thus, the strong gauge group arises in this scheme naturally

It is also clear that appearance of the Lorentz SO(1,3) is related to some mechanism that is to select 2 more of the remaining 6 directions as special. This mechanism may be dynamical

Further special facts about SO(7,7) setup

Generic 3-form in $\mathbb{R}^7$ defines a metric

of signature all plus or (3,4)

$$g_C(\xi,\eta)\mathrm{vol}_C = i_\xi C \wedge i_\eta C \wedge C$$

This implies that generic Weyl spinor of SO(7,7) defines a metric in $\mathbb{R}^7$

Exceptionally, for a Weyl spinor of SO(7,7) there is an invariant form of degree 8!

Highly non-trivial "mass" term for the spinor

One can then imagine that the Yukawa mass terms of the SM are reproduced by linearising the degree 8 invariant around a non-trivial spinor

Representation theory summary

Generic fermion of the Standard Model breaks SO(7,7) symmetry down to SU(3)

(times in general non-compact group of dimension 20)

Generic fermion of the Standard Model defines a metric in seven dimensions

Extremely rare phenomenon when a spinor defines a metric

There is an SO(7,7) invariant interaction term that can be added to the free fermion Lagrangian

Extremely rare phenomenon that “mass” term for Weyl possible

Outlook

Non-zero SM fermion field defines a metric in seven dimensions

Could it be that gravity is an effective field theory describing fluctuations of this metric? This would answer the question of why metric is non-zero, and also why gravity is a special force

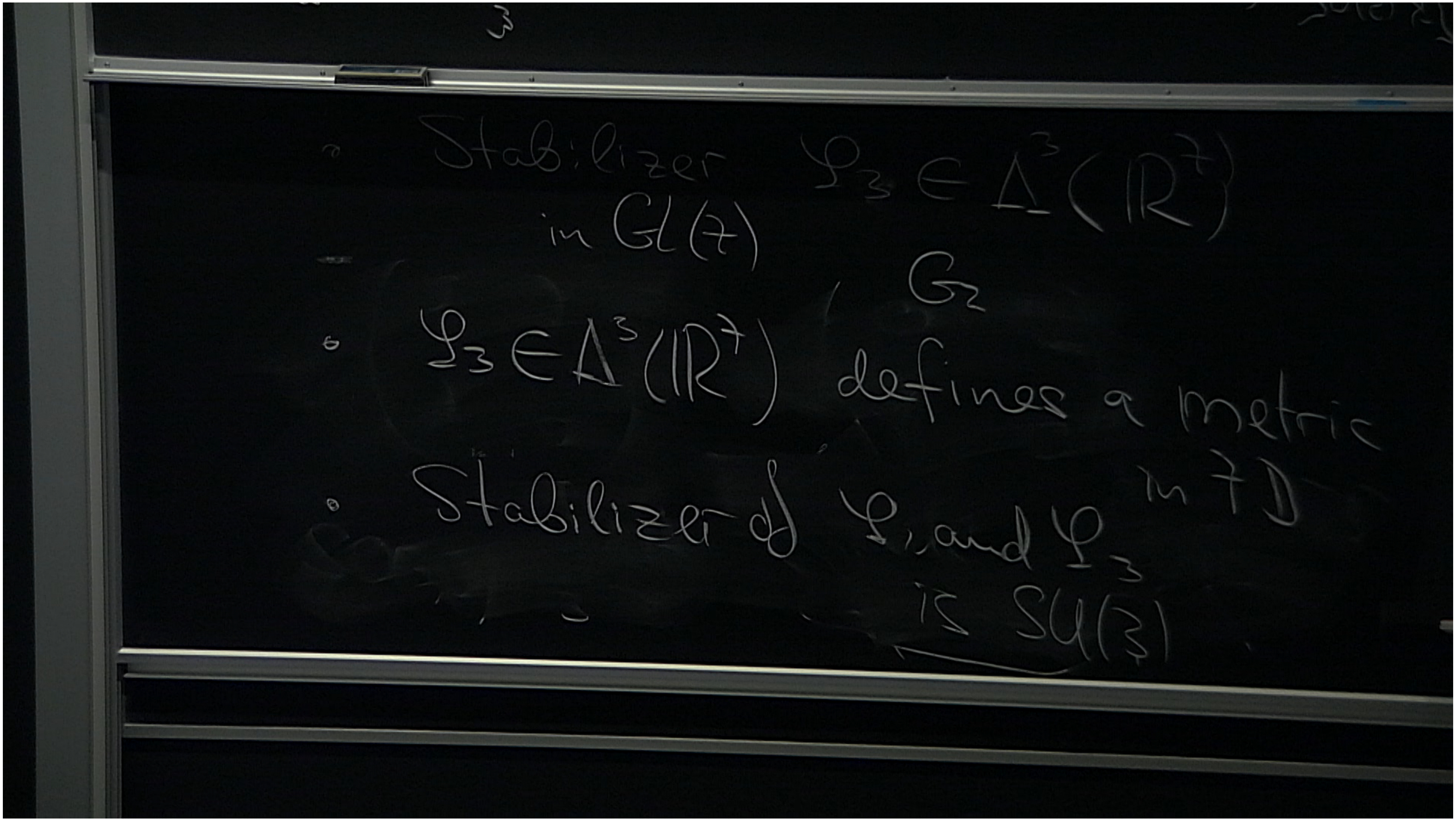
Question that can guide further developments:

$$S[\Psi] = \int_{\mathbb{R}^{7,7}} (\Psi, D\Psi) + V(\Psi)$$

Order 8 invariant (to some appropriate power)

Is there a solution of this theory that "spontaneously compactifies" to 4D and breaks the symmetry to the SM gauge group?

Such Lagrangian only exists in 7+7 dimensions!


Stabilizer $\varphi_3 \in \Lambda^3(\mathbb{R}^7)$
in $GL(7)$
G_2
$\varphi_3 \in \Lambda^3(\mathbb{R}^7)$ defines a metric in 7D
Stabilizer of φ_1 and φ_3
is $SU(3)$

Outlook

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