Title: Real-space recipes for general topological crystalline states

Speakers: Yang Qi

Series: Condensed Matter

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Abstract: Topological crystalline states are short-range entangled states jointly protected by onsite and crystalline symmetries. While the non-interacting limit of these states, e.g., the topological crystalline insulators, have been intensively studied in band theory and have been experimentally discovered, the classification and diagnosis of their strongly interacting counterparts are relatively less well understood. Here we present a unified scheme for constructing all topological crystalline states, bosonic and fermionic, free and interacting, from real-space "building blocks" and "connectors". Building blocks are finite-size pieces of lower dimensional topological states protected by onsite symmetries alone, and connectors are "glue" that complete the open edges shared by two or multiple pieces of building blocks. The resulted assemblies are selected against two physical criteria we call the "no-open-edge condition" and the "bubble equivalence", which, respectively, ensure that each selected assembly is gapped in the bulk and cannot be deformed to a product state. The scheme is then applied to obtaining the full classification of bosonic topological crystalline states protected by several onsite symmetry groups and each of the 17 wallpaper groups in two dimensions and 230 space groups in three dimensions. We claim that our real-space recipes give the complete set of topological crystalline states for bosons and fermions, and prove the boson case analytically using a spectral sequence expansion of group cohomology.

Real-Space Recipes for General Topological Crystalline States

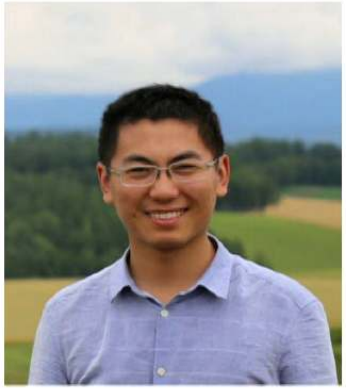
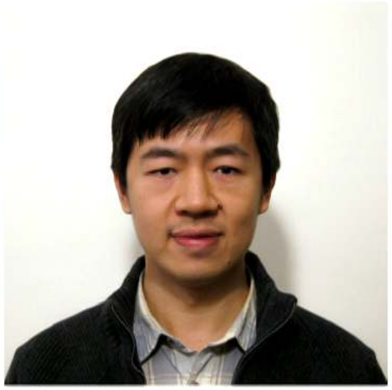
Yang Qi

Department of Physics, Fudan University

Perimeter Institute, Sept. 3rd, 2019.

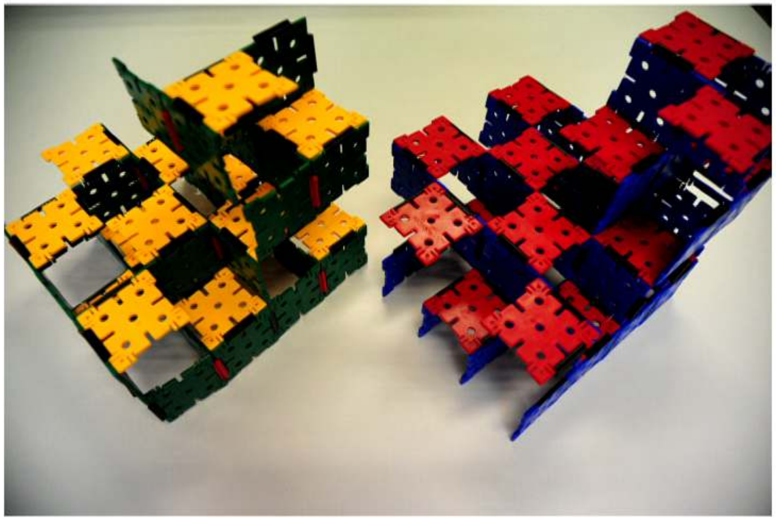
References

- Zhi-Da Song: Princeton University.
- Chen Fang: Institute of Physics, Beijing.

- Zhida Song, Chen Fang and Yang Qi, arXiv:1810.11013

Outline



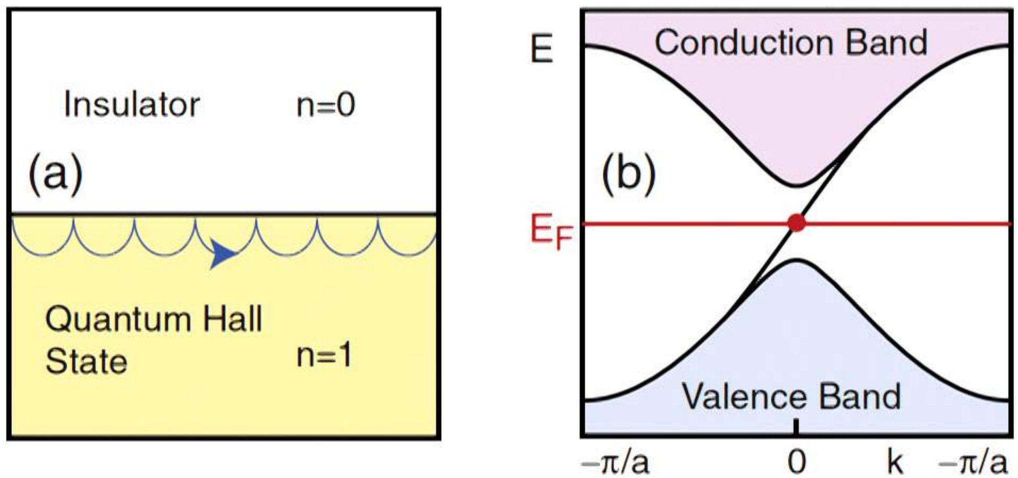
1. Introduction to SPT States
2. Assymbling Topological Crystalline States.
3. Full Version: Higher Pages
4. Conclusion

Symmetry-Protected Topological (SPT) states

- SPT: gapped topological phases beyond Landau paradiam.
- Gapped bulk : cannot be smoothly connected to a trivial state without closing gap or breaking symmetry.
- Symmetry-protected gapless surface states.
- Free-fermion states: topological insulators, topological superconductors: K-theory.
- Bosonic SPTs: Haldane chain, CZX/Levin-Gu state, etc: Group Cohomology.
- Interacting fermionic SPTs.

Abelian-group classification

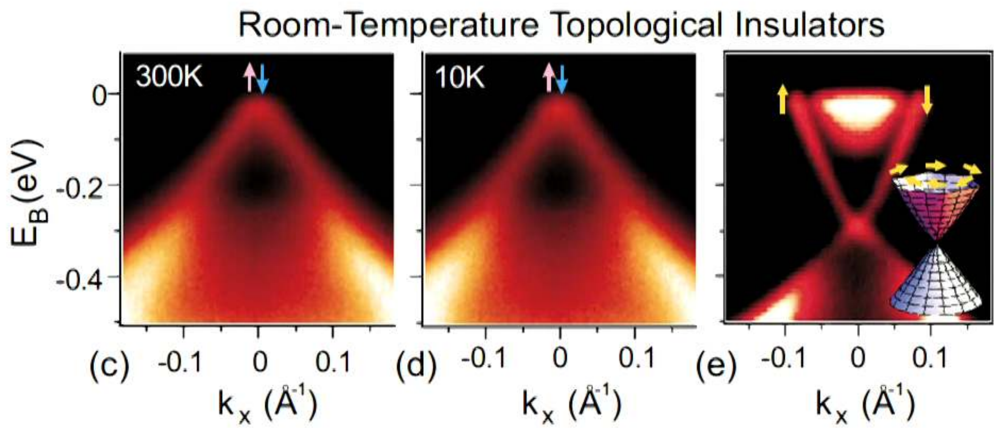
- SPT phases and boundary anomalies are classified by Abelian groups ($\mathbb{Z}$ or $\mathbb{Z}_n$).
  - Addition: stacking of phases/gapless boundaries.
  - 0: The trivial phase/gapped boundary.
- 2D Chern-insulators (Integer Quantum Hall):



Classified by $\mathbb{Z}$: $[n] + [m] = [n + m]$; $[n] + [-n] = 0$.

Abelian-group classification

- 3D Topological Insulators:



Classified by $\mathbb{Z}_2$: $[1] + [1] = 0$.

- 1D Haldane chain:

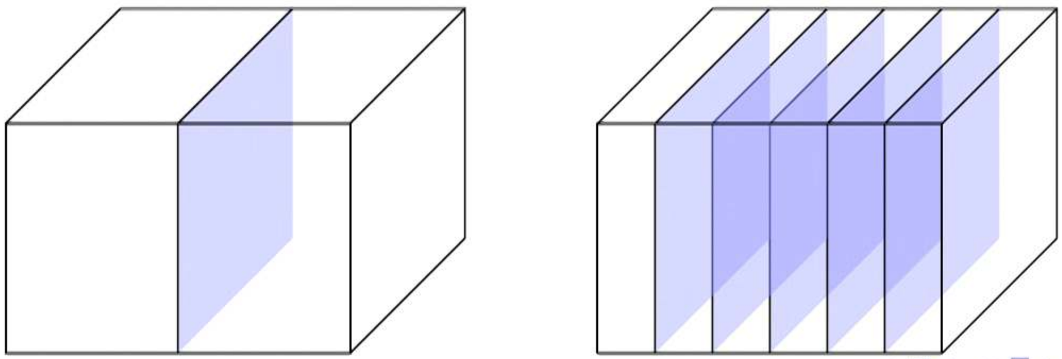
Classified by $\mathbb{Z}_2$: $[1] + [1] = 0$.

Space-group SPT

- We consider $G/G_0 = SG$.
- Thorngren and Else (2018): the crystalline equivalence principle

$$H^{d+1}[G, \mathrm{U}(1)_{PT}].$$

- Dimensional reduction: Liang Fu, Michael Hermele et al.
  Examples: mirror SPT, weak SPT (translation symmetry).
- Patch construction: Zhida Song, Shengjie Huang, Yang Qi, Chen Fang and Michael Hermele, arXiv:1810.02330.
- A more general construction for bosonic SPTs w/ all possible G.



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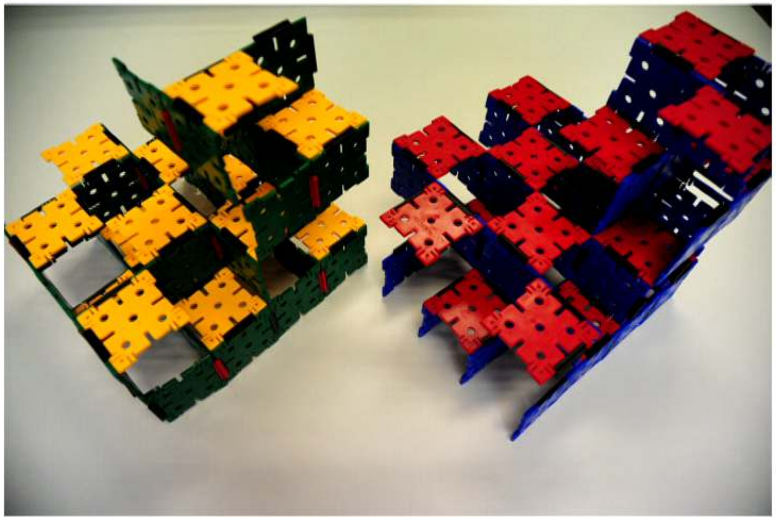

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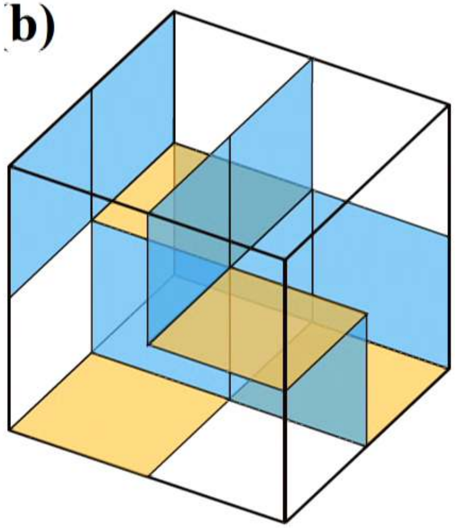
Decomposition of the space

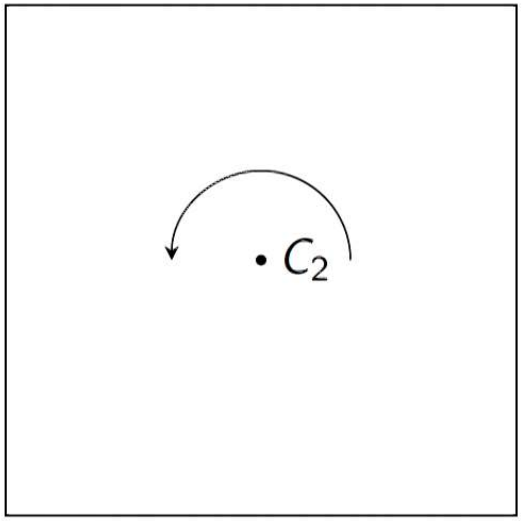
We divide the space into finer cells such that

1. A cell σ is maped to one single cell σ' under SG-action.
2. $G_\sigma = \{g \in G | g : \sigma \to \sigma\}$ acts on σ as onsite symmetry.

A G-complex $Y \simeq \mathbb{R}^d$. Each cell hosts only onsite-symmetry SPTs.

b)





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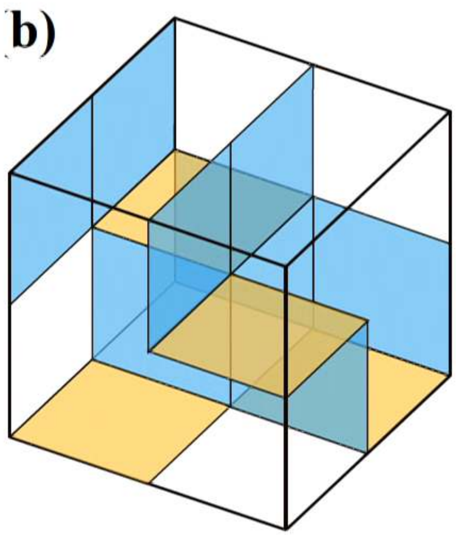
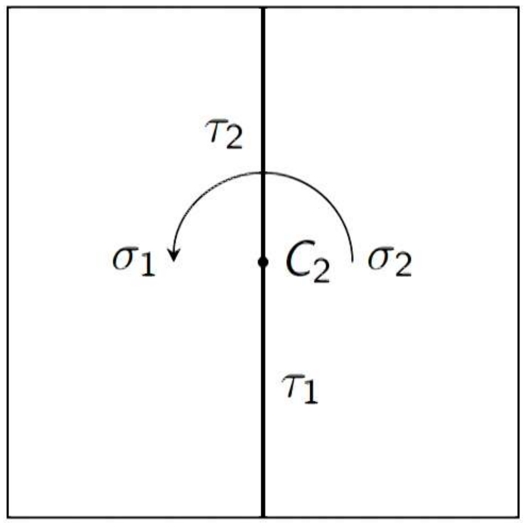
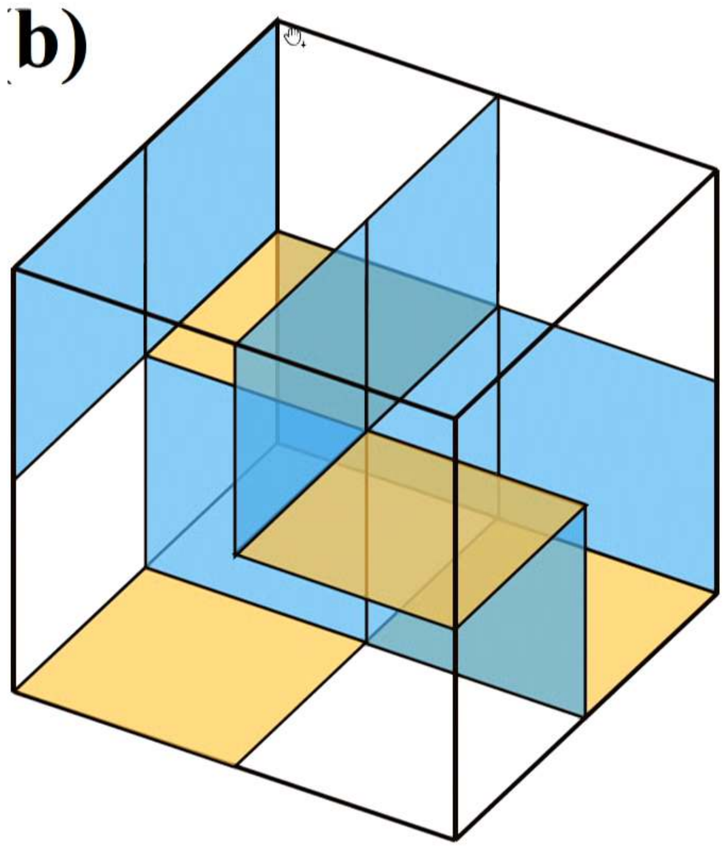




Topological crystalline states are made of building blocks

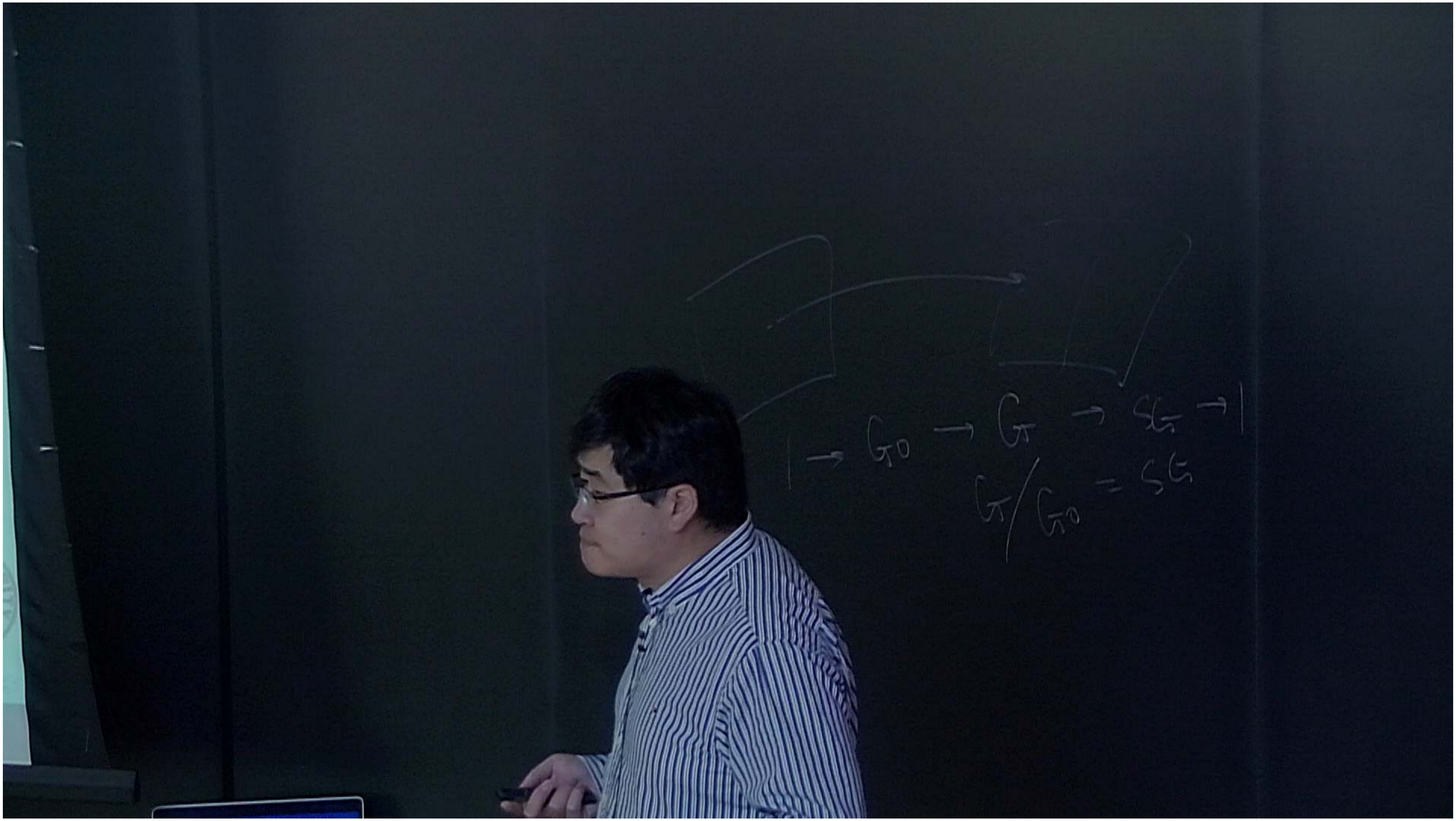


- We divide the space into cells compatible with the space-group symmetry.
- On a p-cell σ, the SPT state is protected only by G_σ.

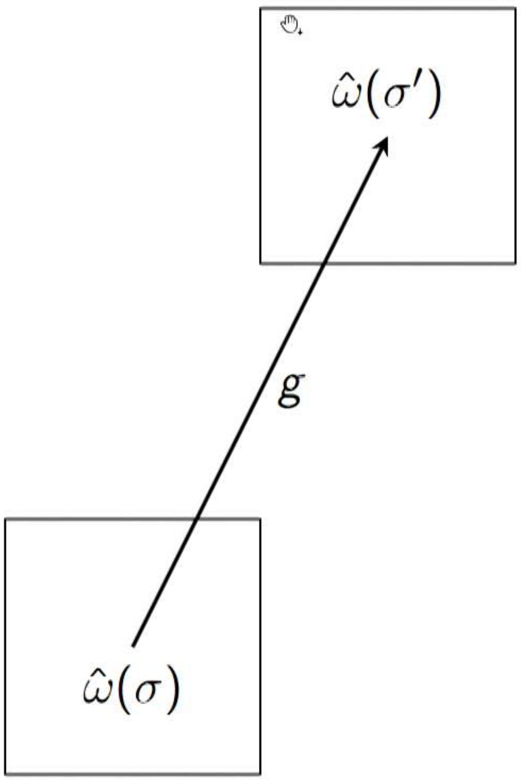
$$\hat{\omega}(\sigma) \in \Phi^p(G_\sigma) = H^{p+1}[G_\sigma, \mathrm{U}(1)_T].$$

- G_σ acts as onsite symmetries.
- Decorate 3d SPT on 3-cells; 2d SPT on 2-cells; 1d SPT on 1-cells; 0d SPT on 0-cells;
- p-block states: $E^p_{p,\infty}$.

$$\text{TCSs} = \text{``}\bigoplus_p\text{''}\, E^p_{p,\infty}.$$


1 → G0 → G → SG → 1
G/G0 = SG

Symmetric conditions



- If $g : \sigma \to \sigma'$, then the cochains attached must be "identical".
- $G_\sigma \neq G_{\sigma'}$, but they are isomorphic:

$$G_{\sigma'} = g G_\sigma g^{-1} \simeq G_\sigma.$$

- This induces another isomorphism:

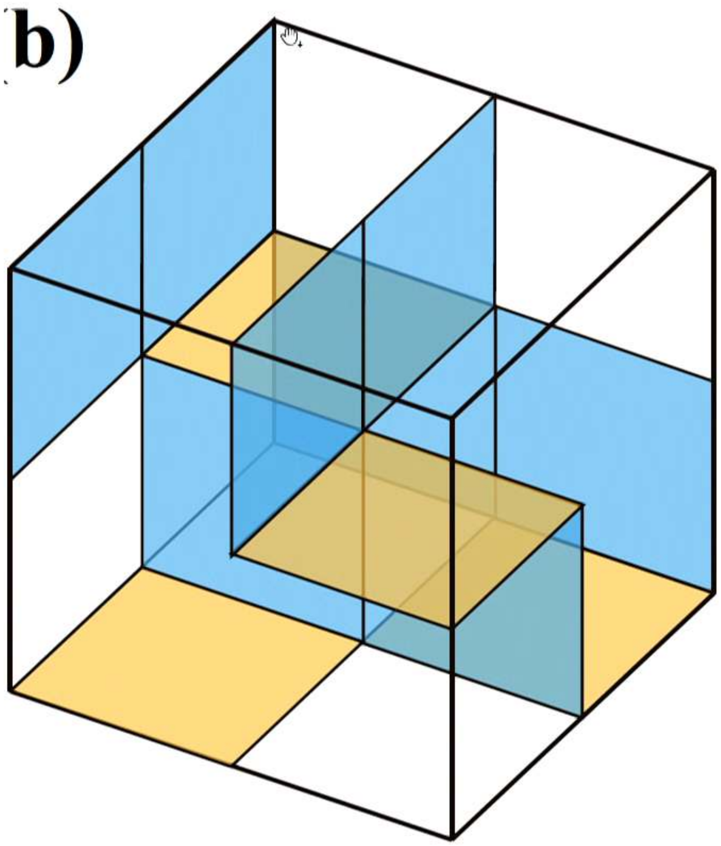
$$H^{p+1}[G_{\sigma'}, \mathrm{U}(1)_T] \simeq H^{p+1}[G_\sigma, \mathrm{U}(1)_T]$$

- $\hat{\omega}(\sigma)$ and $\hat{\omega}(\sigma')$ are related by this isomorphism

$$\hat{\omega}(\sigma') = g \cdot \hat{\omega}(\sigma).$$

- Only decorations on symmetry-unrelated cells are independent: finite # of them.

The 1st page



- 1st page = a collection of SPTs:

$$E^p_{p,1} = \bigoplus_{\sigma \in Y_p/G} \Phi^p(G_\sigma),$$

$$\Phi^p(G_\sigma) = H^{p+1}(G_\sigma, \mathrm{U}(1)_T).$$

- We can generalize this to

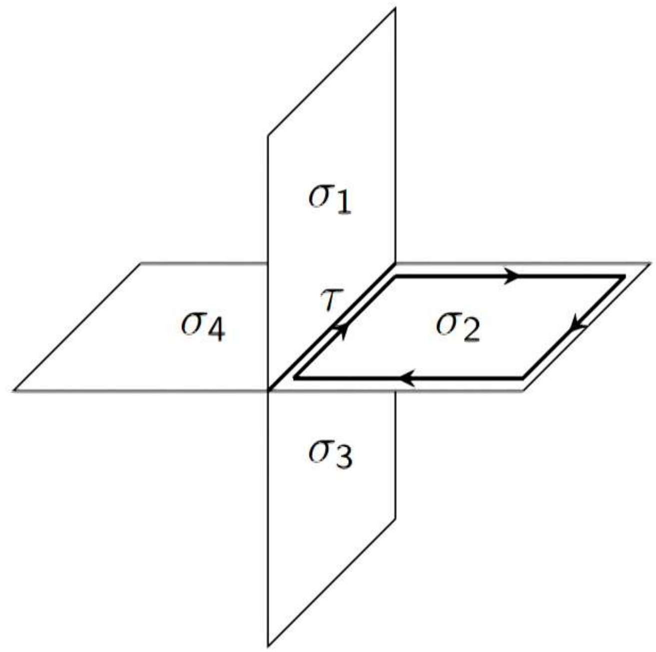
$$E^q_{p,1} = \bigoplus_{\sigma \in Y_p/G} \Phi^q(G_\sigma).$$

No-Open-Edge Conditions

- SPT blocks have nontrivial boundary states.
- Boundary anomaly must cancel on $(p-1)$-cells.
- Naively:

$$\hat{\omega}(\sigma_1) + \hat{\omega}(\sigma_2) + \hat{\omega}(\sigma_3) + \hat{\omega}(\sigma_4)$$

- A subtlety: τ has more symmetry operations than σ: $G_\tau \supseteq G_\sigma$
- Hence, $\hat{\omega}(\sigma_i)$ is not G_τ-symmetric.
- When $[G_\tau : G_\sigma] > 1$: τ borders $[G_\tau : G_\sigma]$ symmetry-related copies of σ_i.
- $\hat{\omega}(\sigma_1) + \hat{\omega}(\sigma_2) + \hat{\omega}(\sigma_3) + \hat{\omega}(\sigma_4) \in \Phi^p(G_\tau)$.
- A transfer map: $\Phi^p(G_\sigma) \to \Phi^p(G_\tau)$.



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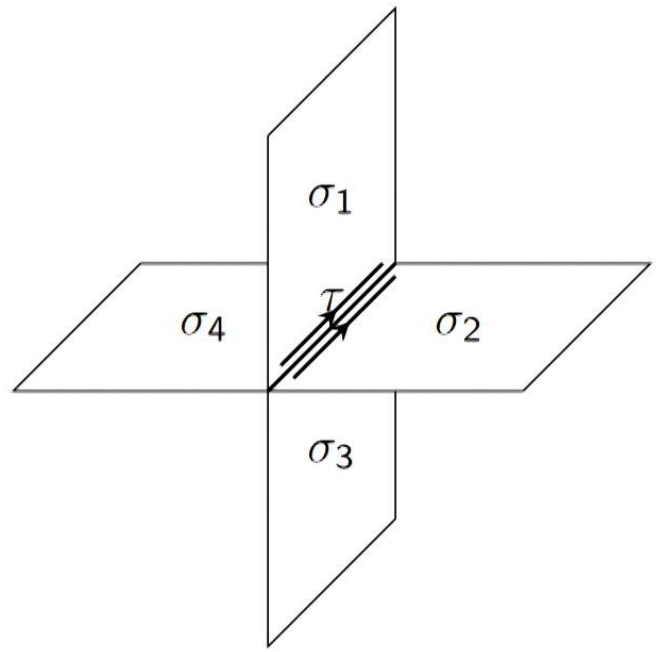

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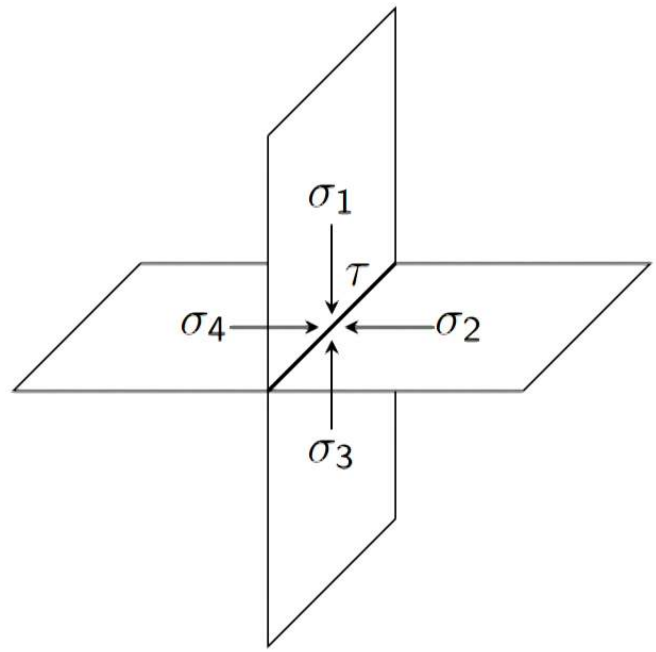


No-Open-Edge Conditions

- We can define a boundary-transfer operation $\partial\omega$:

$$(\partial\hat{\omega})(\tau) = \hat{\omega}(\sigma_1) + \hat{\omega}(\sigma_2) + \hat{\omega}(\sigma_3) + \hat{\omega}(\sigma_4).$$

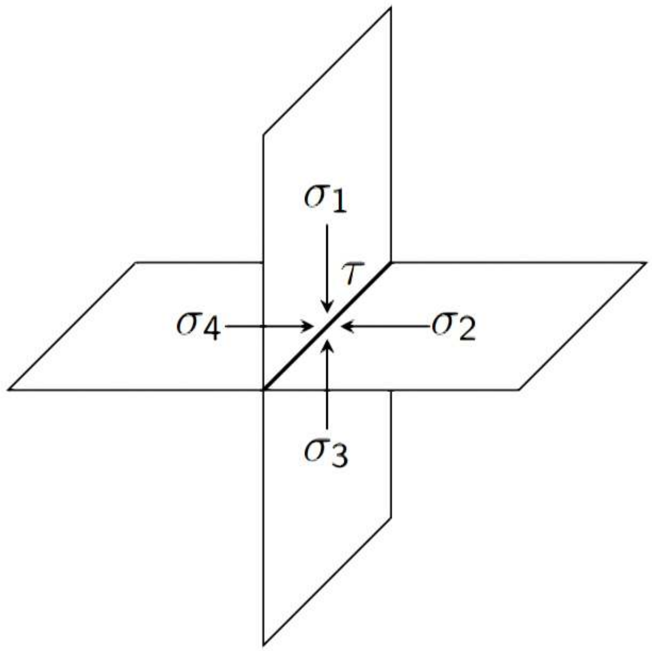
- Cocycle equation: $\partial\omega \simeq 0$.
- Define $d_1 : E^p_{p,1} \to E^p_{p-1,1}$:

$$d_1\hat{\omega} = \partial\hat{\omega}.$$

- First-page no-open-edge condition:

$$d_1\hat{\omega} \simeq 0.$$

- $E^{p+1}_{p,r}$: anomaly pattern.



Bubbling Equivalence

- Coboundaries: $\hat{\omega} \simeq 0$.
- A coboundary: attaching the same SPT state to $\partial\sigma$.
- The bubbling process can also expressed by ∂:

$$\hat{\omega} = \partial\hat{\mu} \simeq 0.$$

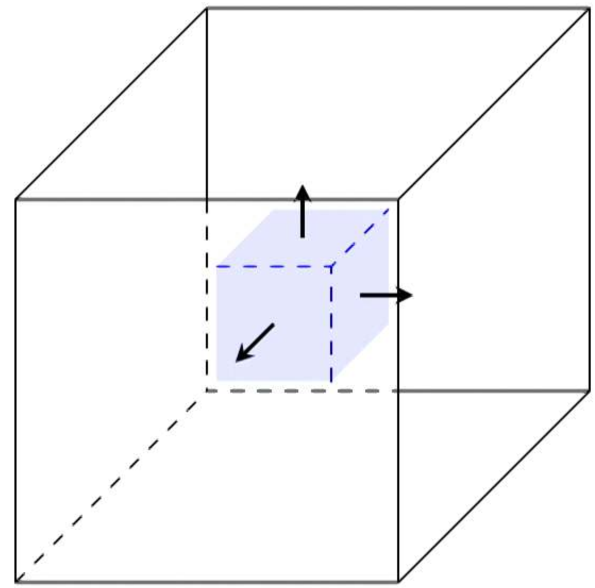
- We define $d_1 : E^p_{p+1,1} \to E^p_{p,1}$:

$$d_1\hat{\mu} = \partial\hat{\mu}.$$

- The first-page bubbling equivalence:

$$d_1\hat{\mu} \simeq 0.$$

- $E^{p-1}_{p,r}$: Bubbling patterns.



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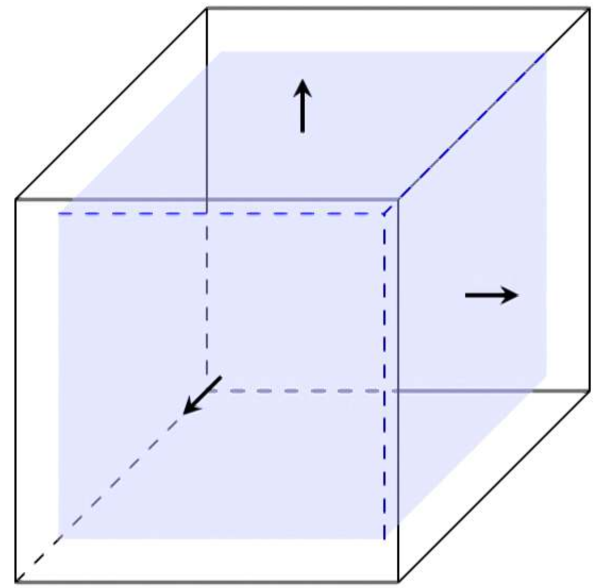


2nd page: a homology-group calculation

- No-open-edge conditions:

$$d_1\hat{\omega} \simeq 0, \hat{\omega} \in E^{p}_{p,1}.$$

- Bubbling equivalence:

$$d_1\hat{\mu} \simeq 0, \hat{\mu} \in E^{p}_{p+1,1}.$$

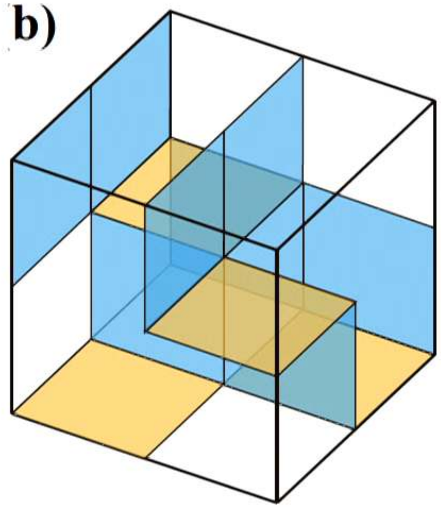
- A homology-group calculation:

$$E^{p}_{p+1,1} \xrightarrow{d_1} E^{p}_{p,1} \xrightarrow{d_1} E^{p}_{p-1,1},$$

$$E^{p}_{p,2} = \frac{\ker d^{p}_{p+1,1}}{\mathrm{img}\, d^{p}_{p,1}}.$$

An example of a nontrivial building block

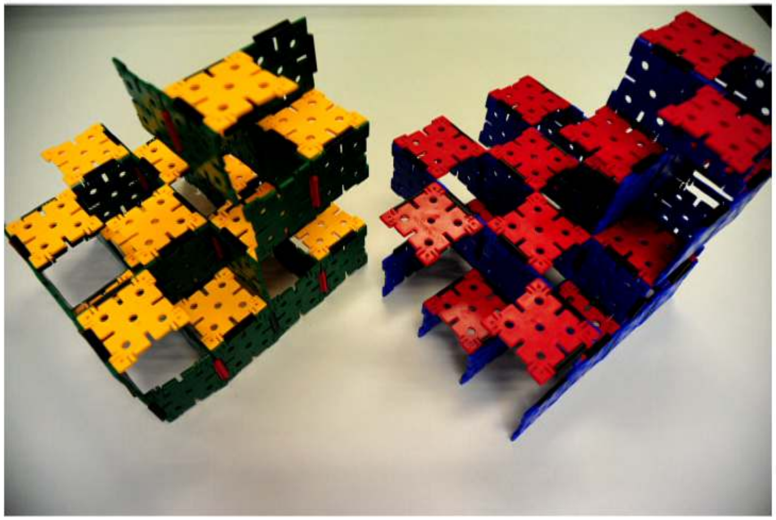
Consider $G = SG \times \mathbb{Z}_2$, $SG = P4_22_12$ (#94 space group).



Each colored 2-cell is decorated with a Levin-Gu or CZX state protected by the onsite $\mathbb{Z}_2$.

Adapted from Z Song, Y Huang, YQ, C Fang and M Hermele, arXiv:1810.02330.

Outline



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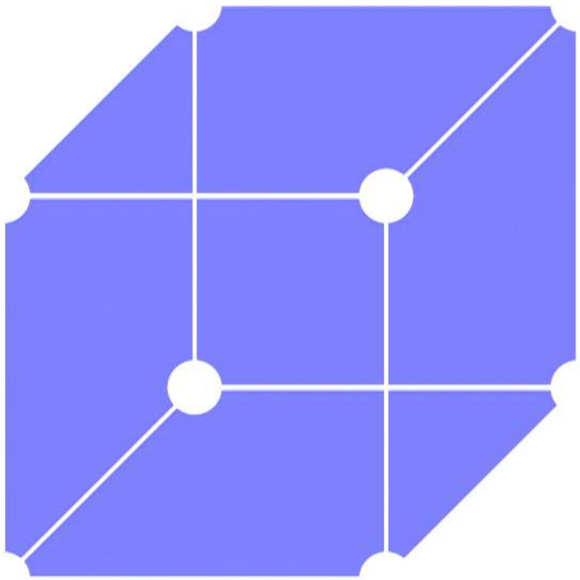
4. Conclusion

High-Order No-Open-Edge Conditions

- Consider $\hat{\omega}_{2+} = \hat{\omega}_2 + \hat{\omega}_1 + \hat{\omega}_0$.
- Cocycle condition: $(d - \partial)\hat{\omega} = 0$:

$$d\hat{\omega}_2 = 0;$$
$$d\hat{\omega}_1 = \partial\hat{\omega}_2 \simeq 0;$$
$$d\hat{\omega}_0 = \partial\hat{\omega}_1 \simeq 0.$$

1. Choose a cocycle for each 2-cell σ.
   Check $\partial\hat{\omega}_2(\tau) \simeq 0$ for each 1-cell τ.
   1st-page no-open-edge condition.
2. Choose a cochain $\hat{\omega}_1$ for each τ.
   Check $\partial\hat{\omega}_1(\lambda) \simeq 0$ for each 0-cell λ.
   2nd-page no-open-edge condition.
3. Choose a cochain $\hat{\omega}_0$ for each λ.



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$$
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d\hat{\omega}_2 &= 0; \\
d\hat{\omega}_1 &= \partial\hat{\omega}_2 \simeq 0; \\
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\end{aligned}
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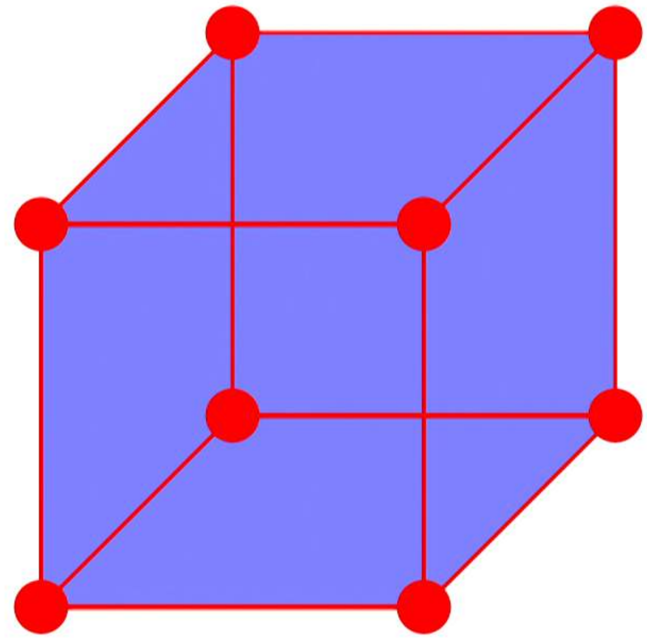
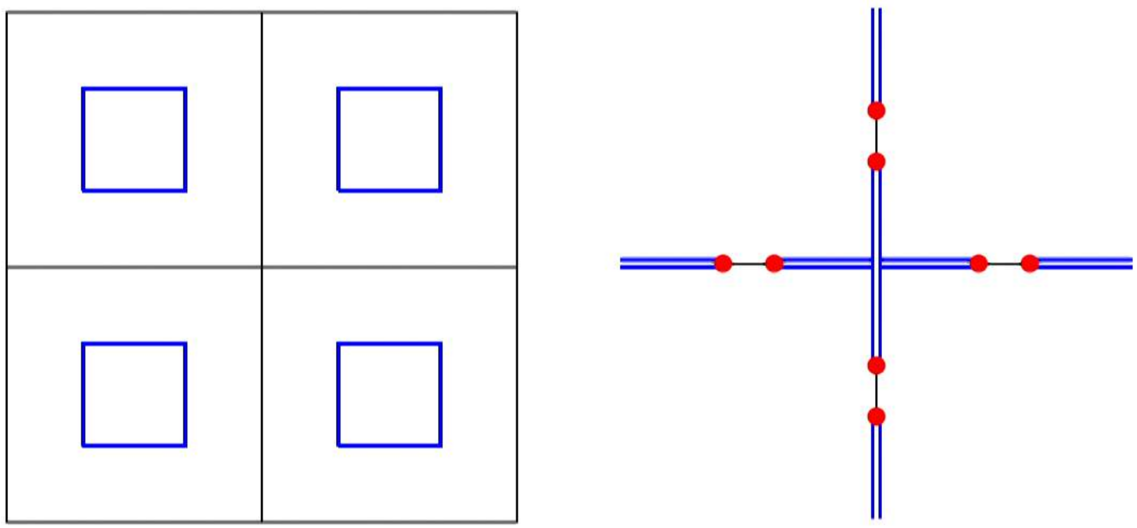


High-Order Bubbling Equivalences

- There is a similar algorithm for the bubbling equivalence.
- Assign 1d SPT states ($d\hat{\mu}_1 = 0$) to 3-cells: $\partial\hat{\mu}_2$ trivializes $\hat{\omega}_1$ building blocks.
- If $\partial\hat{\mu}_2 \simeq 0$, choose $d\hat{\mu}_1 + \partial\hat{\mu}_2 = 0$, then $\partial\hat{\mu}_1$ trivializes $\hat{\omega}_0$ building blocks.
- ......



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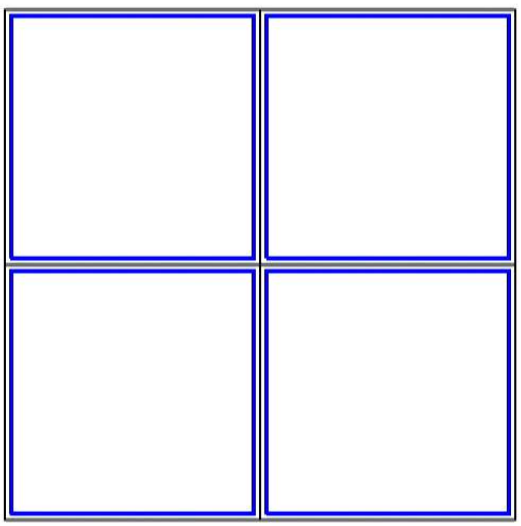

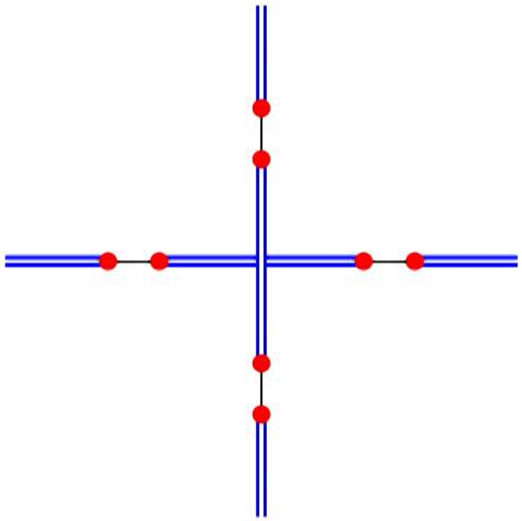

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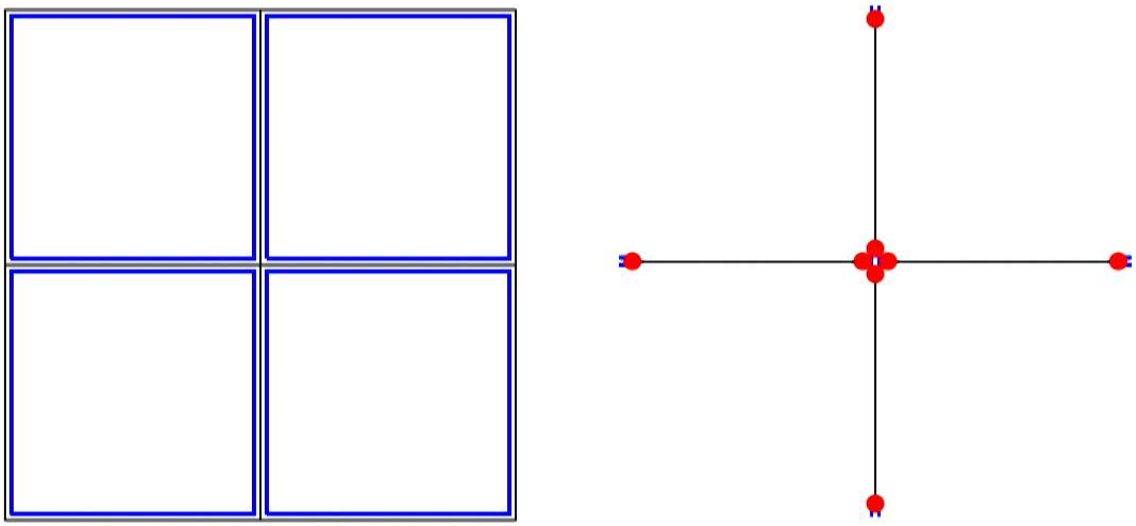


A spectral sequence

We compute in the following sequence:

1. 0st-page no-open-edge condition + 0st-page bubbling equivalence.

$$E^{p}_{p,1} = \frac{\ker d_0}{\operatorname{img} d_0}.$$

2. 1st-page no-open-edge condition + 1st-page bubbling equivalence.

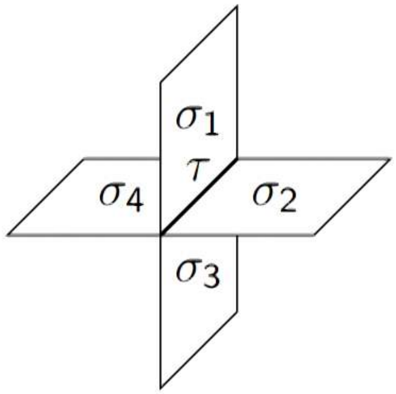
$$E^{p}_{p,2} = \frac{\ker d_1}{\operatorname{img} d_1}.$$

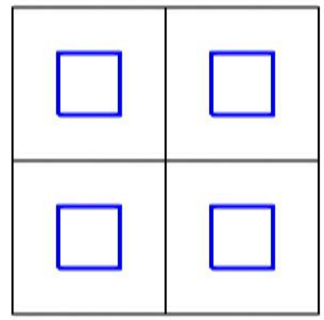
3. 2nd-page no-open-edge condition + 2st-page bubbling equivalence.

$$E^{p}_{p,3} = \frac{\ker d_2}{\operatorname{img} d_2}.$$

$$E^{p}_{p,1} \supseteq E^{p}_{p,2} \supseteq E^{p}_{p,3} \supseteq \cdots = E^{p}_{p,\infty}.$$

Second Page

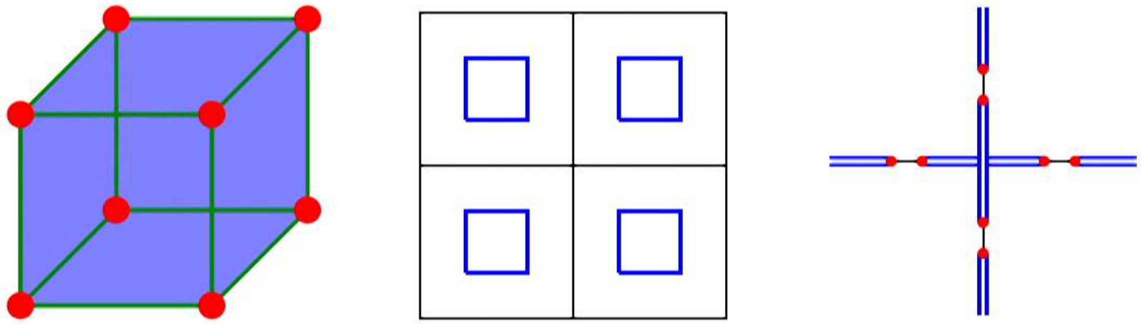




- No-open-edge condition: $\partial\hat{\omega}_2 \sim 0$.
- Bubbling equivalence: $\hat{\omega}_2 \to \hat{\omega}_2 + \partial\hat{\mu}_3$.
- Define $d_1 = \partial : E^q_{p,1} \to E^q_{p-1,1}$.

$$E^p_{p,2} = \frac{\ker d_1}{\operatorname{img} d_1}.$$

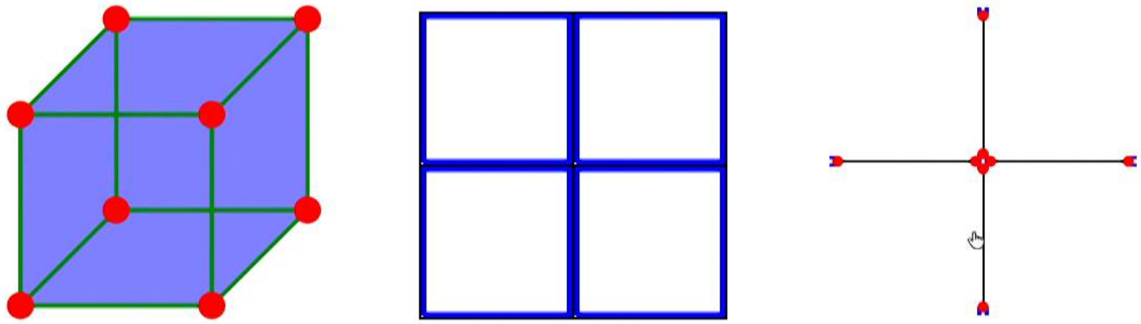
Third Page



- Define $d_2 : E^q_{p,2} \to E^{q-1}_{p-2,2}$:
  1. Start from $\hat{\omega}_p$.
  2. Find the connectors $\hat{\omega}_{p-1}$, s.t. $d\hat{\omega}_{p-1} = \partial\hat{\omega}_p$.
  3. Compute $d_2\hat{\omega}_p = \partial\hat{\omega}_{p-1}$.
- No-open-edge condition: $d_2\hat{\omega}_p \sim 0$.
- Bubbling equivalence: $\hat{\omega}_p \to \hat{\omega}_p + d_2\hat{\omega}_{p+2}$.

$$E^p_{p,3} = \frac{\ker d_2}{\mathrm{img}\, d_2}.$$

Third Page



- Define $d_2 : E^q_{p,2} \to E^{q-1}_{p-2,2}$:
  1. Start from $\hat{\omega}_p$.
  2. Find the connectors $\hat{\omega}_{p-1}$, s.t. $d\hat{\omega}_{p-1} = \partial\hat{\omega}_p$.
  3. Compute $d_2\hat{\omega}_p = \partial\hat{\omega}_{p-1}$.
- No-open-edge condition: $d_2\hat{\omega}_p \sim 0$.
- Bubbling equivalence: $\hat{\omega}_p \to \hat{\omega}_p + d_2\hat{\omega}_{p+2}$.

$$E^p_{p,3} = \frac{\ker d_2}{\operatorname{img} d_2}.$$

Final step: the group extension problem

- Assume we have computed the classification $E^p_{p,\infty}$.
- The classification may not be simply $E^3_{3,\infty} \oplus E^2_{2,\infty} \oplus E^1_{1,\infty} \oplus E^0_{0,\infty}$.
- Example: consider two $E^2_{2,\infty}$ blocks

$$\hat{\omega}_{2+} = \hat{\omega}_2 + \hat{\omega}_1 + \hat{\omega}_0;$$
$$\hat{\omega}'_{2+} = \hat{\omega}'_2 + \hat{\omega}'_1 + \hat{\omega}'_0.$$

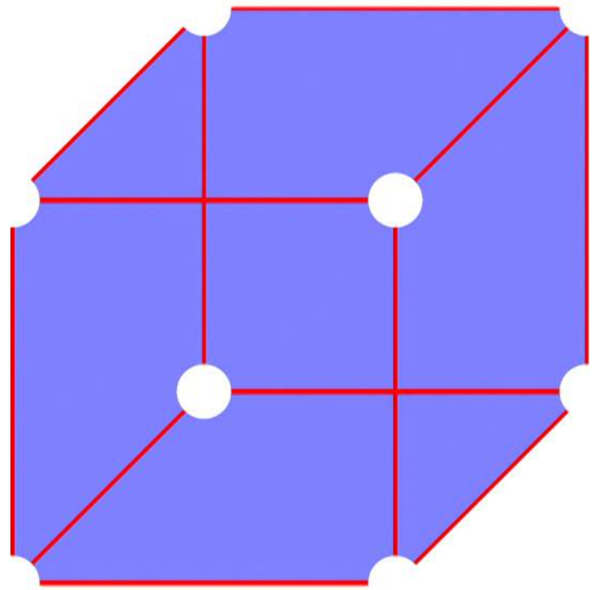
- If the sum is trivial in $E^2_{2,\infty}$: $\hat{\omega}_2 + \hat{\omega}'_2 \simeq 0$.
- The subleading term may be nontrivial: $\hat{\omega}_1 + \hat{\omega}'_1$. So this is a nontrivial term in $E^1_{1,\infty}$.
- Need to compute the group-extension problem: can be done if we know all $\hat{\omega}_i$ explicitly.

High-Order No-Open-Edge Conditions

- Consider $\hat{\omega}_{2+} = \hat{\omega}_2 + \hat{\omega}_1 + \hat{\omega}_0$.
- Cocycle condition: $(d - \partial)\hat{\omega} = 0$:

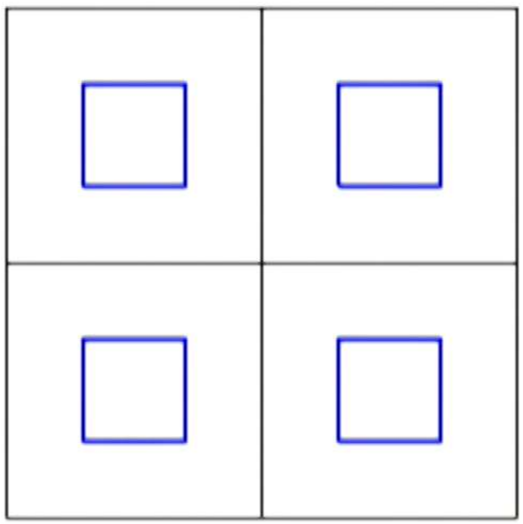
$$d\hat{\omega}_2 = 0;$$
$$d\hat{\omega}_1 = \partial\hat{\omega}_2 \simeq 0;$$
$$d\hat{\omega}_0 = \partial\hat{\omega}_1 \simeq 0.$$

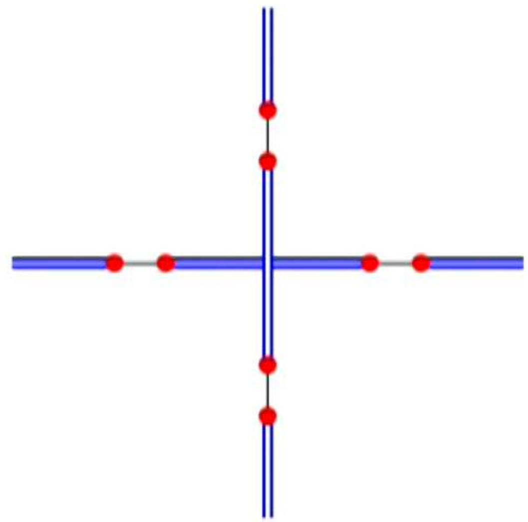
1. Choose a cocycle for each 2-cell σ.
   Check $\partial\hat{\omega}_2(\tau) \simeq 0$ for each 1-cell τ.
   1st-page no-open-edge condition.
2. Choose a cochain $\hat{\omega}_1$ for each τ.
   Check $\partial\hat{\omega}_1(\lambda) \simeq 0$ for each 0-cell λ.
   2nd-page no-open-edge condition.
3. Choose a cochain $\hat{\omega}_0$ for each λ.



High-Order Bubbling Equivalences

- There is a similar algorithm for the bubbling equivalence.
- Assign 1d SPT states ($d\hat{\mu}_1 = 0$) to 3-cells: $\partial\hat{\mu}_2$ trivializes $\hat{\omega}_1$ building blocks.
- If $\partial\hat{\mu}_2 \simeq 0$, choose $d\hat{\mu}_1 + \partial\hat{\mu}_2 = 0$, then $\partial\hat{\mu}_1$ trivializes $\hat{\omega}_0$ building blocks.
- ……





A spectral sequence

We compute in the following sequence:

1. 0st-page no-open-edge condition + 0st-page bubbling equivalence.

$$E^p_{p,1} = \frac{\ker d_0}{\text{img } d_0}.$$

2. 1st-page no-open-edge condition + 1st-page bubbling equivalence.

$$E^p_{p,2} = \frac{\ker d_1}{\text{img } d_1}.$$

3. 2nd-page no-open-edge condition + 2st-page bubbling equivalence.

$$E^p_{p,3} = \frac{\ker d_2}{\text{img } d_2}.$$

$$E^p_{p,1} \supseteq E^p_{p,2} \supseteq E^p_{p,3} \supseteq \cdots = E^p_{p,\infty}.$$

First Page

$$\boxed{\hat{\omega}(\sigma)}$$

- The first page concerns only the building blocks.
- $d\hat{\omega}(\sigma) = 0$.
- $d\hat{\mu}(\sigma) \sim 0$.
- Define $d_0 = d$.

$$E^p_{p,1} = \frac{\ker d_0}{\text{img } d_0} = \bigoplus_{\sigma \in Y_p/G} H^p(G_\sigma, U(1)_T).$$

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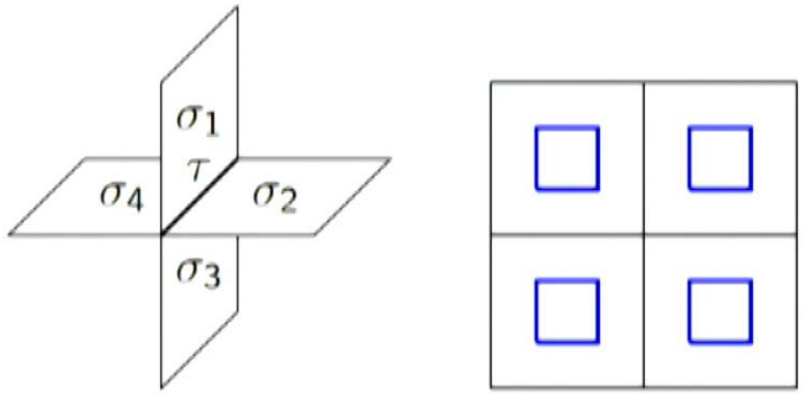
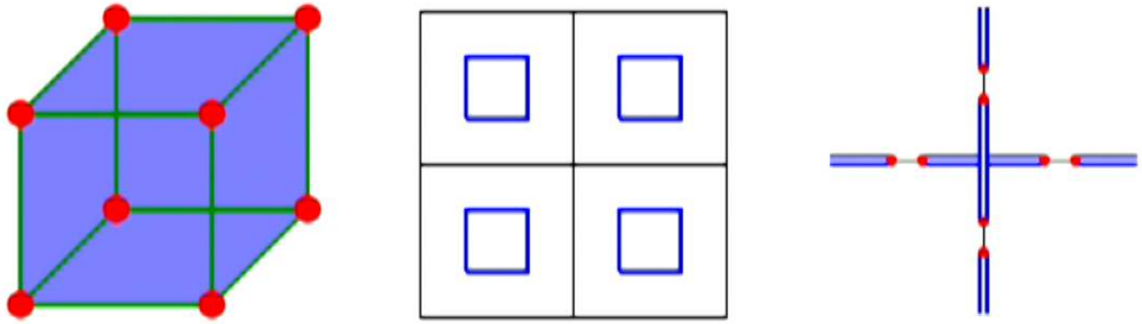
Second Page



- No-open-edge condition: $\partial\hat{\omega}_2 \sim 0$.
- Bubbling equivalence: $\hat{\omega}_2 \to \hat{\omega}_2 + \partial\hat{\mu}_3$.
- Define $d_1 = \partial : E^q_{p,1} \to E^q_{p-1,1}$.

$$E^p_{p,2} = \frac{\ker d_1}{\mathrm{img}\ d_1}.$$

Third Page



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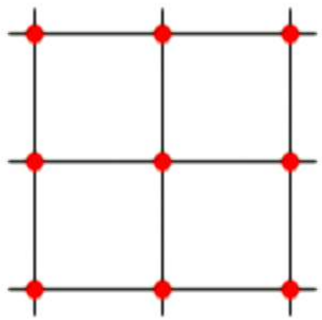
Some simplifications

- If $G = SG \times G_0$, bosonic SPTs $\Phi^d(G) = H^{d+1}[G, \mathrm{U}(1)_T]$.
- If we ignore $E^0_{0,\infty}$: not treating the atomic insulators as nontrivial SPTs.
- Only need to check things on the second page:
  1. $E^2_{2,2} = E^2_{2,\infty}$: assign 2d SPTs to 2-cells, and check-anomaly vanishing on 1-cells. Coboundaries are assigning 2d SPTs to 3-cells.
  2. $E^1_{1,2} = E^1_{1,\infty}$: assign 1d SPTs to 1-cells, and check-anomaly vanishing on 0-cells. Coboundaries are assigning 1d SPTs to 2-cells.
- Trivial group extension: $E^2_{2,2} \oplus E^1_{1,2}$.

An example of 2nd order computation

Adapted from Xu Yang, Shenghan Jiang, Ashvin Vishwanath and Ying Ran, arXiv:1705.09298.

- Consider a magnetic translation symmetry $T_x T_y T_x^{-1} T_y^{-1} = X$, and $G_0 = \mathbb{Z}_2^X \times \mathbb{Z}_2^T$.
- $H^3[G_0, \mathrm{U}(1)_T] = \mathbb{Z}_2 \oplus \mathbb{Z}_2$: the one protected by both $\mathbb{Z}_2^X$ and $\mathbb{Z}_2^T$ is not compatible with the magnetic translation symmetry.
- Try to decorate $\omega \in H^3[G_0, \mathrm{U}(1)_T]$ to all 2-cells: the 1-cells can be gapped out, but the 0-cells cannot.
- There must be a $T^2 = -1$ Kramers doublet at each 0-cell.



Mathematical Proof: Real-Space Construction is Complete

- Equivariant group cohomology:

$$H^{d+1}[G, \mathrm{U}(1)_{PT}]] \simeq H_G^{d+1}[X, \mathrm{U}(1)_{PT}].$$

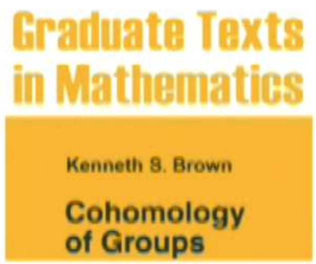
  Here $X \sim$ pt is a (non-free) G-complex. See Thorngren and Else, PRX (2018).

- There is a spectral sequence:

$$E_1^{pq} = \bigoplus_{\sigma \in X_p/G} H^q[G_\sigma, \mathrm{U}(1)_T] \Rightarrow H_G^{p+q}[X, \mathrm{U}(1)_{PT}] \simeq H^{p+q}[G, \mathrm{U}(1)_{PT}]].$$

  See Kenneth S. Brown's book, Chapter VII.

- The topological space Y we used is the Poincaré dual of X: $E_r^{pq} \simeq E_{d-p,r}^{q-1}$



Shapiro's lemma now yields

(7.6) $$H_q(G, C_p(X, M)) \approx \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma, M_\sigma),$$

so that 5.3 takes the form:

(7.7) $$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma, M_\sigma) \Rightarrow H^G_{p+q}(X, M).$$

Suppose, for example, that the G-action is free, so that each $G_\sigma = \{1\}$, and assume for simplicity that $M = \mathbb{Z}$. The spectral sequence then collapses at

Summary

- We develop a way to sysmetically construct space-group SPTs.
- Check cocycle conditions and coboundary equivalences order-by-order.
- For $G = SG \times G_0$, second page is enought.
- Examples beyond simple layered construction.
- Examples where second-page calculation is not enough.

Outlooks

- Outlook: cut-and-glue other things:
  1. Fermionic space-group SPTs.
  2. 2D space-group SETs.
- A way to simplify computations for arbitrary symmetry groups. See M Cheng and C Wang, arXiv:1810.12308.
- Another work about spectral sequence: Ken Shiozaki, Masatoshi Sato and Kiyonori Gomi, arXiv:1802.06694 (momentum-space analysis).
- An independent work: Else and Thorngren, arXiv:1810.10539.
- Related work: K. Shiozaki, C. Zhaoxi Xiong, and K. Gomi, arXiv:1810.00801.