Title: A solvable model for magnetic skyrmions

Speakers: Bernd Schroers

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Abstract: Magnetic skyrmions are topological solitons which occur in a large class of ferromagnetic materials and which are currently attracting much attention, not least because of their potential use for low-energy magnetic information storage and manipulation. The talk is about an integrable model for magnetic skyrmions, introduced in a recent paper (arxiv:1812.07268) and generalised in arxiv:1905.06285. The model is based on a geometrical interpretation of the Dzyaloshinskii-Moriya interaction in terms of a non-abelian gauge field. In the talk will explain the model and the geometry behind its solution, and discuss solutions and their applications.

A Solvable Model for Magnetic Skyrmions

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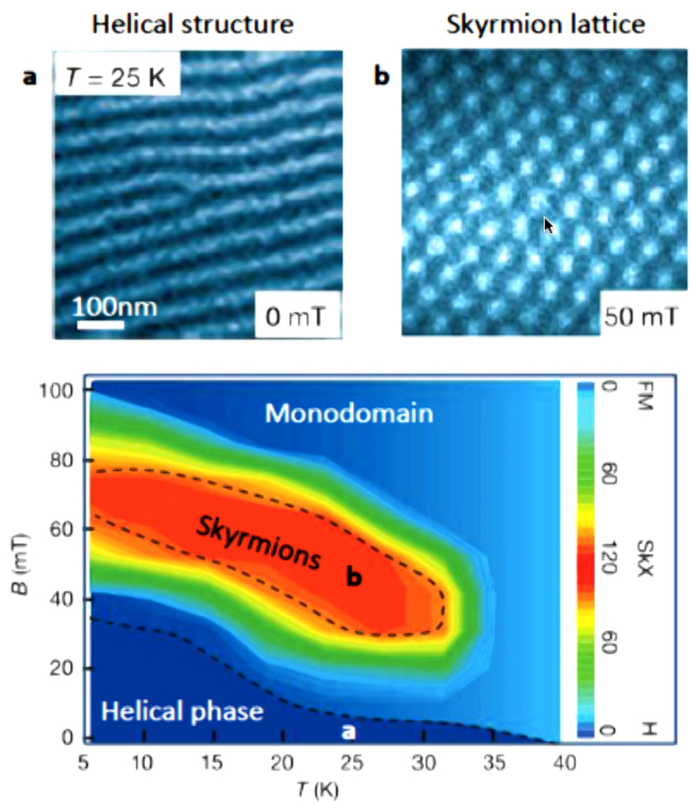
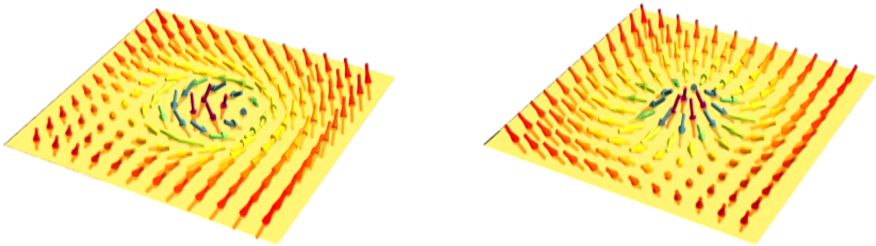
Condensed Matter Seminar, Perimeter Institute,
13 September 2019

Magnetic Skyrmions - **Experiment**

Lorentz TEM images of $Fe_{0.5}Co_{0.5}Si$

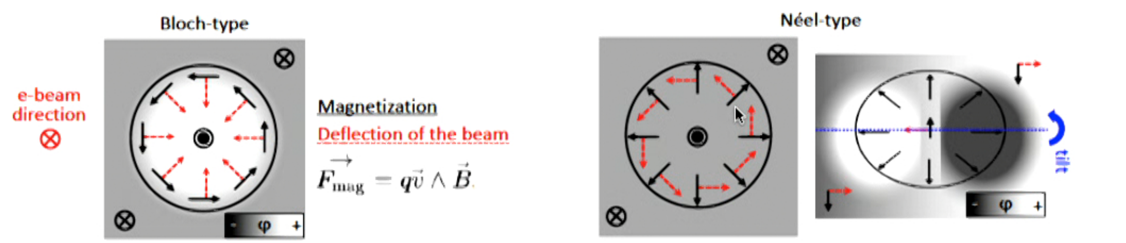


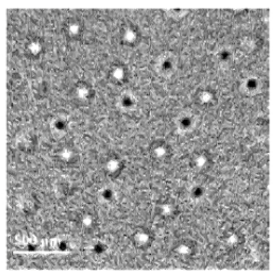
X. Z. Yu et al., Nature 465, 901-904 (2010)

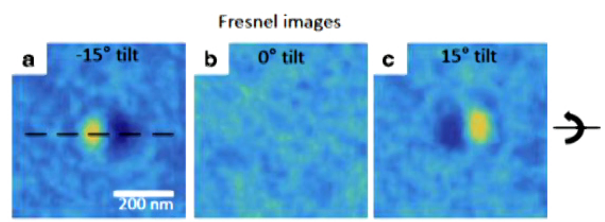


Magnetic Skyrmions - **Experiment**

Imaging Néel-type skyrmions







Pollard S. et al., Nat. Commun. 8, 14761 (2017)

Magnetic Skyrmions - Theory

The energy of the lattice model is

$$E[S] = \sum_{i,j} \underbrace{-J\boldsymbol{S}_i \cdot \boldsymbol{S}_j}_{\text{Heisenberg}} + \underbrace{D_{ij} \cdot \boldsymbol{S}_i \times \boldsymbol{S}_j}_{\text{DMI}} - \underbrace{\sum_i \boldsymbol{B} \cdot \boldsymbol{S}_i}_{\text{Zeeman}} + \underbrace{\sum_j (\boldsymbol{k} \cdot \boldsymbol{S}_i)^2}_{magnetic\ \ anisotropy}$$

The continuum limit is

$$E[\boldsymbol{n}] = \int_{\mathbb{R}^2} \frac{1}{2}(\nabla \boldsymbol{n})^2 + \sum_{a,j} \mathcal{D}_{aj}(\partial_j \boldsymbol{n} \times \boldsymbol{n})_a + \mu^2(1 - n_3) + k(1 - n_3^2)\ dx_1 \wedge dx_2,$$

where the spiralization tensor $\mathcal{D}$ encodes the **Dzyaloshinskii-Moriya (DM)** spin-orbit interaction.

A **very** brief history

- Topological twists in the magnetisation field of real planar magnetic materials (Bogdanov and Jablonskii 1989)
- Past 10 years: technological interest as potential information carriers in low-energy magnetic **racetrack** memory devices.

Pure Heisenberg model revisited

Basic field is the unit magnetisation vector

$$n : \mathbb{R}^2 \to S^2 \subset su(2),$$

with energy

$$E[n] = \frac{1}{2} \int_{\mathbb{R}^2} ((\partial_1 n)^2 + (\partial_2 n)^2) dx_1 \wedge dx_2.$$

For this to be finite, require existence of limit $\lim_{|x| \to \infty} n(x) = n_\infty$, so that n extends to map

$$\tilde{n} : \mathbb{R}^2 \cup \{\infty\} \to S^2,$$

with integer degree

$$\deg[n] = \frac{1}{4\pi} \int n \cdot [\partial_1 n, \partial_2 n] \, dx_1 \wedge dx_2.$$

The Bogomol'nyi argument

Write energy as

$$E[n] = \frac{1}{2}\int_{\mathbb{R}^2} \left((\partial_1 n \pm [n, \partial_2 n])^2 \pm n \cdot [\partial_1 n, \partial_2 n]\right)\, dx_1 \wedge dx_2,$$

to deduce lower bound

$$E[n] \geq 4\pi|\deg[n]|$$

with equality iff the Bogomol'nyi equations holds:

$$\partial_1 n = \mp[n, \partial_2 n].$$

They imply the variational equations

$$[n, (\partial_1^2 + \partial_2^2)n] = 0.$$

Invariant formulation

Consider Riemann surface Σ with local complex coordinates $z = x_1 + ix_2, \quad \partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$.
The Hodge $\star$ operation on 1-forms is a complex structure:

$$\star dz = -idz, \quad \star d\bar{z} = id\bar{z},$$

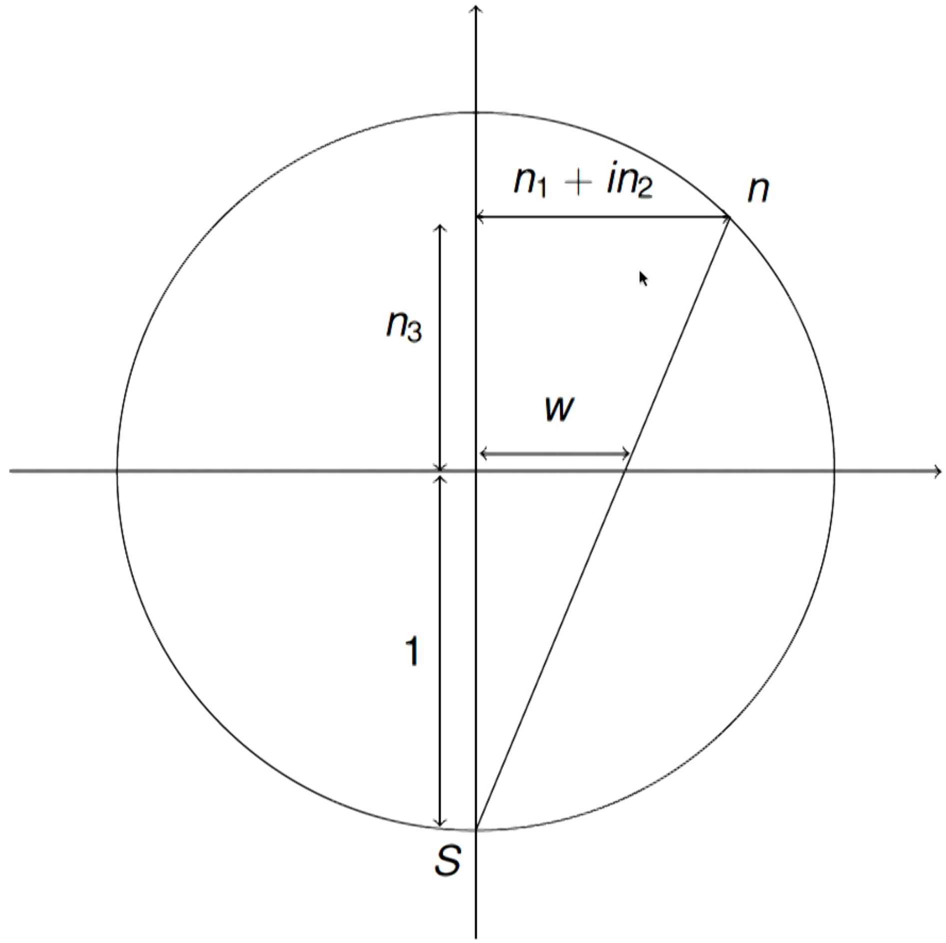
The energy only depends on complex structure:

$$\begin{aligned} E[n] &= \frac{1}{2}\int_\Sigma (dn, \wedge \star\, dn) \\ &= \frac{1}{4}\int_\Sigma ((dn \mp \star[n, dn]), \wedge \star\, (dn \mp \star[n, dn])) \pm \frac{1}{2}\int_\Sigma (n, [dn, dn]), \end{aligned}$$

and the Bogomol'nyi equations are

$$\star dn = \pm[n, dn].$$

Stereographic projection



In terms of stereographic coordinate $w \in \mathbb{C} \cup \{\infty\}$:

$$E[w] = 2\int_\Sigma \frac{dw \wedge \star d\bar{w}}{(1+|w|^2)^2}$$
$$= 2\int_\Sigma \frac{(dw \pm i \star dw) \wedge \star\overline{(dw \pm i \star dw)}}{(1+|w|^2)^2} \mp 2\int_\Sigma \frac{dw \wedge d\bar{w}}{(1+|w|^2)^2}$$

Bogomol'nyi equations are

$$dw = \pm i \star dw.$$

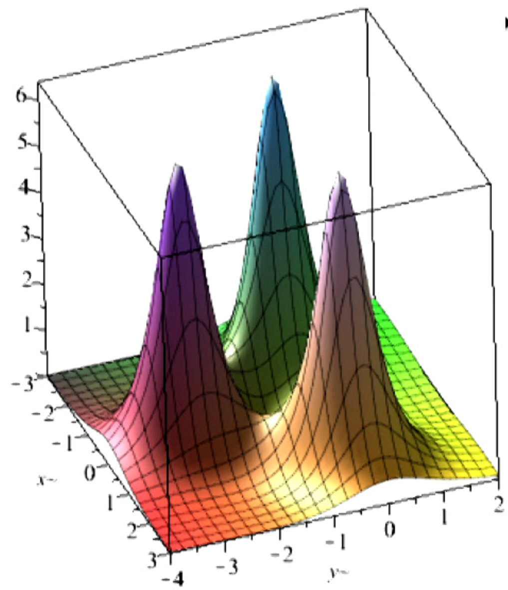
This is equivalent to

$$\partial_{\bar{z}} w = 0 \quad \text{or} \quad \partial_z w = 0.$$

Belavin-Polyakov instantons

General solution with $w_\infty = 0$ for degree $n > 0$ is holomorphic map of degree n, so a rational map of the form

$$w = \frac{a_0 + a_1 z + \ldots a_{n-1} z^{n-1}}{b_0 + b_1 z + \ldots b_{n-1} z^{n-1} + z^n}.$$



Baby Skyrmions

Can construct a toy-model for 3d Skyrmions by breaking scale invariance:

$$E[n] = \frac{1}{2}\int_{\mathbb{R}^2} ((\partial_1 n)^2 + (\partial_2 n)^2 + \kappa[\partial_1 n, \partial_2 n]^2 + \mu^2(1 - n_3))dx_1 \wedge dx_2.$$

- Energy still bounded by $4\pi \times$ |degree|, but bound not attained by solutions
- Solutions exponentially localised
- Baby skyrmions exert orientation-dependent forces on each other.
- Need numerical methods for detailed study.

Magnetic skyrmions at critical coupling

Critical combination of Zeeman energy and easy plane potential:

$$\frac{1}{2}(1-n_3)^2 = (1-n_3) - \frac{1}{2}(1-n_3^2)$$

leads to energy

$$E_S[\boldsymbol{n}] = \int_{\mathbb{R}^2} \frac{1}{2}(\nabla \boldsymbol{n})^2 + \kappa \boldsymbol{n}\cdot\nabla^{-\alpha}\times\boldsymbol{n} + \frac{\kappa^2}{2}(1-n_3)^2 \, dx_1 \wedge dx_2, \textit{ where}$$

where $\nabla^{-\alpha}\times\boldsymbol{n} = R_3(-\alpha)\boldsymbol{e}_i \times \partial_i \boldsymbol{n}$ so that spirality tensor is rotation and DMI terms is

$$\begin{aligned}&\kappa\cos\alpha(n_1\partial_2 n_3 - n_2\partial_1 n_3 + n_3(\partial_1 n_2 - \partial_2 n_1)) \\ &\qquad + \kappa\sin\alpha(-n_1\partial_1 n_3 - n_2\partial_2 n_3 + n_3(\partial_1 n_1 + \partial_2 n_2).\end{aligned}$$

Variational equation is

$$2\kappa(\boldsymbol{n}\cdot\nabla^{-\alpha})\boldsymbol{n} = \left(\Delta\boldsymbol{n} + \kappa^2(1-n_3)\boldsymbol{e}_3\right)\times\boldsymbol{n}.$$

A gauged sigma model

Consider principal $SU(2)$ bundle over Σ with connection A and associated adjoint vector bundle with section n valued in unit sphere. With

$$Dn = dn + [A, n] \qquad F_A = dA + A \wedge A,$$

consider the energy functional

$$E[A, n] = \int_\Sigma \frac{1}{2}(Dn, \wedge \star Dn) - (F, n).$$

Use $\frac{1}{2}(n, [Dn, Dn]) = \frac{1}{2}(n, [dn, dn]) + (F, n) - d(A, n)$ to write

$$E[A, n] = \frac{1}{4}\int_\Sigma ((Dn - \star[n, Dn]), \wedge \star (Dn - \star[n, Dn])) + \frac{1}{2}\int_\Sigma (n, [dn, dn]) - \int_{\partial\Sigma} (A, n).$$

A modified energy functional

Consider

$$\tilde{E}[A,n] = E[A,n] + \int_{\partial\Sigma}(A,n),$$

so that

$$\tilde{E}[A,n] = \frac{1}{4}\int_{\Sigma}((Dn - \star[n,Dn]), \wedge \star(Dn - \star[n,Dn])) + \frac{1}{2}\int_{\Sigma} n\cdot[dn,dn].$$

Now fix A and impose Bogomol'nyi equation in the boundary region. Then

$$\delta\tilde{E}[A,n] = -\int_{\Sigma}((D\wedge\star Dn + F),\delta n) + \int_{\partial\Sigma}(\epsilon, dn).$$

So variational problem for $\tilde{E}[A,n]$ with respect to n is well-defined even for variations $\delta n = [\epsilon, n]$ which vanish slowly as we approach $\partial\Sigma$.

Unitary versus holomorphic structures and a useful formula

- **Any** unitary connection on a $\mathbb{C}^2$-bundle over a Riemann surface Σ, has curvature of the form

$$F_{z\bar{z}} dz \wedge d\bar{z}$$

i.e. of type $(1,1)$.

- By Atiyah, Hitchin, Singer 1978 this means that the connection A defines a holomorphic structure and that one can choose a holomorphic gauge where $A_{\bar{z}} = 0$, i.e. $D_z = \partial_z$.
- In a unitary gauge, the connection can locally be written in the form

$$A = g\bar{\partial}_{\bar{z}} g^{-1} d\bar{z} + (g^{-1})^{\dagger}\partial_z g^{\dagger} dz, \qquad g : U \subset \Sigma \to SL(2,\mathbb{C})$$

See also Karabali and Nair, 1996.

Solving gauged $\sigma-$models

In terms of stereographic coordinates on S^2 and complex coordinates z on Σ, the Bogomol'nyi equation is

$$Dw = i \star Dw \Leftrightarrow (\partial_{\bar{z}} + A_{\bar{z}})w = 0.$$

If $A_{\bar{z}} = g\partial_{\bar{z}}g^{-1}$, can solve this explicitly by going to the holomorphic gauge in terms of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : U \to SL(2, \mathbb{C}),$$

via

$$\boxed{w = \frac{c + df}{a + bf}}$$

for any meromorphic function f.

Magnetic skyrmions from Gauged $\sigma-$models

Consider $\Sigma = \mathbb{C}$ and Cartan's 'helical staircase connection'

$$A_S = -\kappa(t_1 dx_1 + t_2 dx_2), \qquad F_S = \kappa^2 t_3 dx_1 \wedge dx_2.$$

in terms of basis t_1, t_2, t_3 of $su(2)$. Recall

$$E[A, n] = \int_\Sigma \frac{1}{2}(Dn, \wedge \star Dn) - (F, n),$$

and, for $\alpha = 0$,

$$E_S[\boldsymbol{n}] = \int_{\mathbb{R}^2} \frac{1}{2}(\nabla \boldsymbol{n})^2 + \kappa \boldsymbol{n} \cdot \nabla \times \boldsymbol{n} + \frac{\kappa^2}{2}(1 - n_3)^2 \, dx_1 \wedge dx_2.$$

After some calculation,

$$\boxed{E[A_S, n] = E_S[n].}$$

where we replaced $\boldsymbol{n} \to n$.

Modified energy

The modified energy

$$\tilde{E}[A, n] = \int_\Sigma \frac{1}{2}(Dn, \wedge \star Dn) - (F, n) + \int_{\partial\Sigma} (A, n)$$

reproduces the energy functional proposed in L Döring, C Melcher, Calculus of Variations 2017:

$$\tilde{E}_S[\boldsymbol{n}] = \int_{\mathbb{R}^2} \frac{1}{2}(\nabla \boldsymbol{n})^2 + \kappa(\boldsymbol{n} - \boldsymbol{e}_3) \cdot \nabla \times \boldsymbol{n} + \frac{\kappa^2}{2}(1 - n_3)^2 \, dx_1 \wedge dx_2.$$

In other words

$$\boxed{\tilde{E}[A_S, n] = \tilde{E}_S[n].}$$

Harmonic magnetic skyrmions

To solve the Bogomol'nyi equation we note

$$(A_S)_{\bar{z}} = g\partial_{\bar{z}} g^{-1}, \qquad g = \begin{pmatrix} 1 & -\frac{i}{2}\kappa e^{i\alpha}\bar{z} \\ 0 & 1 \end{pmatrix}.$$

The general solution of the gauged sigma model in this case is

$$\boxed{w = \frac{1}{v}, \qquad v(z,\bar{z}) = -\frac{i}{2}\kappa e^{i\alpha}\bar{z} + f(z),}$$

with $f : \mathbb{C} \to \mathbb{CP}^1$ holomorphic.

Reconstruct magnetisation field via

$$n_1 + in_2 = \frac{2\bar{v}}{|v|^2+1}, \qquad n_3 = \frac{|v|^2-1}{|v|^2+1}.$$

Hedgehogs and line defects

From $v = -\frac{i}{2}\kappa e^{i\alpha}\bar{z}$ obtain hedgehog field

$$\boldsymbol{n} = \begin{pmatrix} \sin\theta(r)\cos(\phi+\gamma) \\ \sin\theta(r)\sin(\phi+\gamma) \\ \cos\theta(r) \end{pmatrix}, \qquad z = re^{i\phi},$$

with

$$\gamma = \frac{\pi}{2} - \alpha, \qquad f(r) = 2\tan^{-1}\left(\frac{2}{\kappa r}\right).$$

(also L Döring, C Melcher, Calculus of Variations 2017)

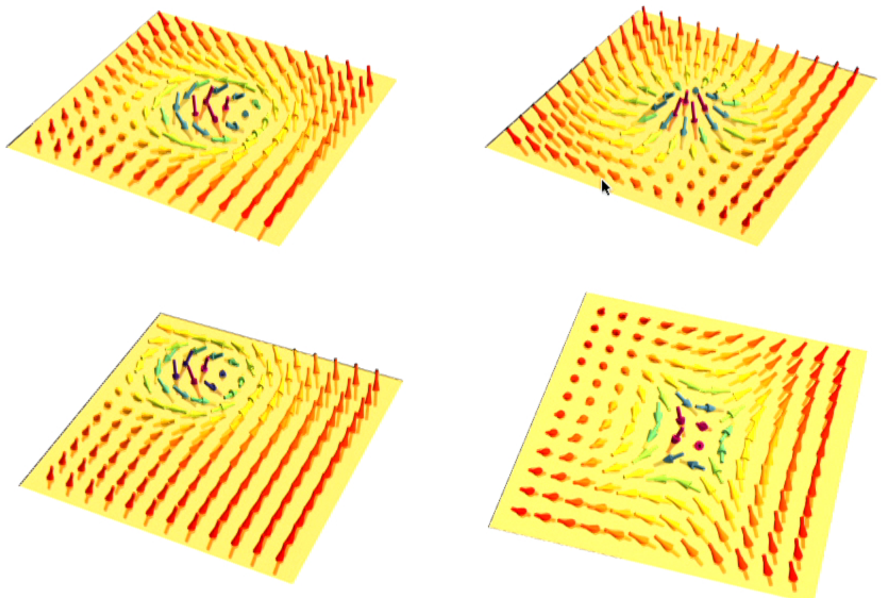


Figure: Top from left to right: Bloch skyrmion $v = -\frac{i}{2}\bar{z}$ and Néel skyrmion $v = \frac{1}{2}\bar{z}$. Bottom from left to right: a shifted Bloch skyrmion $v = -\frac{i}{2}\bar{z} + \frac{1}{2}(3 - 2i)$ and the anti-skyrmion configuration $v = -\frac{i}{2}\bar{z} + 3iz$.

Hedgehogs and line defects

From $v = -\frac{i}{2}\kappa e^{i\alpha}\bar{z}$ obtain hedgehog field

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(also L Döring, C Melcher, Calculus of Variations 2017)

From $v = -\frac{i}{2}\kappa(\bar{z} + z)$ find defect line along $x = 0$:

$$\boldsymbol{n} = \begin{pmatrix} 0 \\ -\frac{2\kappa x}{\kappa^2 x^2 + 1} \\ \frac{\kappa^2 x^2 - 1}{\kappa^2 x^2 + 1} \end{pmatrix}.$$

Energy, degree and vorticity

The energy of solutions can be written as

$$E_S[n] = 4\pi \deg[n] - \int_{\partial\Sigma} (A_S, n)$$

where

$$(A_S, n) = -\kappa(n_1 dx_1 + n_2 dx_2).$$

For which configurations is this well-defined?

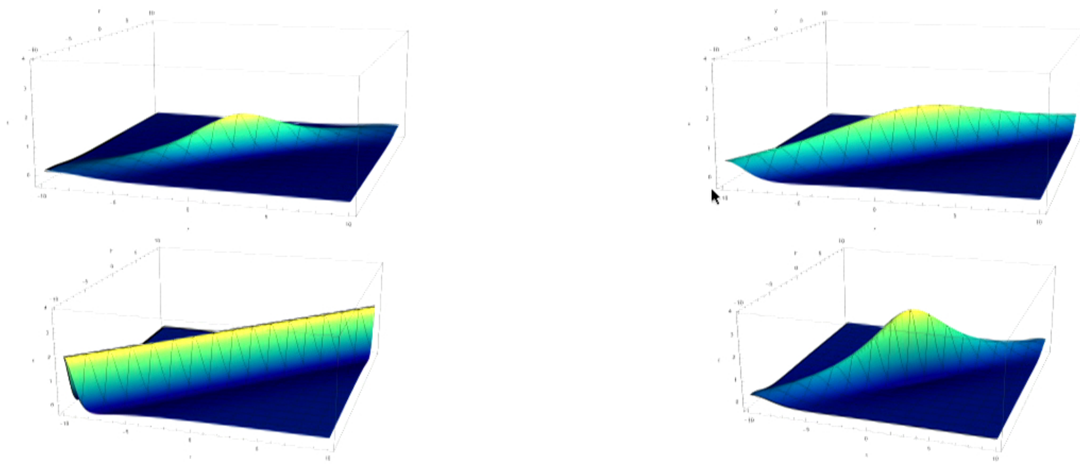


Figure: Stretching and squeezing for the configuration $v = -\frac{i}{2}\bar{z} + az$ with $a = 0.3$ (top left), $a = 0.4$ (top right), $a = 0.5$ (bottom left) and $a = 0.7$ (bottom right).

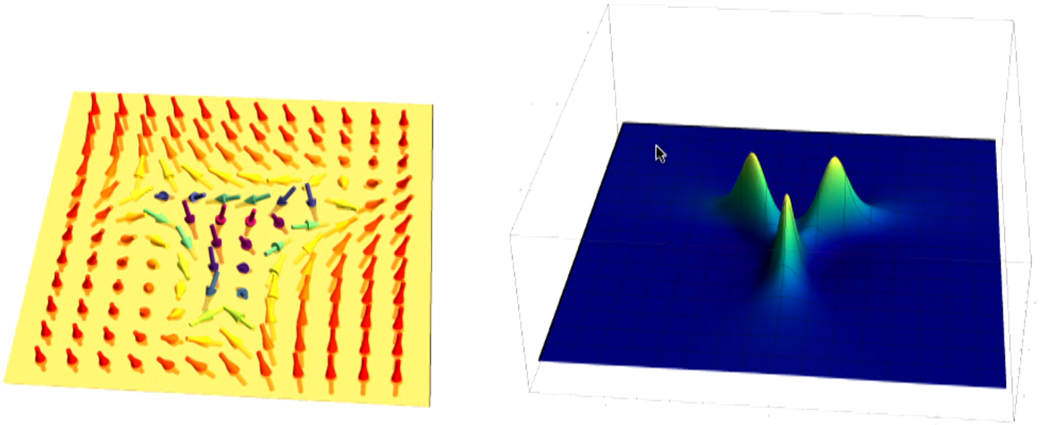
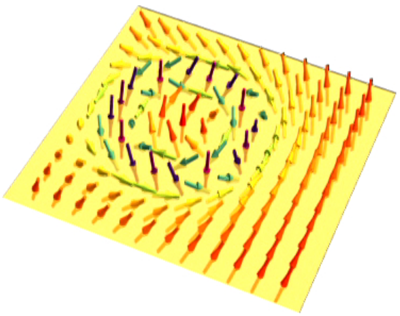


Figure: Magnetisation and energy density for $N = 2$ solution $v = \frac{i}{2}\bar{z} + \frac{1}{2}z^2$. This is an example of a configuration involving a skyrmion and three anti-skyrmions.



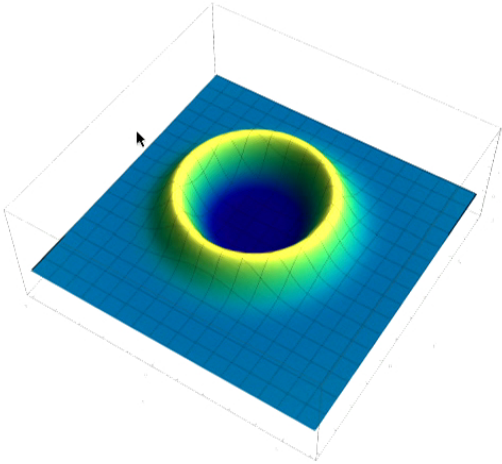
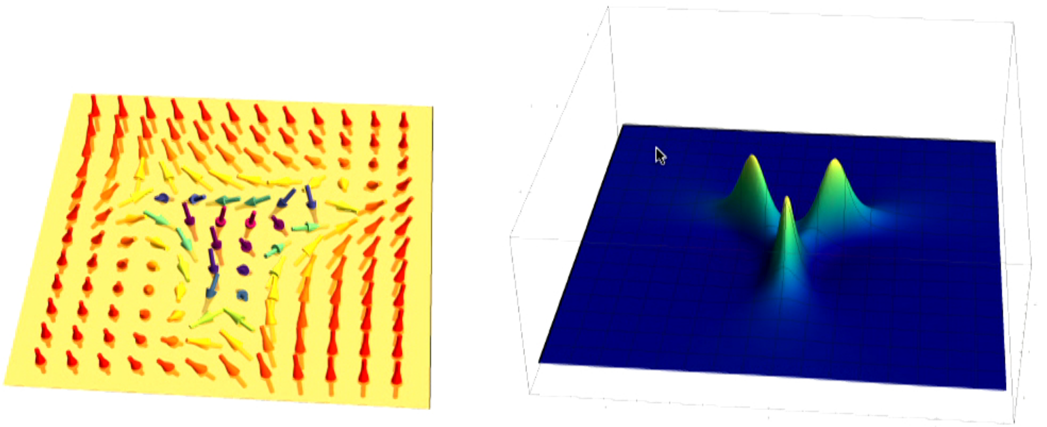


Figure: Magnetisation and energy density for the skyrmion bag defined by $v = -\frac{i}{2}\bar{z} + \frac{z+2i}{z-2i}$.



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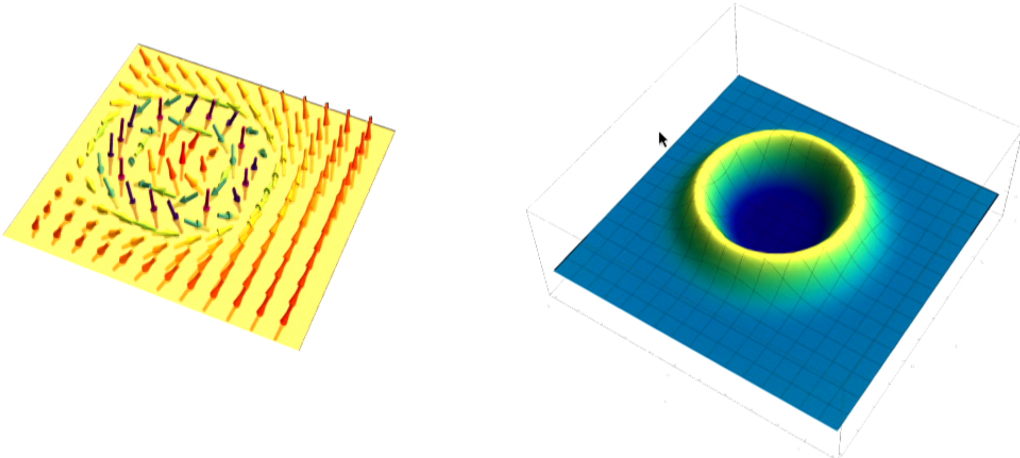

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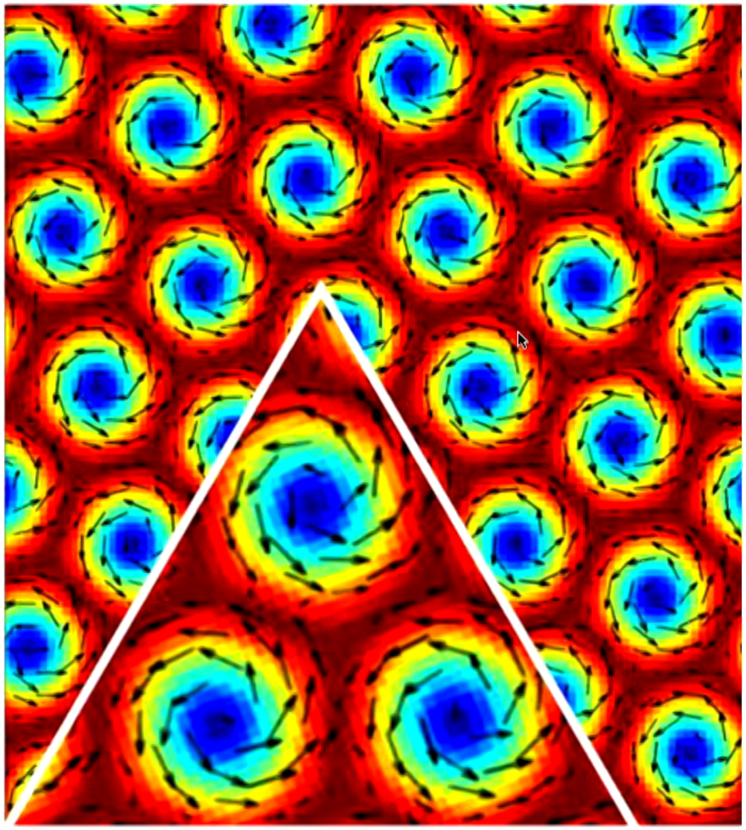


Figure: The numerically computed ground state: an infinite skyrmion lattice. From Lin, Saxena and Batista, Phys Rev B 91 (2015) 224407)

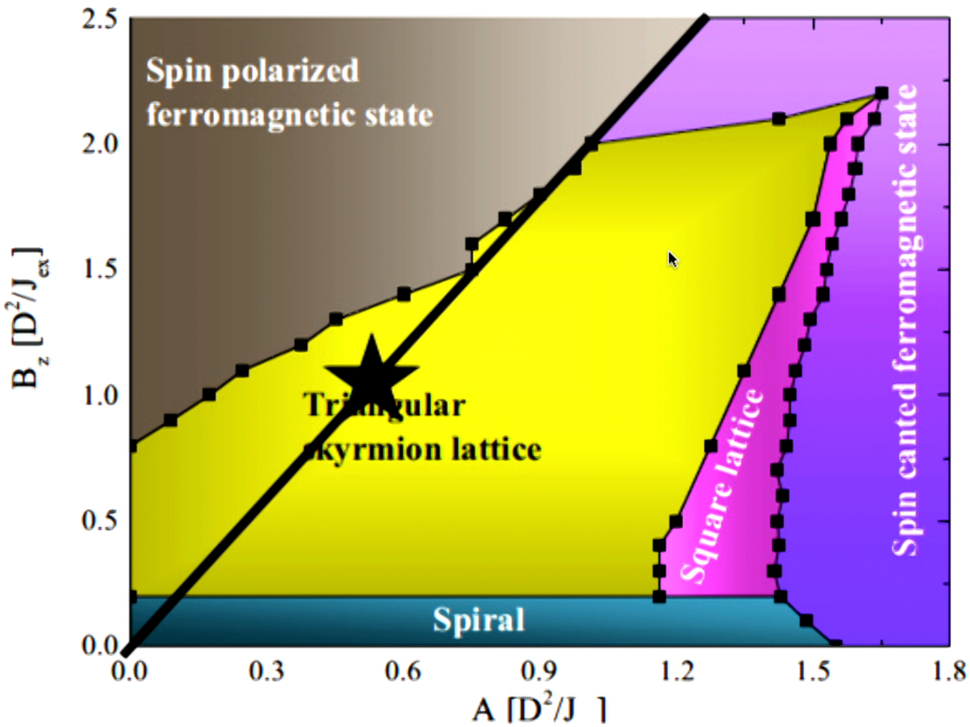


Figure: The numerically computed phase diagram. From Lin, Saxena and Batista, Phys Rev B 91 (2015) 224407)

Rank 1 magnetic skyrmions

Translation of DMI term into gauge theory according to

$$\mathcal{D}_{ai}(\partial_i \boldsymbol{n} \times \boldsymbol{n})_a = (A_i, [\partial_i n, n]).$$

Rank 1 materials correspond to flat connections

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \Leftrightarrow A = a t_3 dx_1 .$$

The energy is

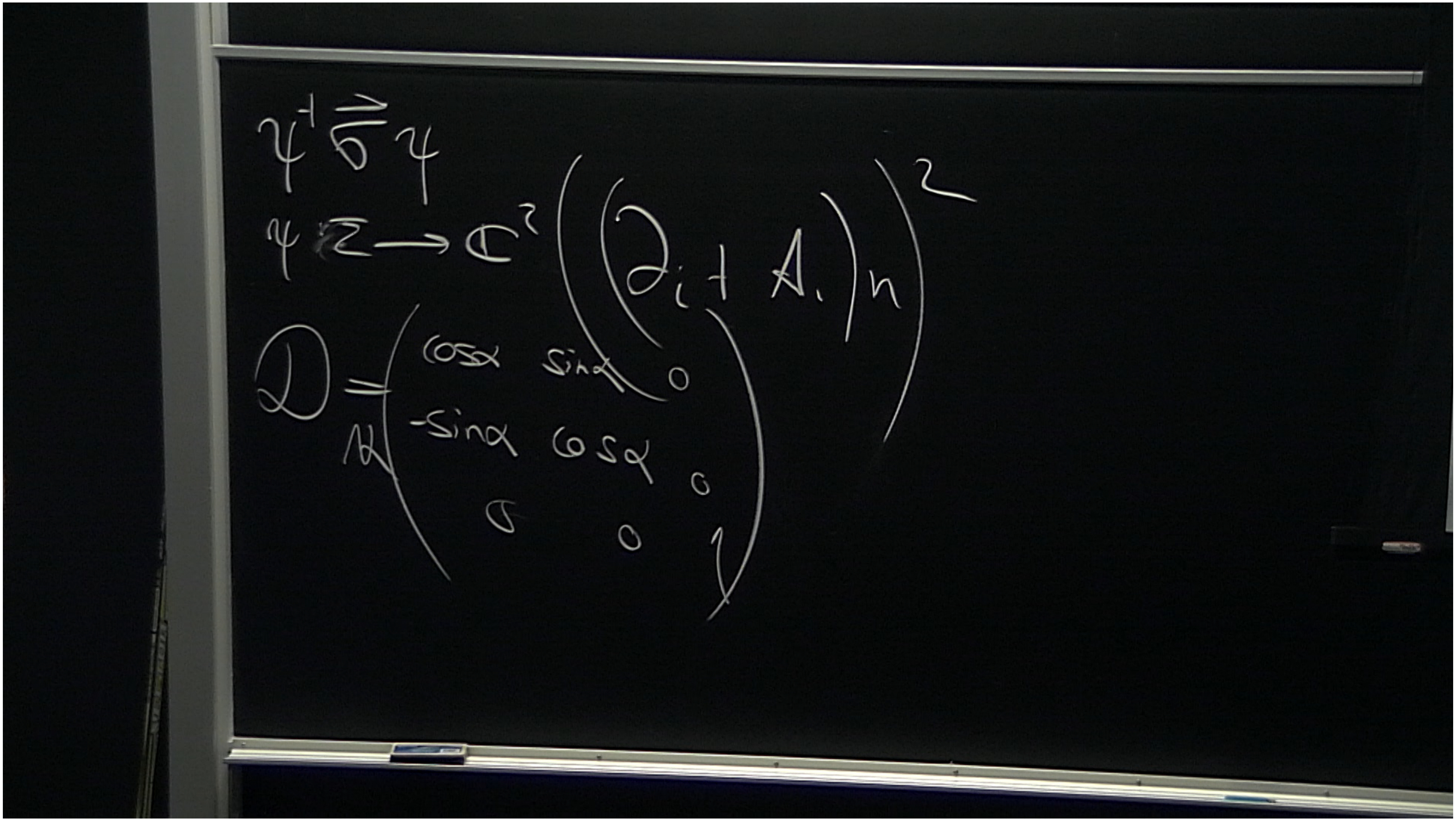
$$\int_{\mathbb{R}^2} \left(\frac{1}{2}|\partial_1 n|^2 + \frac{1}{2}|\partial_2 n|^2 - a(n_2 \partial_1 n_1 - n_1 \partial_1 n_2) - \frac{1}{2}(1 - n_3^2) \right) dx_1 dx_2,$$

and solutions of the Bogomol'nyi equations include skymions, anti-skyrmions and domain walls

$$w = e^{-iax} \frac{p(\bar{z})}{q(\bar{z})} \quad \text{or} \quad w = e^{-iax} \frac{p(z)}{q(z)} \quad \text{or} \quad w = e^{-ay}.$$

Conclusion and Questions

- Magnetic skyrmions at critical coupling are holomorphic sections of $\mathbb{CP}^1$-bundle with connection determined by the DMI term.
- Exact solutions predict unexpected multi-soliton configurations.
- Lattice version?
- Stability of multi-solitons?
- What is the time evolution?


ψ† σ⃗ ψ
ψ: Σ → ℂ²
((∂ᵢ + Aᵢ) n)²
D =
cos α sin α 0
-sin α cos α 0
0 0 1