Title: PSI 2019/2020 - Quantum Theory (Dupuis) - Lecture 14

Speakers: MaÃ¯tÃ© Dupuis

Collection: PSI 2019/2020 - Quantum Theory (Branczyk/Dupuis)

Date: September 20, 2019 - 9:00 AM

URL: http://pirsa.org/19090056

Time-evolution operator

$$U(t) = e^{-\frac{i}{\hbar} H t}$$

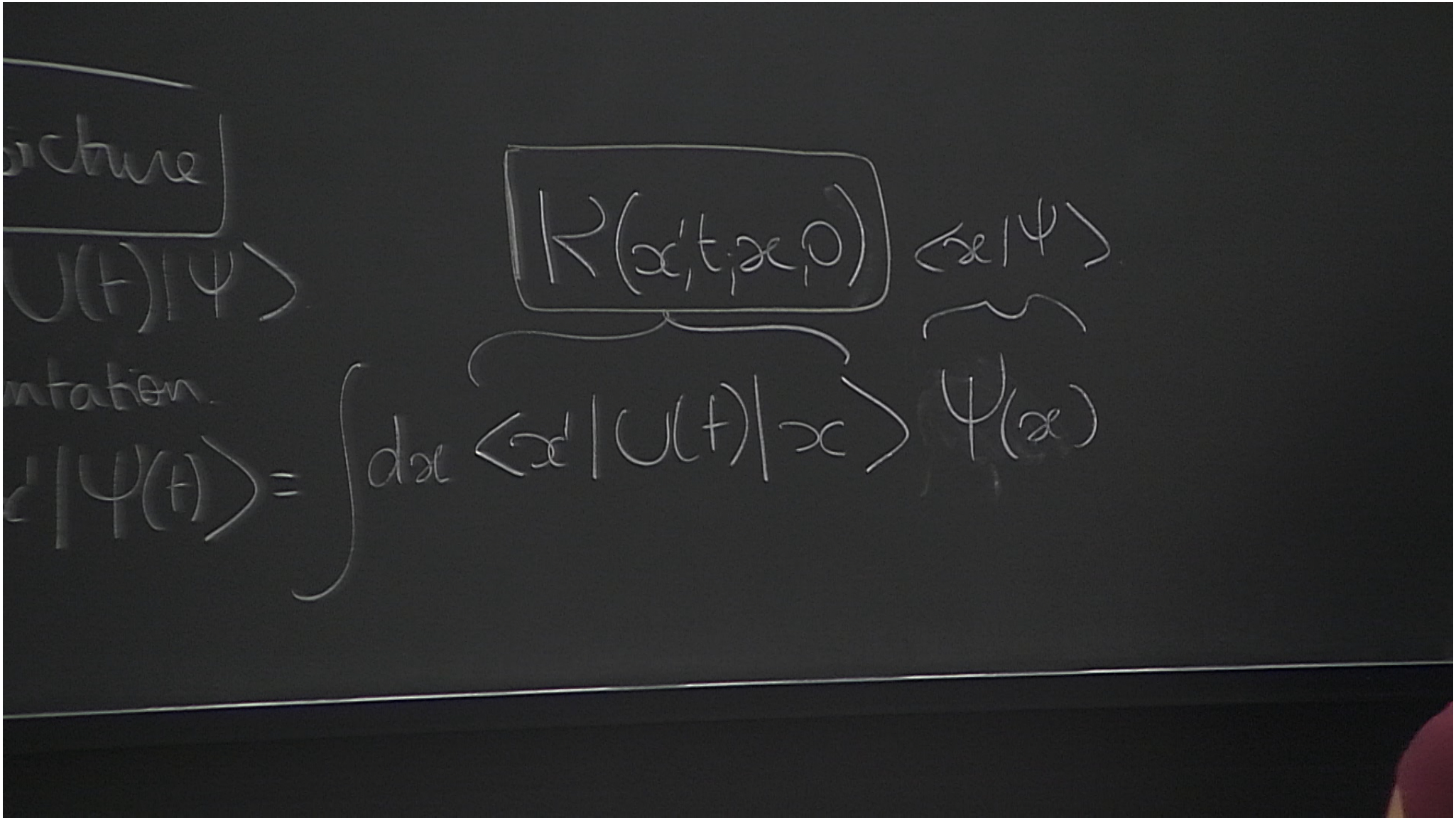
$$U(t) = U(t, t_1) U(t_1)$$

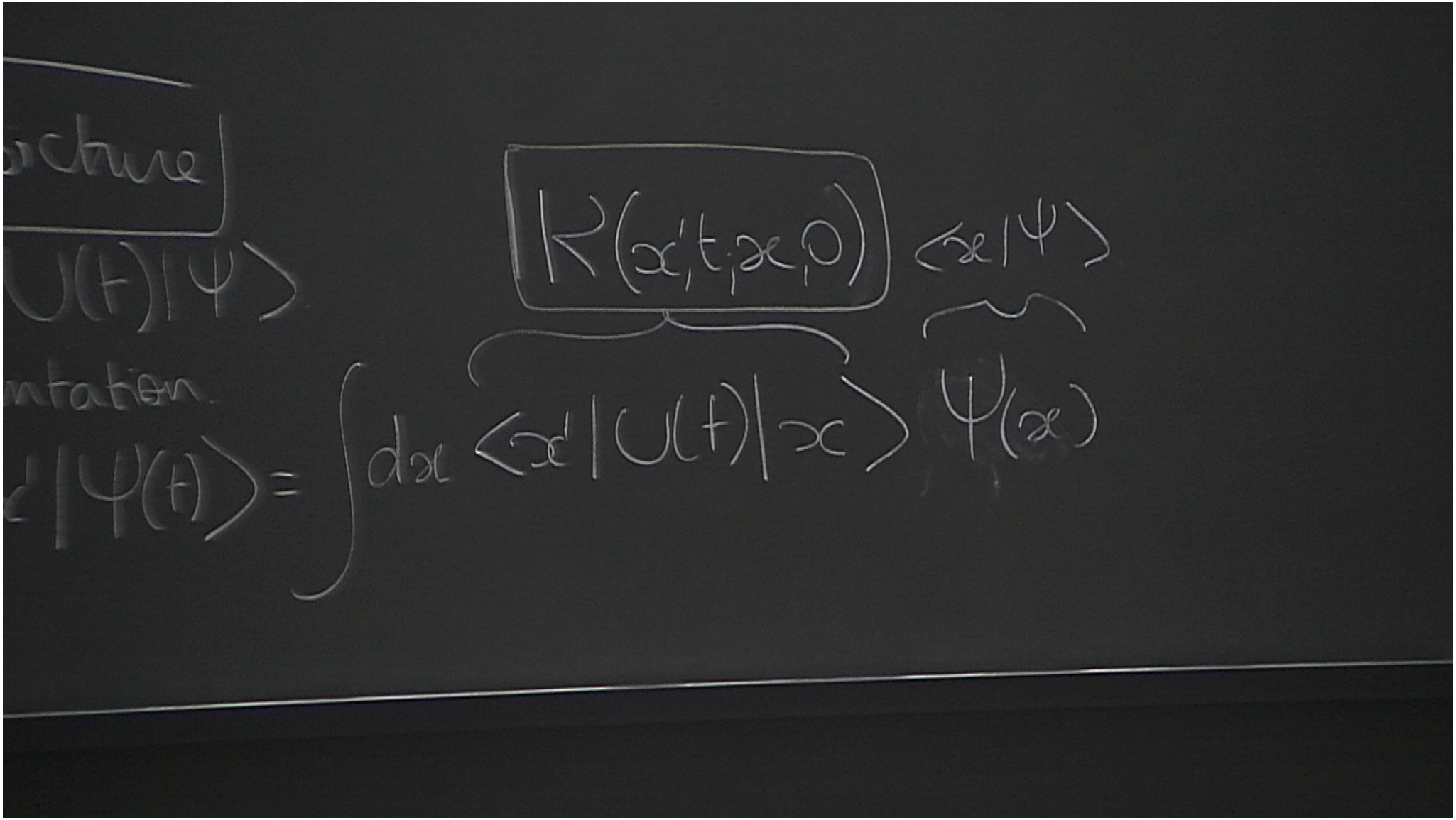
Schrödinger picture

$$|\psi(t)\rangle = U(t)|\psi\rangle$$

position representation.

$$\psi(x',t) = \langle x'|\psi(t)\rangle = \int dx$$


ɔicture
U(t)|ψ⟩
ntation.
d|ψ(t)⟩ = ∫dx ⟨x'|U(t)|x⟩ ψ(x)
K(x',t;x,0)
⟨x|ψ⟩


ture
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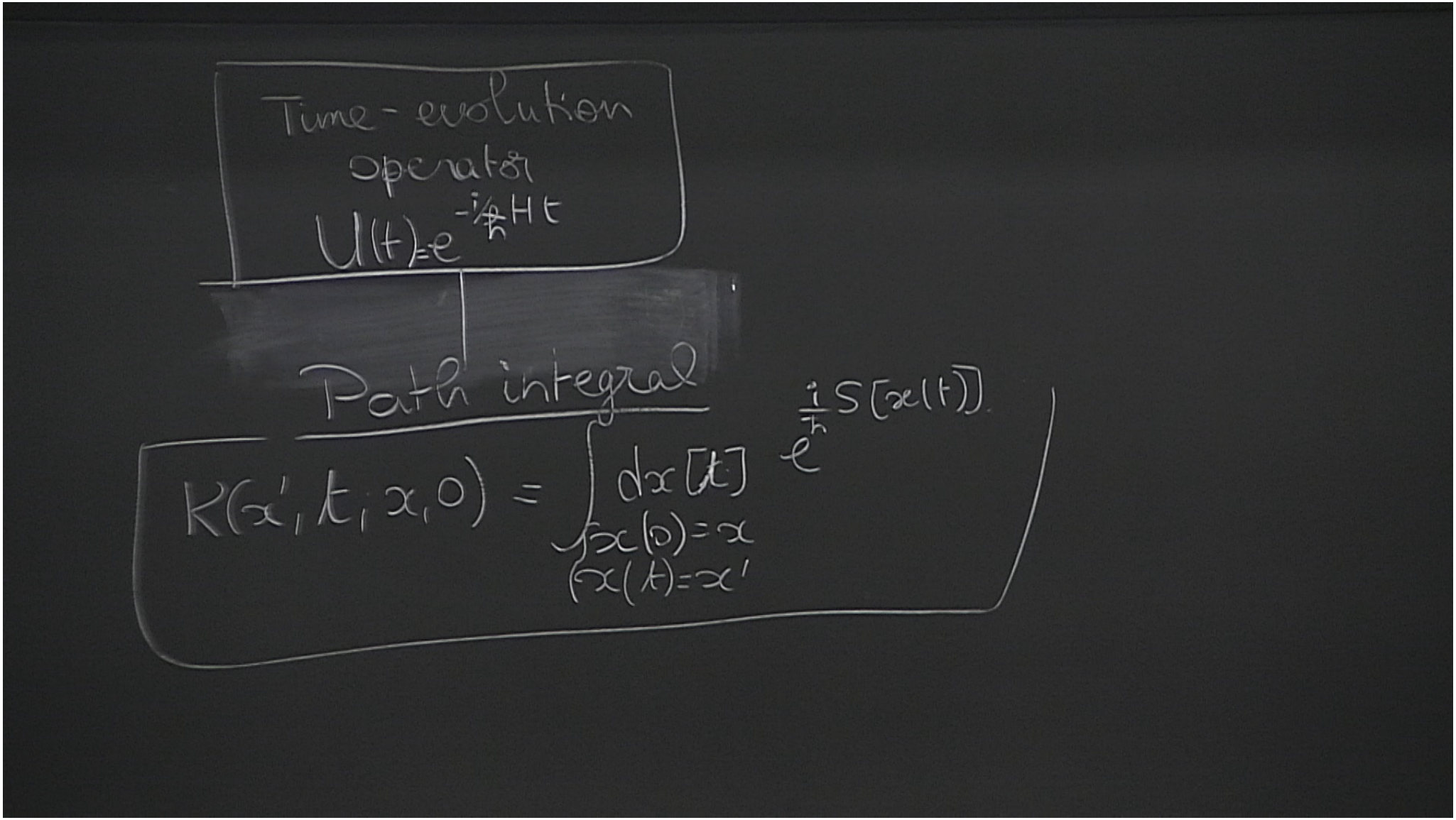
Heisenberg picture

$$\hat{A}(t) = U^{\dagger}(t)\,\hat{A}\,U(t)$$

$$\hat{x}(t) \longrightarrow |x,t\rangle$$

$$K(x',t;x,0) = \langle x'| e^{-iHt/\hbar} |x\rangle$$

$$\boxed{K(x',t,x,0) = \langle x,t | x,0\rangle}$$


Time-evolution operator
$U(t) = e^{-\frac{i}{\hbar} H t}$
Path integral
$K(x', t; x, 0) = \int_{\substack{x(0)=x \\ x(t)=x'}} dx[t]\, e^{\frac{i}{\hbar} S[x(t)]}$

Heisenberg picture

$$\hat{A}(t) = U^{\dagger}(t)\,\hat{A}\,U(t)$$

$$\hat{x}(t) \longrightarrow |x,t\rangle$$

$$K(x',t;x,0) = \langle x'|\, e^{-iHt/\hbar}\,|x\rangle$$

$$K(x',t;x,0) = \langle x',t|x,0\rangle$$

Path integral

$$K(x', t; x, 0) = \int_{\substack{x(0)=x \\ x(t)=x'}} dx[t]\; e^{\frac{i}{\hbar} S[x(t)]}$$

Wick rotation
$t \to -i\tau$

precise meaning

t/N
$\lim N \to \infty$

ex: Free particle

$\left\{ \begin{array}{l} \epsilon/N \\ \lim N \to \infty \end{array} \right.$

Ex: Free particle

semi-classical expansion $\downarrow$ x_c?

$$K = \underbrace{e^{-S_E[x_c]/\hbar}}_{} \underbrace{\int\int_{u(\tau')=u(\tau'')=0} du(\tau)\; e^{-\frac{1}{2}\int\int d\tau_1 d\tau_2\, u(\tau_1) A(\tau_1,\tau_2) u(\tau_2)}}_{}$$

$$\int_{u(\tau)=u(t')=0} \mathcal{D}u(\tau)\, e^{\frac{i}{\hbar}\int_{t'}^{t} d\tau \left[\frac{1}{2}m\dot{u}^2 + \frac{1}{3!}V'''(x_{cl})u^3\right]}$$

$$A(\tau_1,\tau_2) = \left[-m\partial_{\tau_1}^2 + \partial_x^2 V(x)\right]\delta(\tau_1-\tau_2)$$

↳ ex to compute exactly K as a PI for H.O.

$$\frac{1}{2}\iint d\tau_1 d\tau_2\, M(\tau_1) A(\tau_1,\tau_2) M(\tau_2) \left(1+\frac{\sqrt{\hbar}}{3!}\int d\tau\, M^3(\tau)\frac{\partial^3 V}{\partial x^3}+O(\hbar)\right)$$

$$\tau_2) = \left[-m\partial_{\tau_1}^2 + \partial_x^2 V(x)\right]\delta(\tau_1-\tau_2)$$

compute exactly K as a PI for H.O.

HO + source term: A is unchanged

Free particle

semi-classical expansion ↓ x_c ?

$$K = \underbrace{e^{-S_E[x_c]/\hbar}}_{\text{compute } x_c} \int_{u(\tau)=u(\tau')=0} du(\tau)\, e^{-\frac{1}{2}\iint d\tau_1 d\tau_2\, u(\tau_1) A(\tau_1,\tau_2) u(\tau_2)}$$

$$A(\tau_1,\tau_2) = \left[-m\partial_{\tau_1}^2 + \partial_x^2 V(x)\right]\delta(\tau_1-\tau_2)$$

↳ ex: to compute exactly K as a

HO +

$$Z[\beta] = \mathrm{Tr}\, U(\beta/2, -\beta/2) = \int_{x(\beta/2)=x(-\beta/2)} dx(\tau)\, e^{-S_E(x)} = \sum_n e^{-\beta E_n}$$

$$K[0,\beta,0,0] = \int_{\substack{x'=0 \\ x'=0}} dx[\tau]\, e^{-S_E(x)} = \sum_n e^{-\beta E_n} \psi_n^*(0)\psi_n(0)$$

$$E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \log \left| \frac{Z}{K} \right.$$

$$\frac{-\beta E_0}{\beta} + \frac{\log \psi_0^*(0)\psi_0(0)}{\beta}$$

energy
spectrum?
ground state energy E_0

$$Z[\beta] = \mathrm{Tr}\, U(\beta/2, -\beta/2) = \int_{x(\beta/2)=x(-\beta/2)} dx(\tau)\, e^{-S_E(x)} = \sum_n e^{-\beta E_n}$$

$$K[0,\beta,0,0] = \int_{\substack{x'=0\\ x'=0}} dx(\tau)\, e^{-S_E(x)} = \sum_n e^{-\beta E_n} \psi_n^*(0)\psi_n(0)$$

$$E_0 = -\lim_{\beta\to\infty}\left(\frac{1}{\beta}\log\left|\frac{Z}{K}\right.\right)$$

- Perturbation theory: $H = H_0 + \lambda H_I$

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$$K[0,\beta,0,0] = \int_{\substack{x'=0 \\ x''=0}} dx[\tau]\, e^{-S_E(x)} = \sum_n e^{-\beta E_n} \psi_n^*(0)\psi_n(0)$$

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- Perturbation theory: $H = H_0 + \lambda H_I$

$$Z^\lambda[\beta] = e^{-S_I\left[\frac{\delta}{\delta J}\right]} Z^0[\beta, J]\Big|_{J=0}$$

$$K^\lambda[\beta] = e^{-S_I\left[\frac{\delta}{\delta J}\right]} K^0[\beta, J]\Big|_{J=0}$$

spectrum

ground state energy E_0

$S[x(t)]$

$x(\beta/2) = x(-\beta/2)$

$$K[0,\beta,0,0] = \int_{\substack{x'=0\\ \dot{x}'=0}} dx\, e^{[\tau]}\, e^{-S_E(x)} = \sum_n e^{-\beta E_n} \psi_n^*(0)\psi_n(0)$$

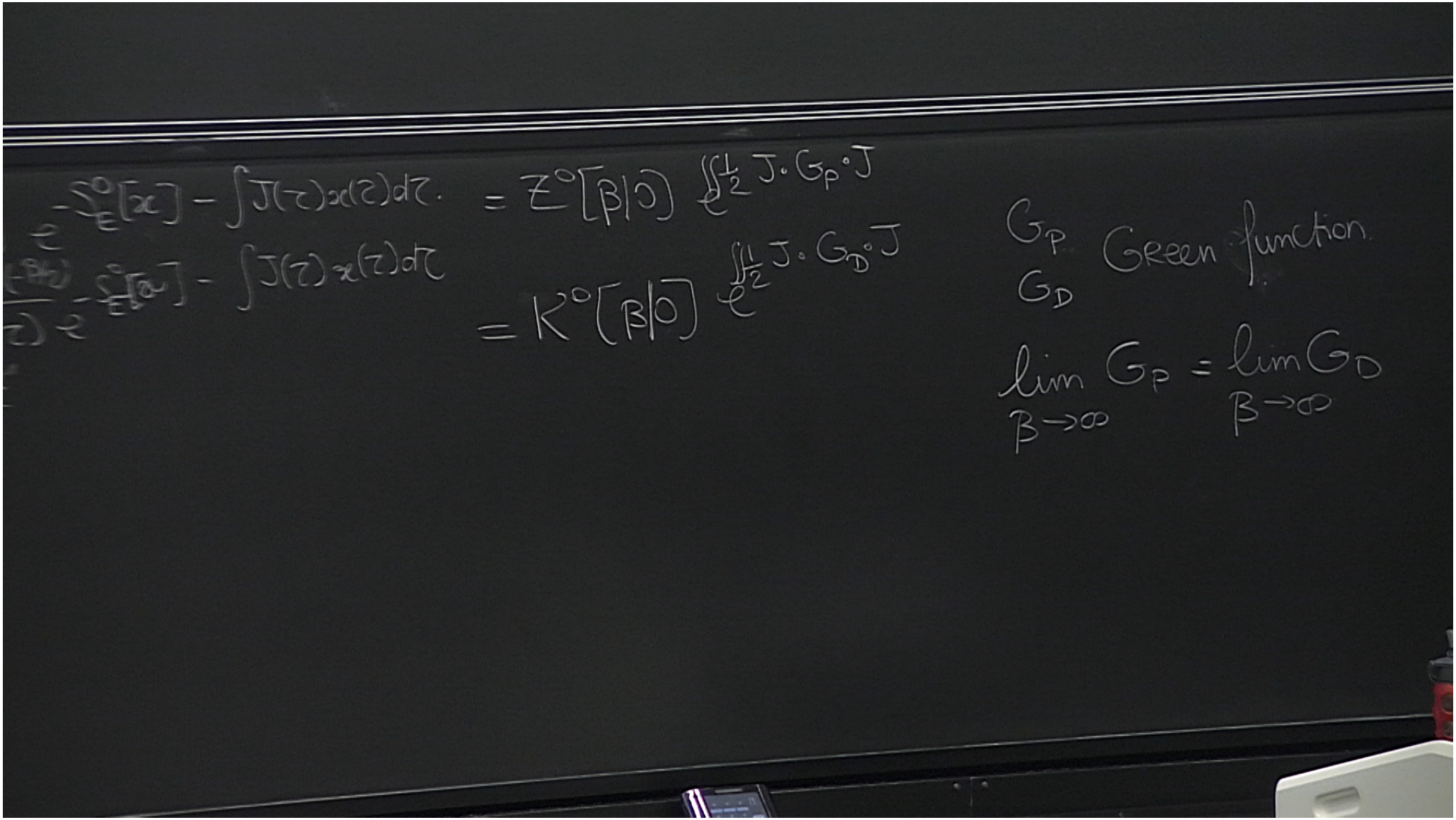
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$$-\frac{1}{2}\iint d\tau_1 d\tau_2\, M(\tau_1) A(\tau_1,\tau_2) M(\tau_2) \left(1 + \frac{\sqrt{\hbar}}{3!}\int d\tau\, M^3(\tau)\frac{\partial^3 V}{\partial x^3} + O(\hbar)\right)$$


$-S_E^0[x] - \int J(\tau)x(\tau)d\tau$ $= Z^0[\beta|0]\, e^{\frac{1}{2}\int\int J \cdot G_P \cdot J}$
$-S_E^0[x] - \int J(\tau)x(\tau)d\tau$ $= K^0[\beta|0]\, e^{\frac{1}{2}\int\int J \cdot G_D \cdot J}$
G_P Green function
G_D
$\lim_{\beta\to\infty} G_P = \lim_{\beta\to\infty} G_D$

Vacuum

$$= \lim_{\beta \to \infty} \frac{\int dx[\tau]\, x(\tau_1) \cdots x(\tau_n) e^{-\frac{S_E}{\hbar}}}{\int dx[\tau]\, e^{-S_E(x)}}$$

Time evolution operator

$$U(t) = e^{-\frac{i}{\hbar} H t}$$

Path int

Generating functional

$$\mathcal{Z}[J] = \frac{\langle 0|0\rangle_J}{\langle 0|0\rangle_{J=0}} = \frac{\int dx[\tau]\, e^{-\frac{S(x) - \int_{-\infty}^{+\infty} J(\tau) x(\tau) d\tau}{\hbar}}}{\int dx[\tau]\, e^{-\frac{S(x)}{\hbar}}}$$

$$\mathcal{Z}[J]$$

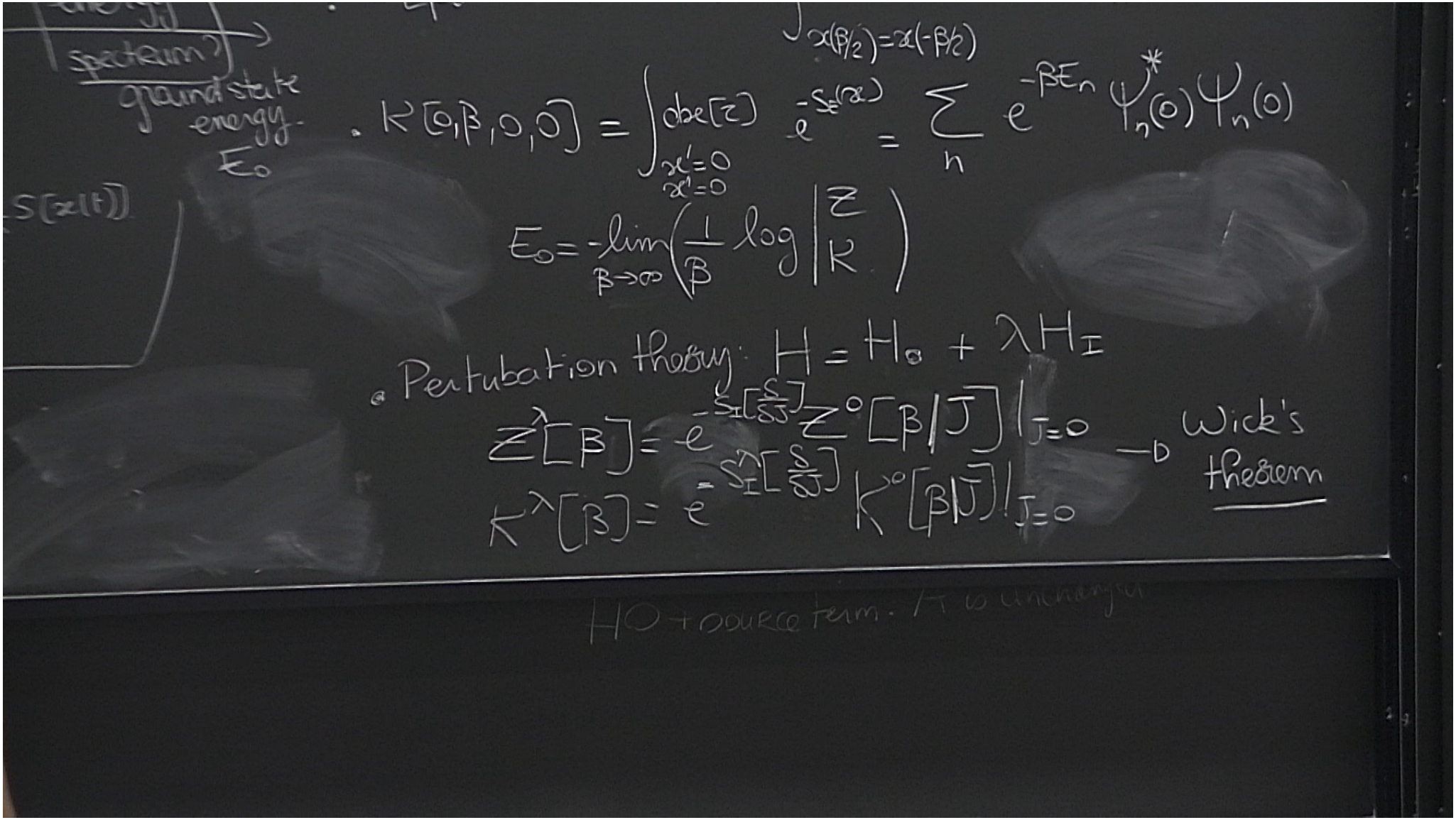
$K(x', t; x, 0) =$

precise meaning

t/N

$\lim N \to \infty$

ex: Free particle


spectrum?
ground state energy
E_0
$S[x(t)]$
$J_{x(\beta/2)=x(-\beta/2)}$
$K[0,\beta,0,0] = \int_{x'=0}^{x'=0} dx\, e^{-S_E(x)} = \sum_n e^{-\beta E_n} \psi_n^*(0)\psi_n(0)$
$E_0 = -\lim_{\beta\to\infty}\left(\frac{1}{\beta}\log\left|\frac{Z}{K}\right.\right)$
Perturbation theory: $H = H_0 + \lambda H_I$
$Z^\lambda[\beta] = e^{-S_I[\frac{\delta}{\delta J}]} Z^0[\beta|J]\,|_{J=0}$
$K^\lambda[\beta] = e^{-S_I[\frac{\delta}{\delta J}]} K^0[\beta|J]\,|_{J=0}$
Wick's theorem
HO + source term. H is unchanged

Fermionic path integral

Bosons

$[p, x] = -i$

operators

$[p, x] = 0$

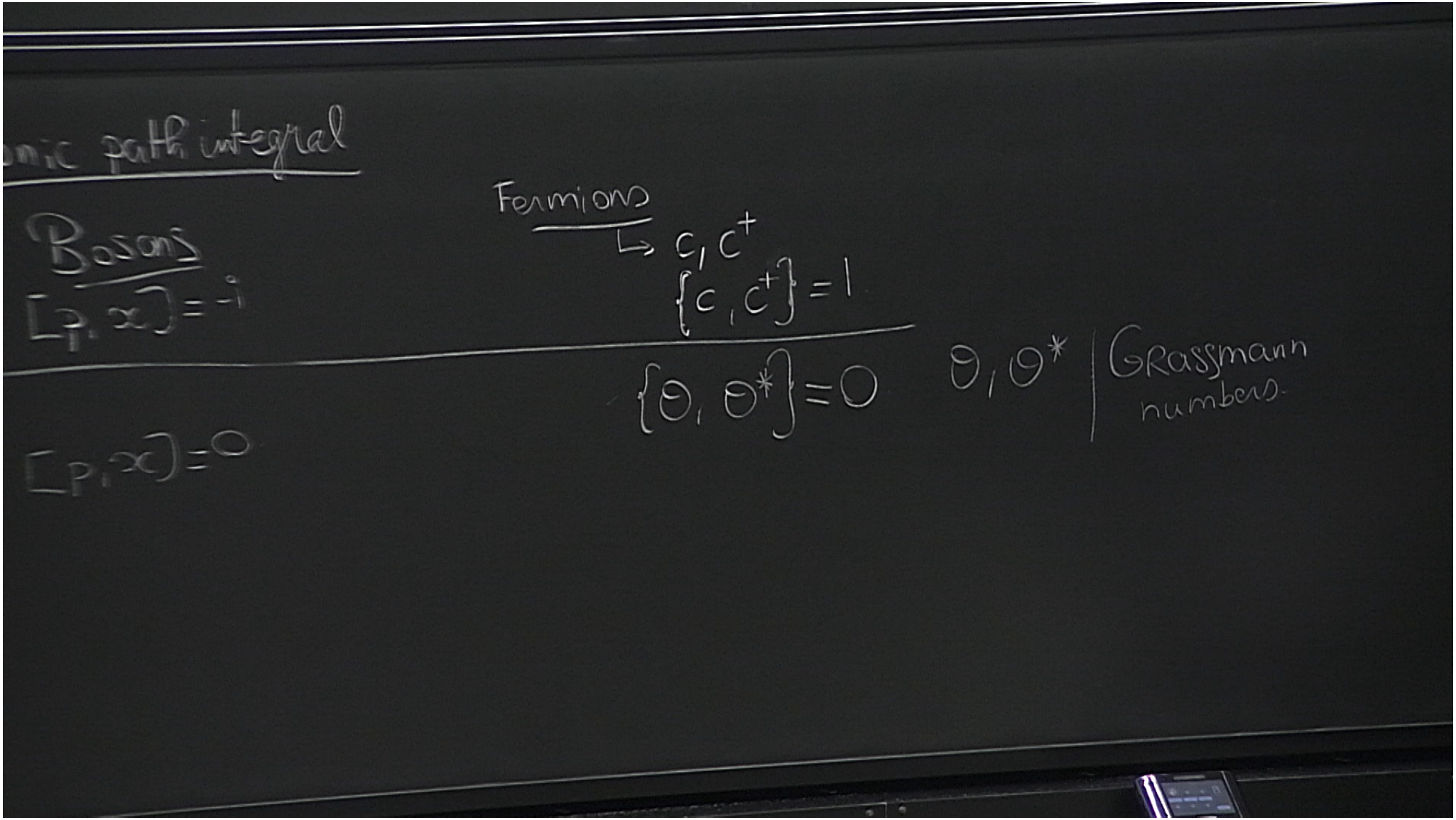
P. I

Fermions

↳ c, c^+

$\{c, c^+\} = 1$

$\{\theta, \theta^*\} = 0$ θ, θ^*


onic path integral
Bosons
$[p, x] = -i$
$[p, x] = 0$
Fermions
↳ $c, c^\dagger$
$\{c, c^\dagger\} = 1$
$\{\theta, \theta^*\} = 0$
θ, θ^* | Grassmann numbers.

One-state hamiltonian - ex of the fermionic H.O

$$H = \frac{1}{2}(c^\dagger c - c\, c^\dagger)\omega$$

$$\{c, c^\dagger\} = 1 \;,\; \{c, c\} = 0 \;,\; \{c^\dagger, c^\dagger\} = 0 \;,\; N = c^\dagger c$$

$\mathcal{H} = \mathrm{Span}\{|0\rangle, |1\rangle\}$

$c^\dagger|0\rangle = |1\rangle$, $c^\dagger|1\rangle = 0$

$c|0\rangle = 0$ $\quad c|1\rangle = |0\rangle$

$|f\rangle = |0\rangle f_0 + |1\rangle f_1 \qquad f_0, f \in \mathbb{C}$

$\{c, c^+\} = 1$, $\{c, c\} = 0$, $\{c^+, c^+\} = 0$, $N = c^+ c$

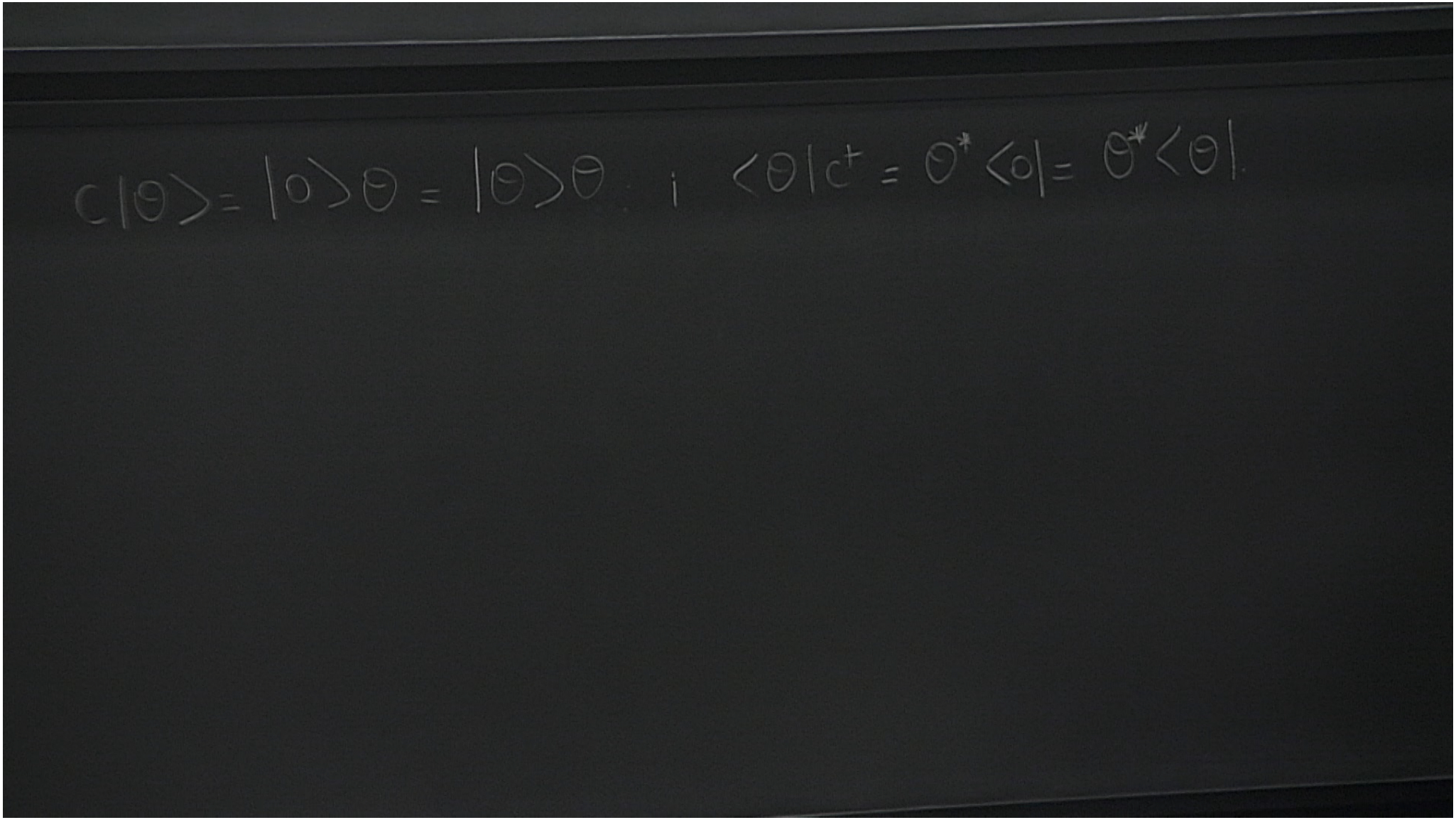
$\mathcal{H} = \mathrm{Span}\{|0\rangle, |1\rangle\}$ $c^+|0\rangle = |1\rangle$, $c^+|1\rangle = 0$

$c|0\rangle = 0$ $c|1\rangle = |0\rangle$

$|f\rangle = |0\rangle f_0 + |1\rangle f_1$ $f_0, f \in \mathbb{C}$.

$|\theta\rangle = |0\rangle + |1\rangle\theta$

$\langle\theta| = \langle 0| + \theta^* \langle 1|$ θ, θ^* | Grassmann numbers.


$C|\theta\rangle = |o\rangle\theta = |\theta\rangle\theta$; $\langle\theta|C^\dagger = \theta^*\langle o| = \theta^*\langle\theta|$.

assmann
umbers.

$\mathcal{H} = \mathrm{Span}\{|0\rangle, |1\rangle\}$

$C|0\rangle = 0 \qquad C|1\rangle = |0\rangle$

$|f\rangle = |0\rangle f_0 + |1\rangle f_1 \qquad f_0, f_1 \in \mathbb{C}$

$|\theta\rangle = |0\rangle + |1\rangle\theta$

$\langle\theta| = \langle 0| + \theta^*\langle 1|$

θ, θ^* | Grassmann numbers.

$\theta^*\langle\theta|$

Ex Prove $\langle\theta'|\theta\rangle = 1 + \theta'^*\theta = e^{\theta'^*\theta}$

$c|\theta\rangle = |0\rangle\theta = |\theta\rangle\theta$; $\langle\theta|c^\dagger = \theta^* \langle$

Path integral? $H_0 = \omega c^\dagger c$, $H_0^2 = \omega H_0$. ◁— check it!

① $U_0(t) = e^{-H_0 t} = \mathbb{1} - H_0 t + \omega H_0 \frac{t^2}{2} - \omega^2 H_0 \frac{t^3}{3!} + \cdots =$

② Matrix elements $\langle\theta|U_0(t)|\theta'\rangle = 1 + \theta^*\theta' e^{-\omega t} = e^{(\theta^*\theta' e^{-\omega t})}$

$|o\rangle\theta = |\theta\rangle\theta$; $\langle\theta|c^\dagger = \theta^*\langle o| = \theta^*\langle\theta|$ Ex

$c^\dagger c$, $H_o^2 = \omega H_o$ ◁— check it!

$$\mathbb{1} - H_o t + \omega H_o \frac{t^2}{2} - \omega^2 H_o \frac{t^3}{3!} + \cdots = \left[\omega \mathbb{1} + (e^{-\omega t} - 1) H_o\right]/\omega$$

$$\langle\theta|U_o(t)|\theta'\rangle = 1 + \theta^*\theta' e^{-\omega t} = e^{(\theta^*\theta' e^{-\omega t})}$$

$c|\theta\rangle = |0\rangle\theta = |\theta\rangle\theta$; $\langle\theta|c^\dagger = \theta^*\langle 0| = \theta^*\langle\theta|$

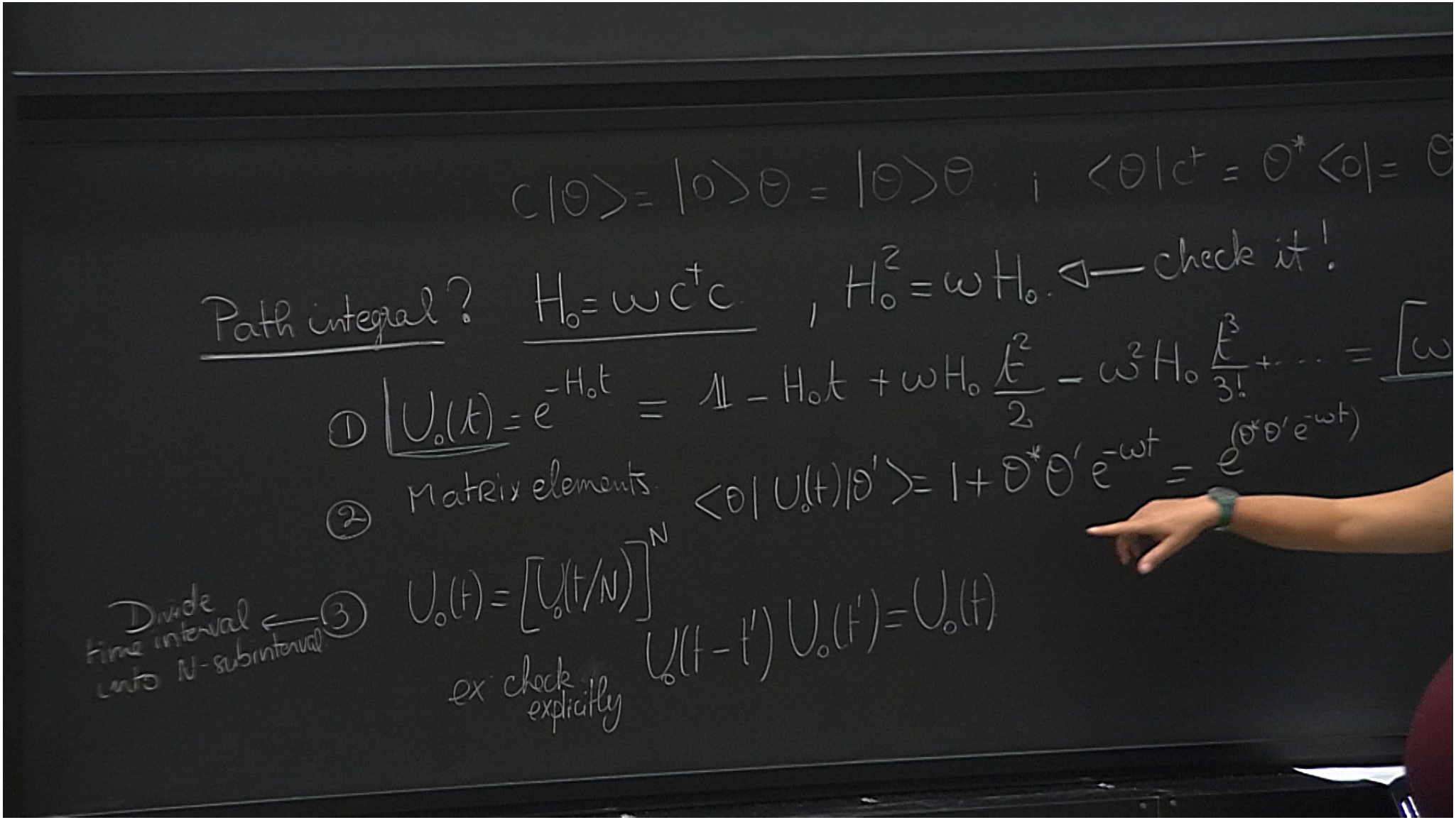
Path integral? $H_0 = \omega c^\dagger c$, $H_0^2 = \omega H_0$ ◁— check it!

① $U_0(t) = e^{-H_0 t} = \mathbb{1} - H_0 t + \omega H_0 \frac{t^2}{2} - \omega^2 H_0 \frac{t^3}{3!} + \cdots = [\omega \mathbb{1} +$

② Matrix elements $\langle\theta|U_0(t)|\theta'\rangle = 1 + \theta^*\theta' e^{-\omega t} = e^{(\theta^*\theta' e^{-\omega t})}$

③ $U_0(t) = [U_0(t/N)]^N$

ex check explicitly $U_0(t-t')\,U_0(t')$


$c|\theta\rangle = |0\rangle\theta = |\theta\rangle\theta$; $\langle\theta|c^\dagger = \theta^*\langle 0| =$
Path integral? $H_0 = \omega c^\dagger c$, $H_0^2 = \omega H_0$ ← check it!
① $U_0(t) = e^{-H_0 t} = \mathbb{1} - H_0 t + \omega H_0 \frac{t^2}{2} - \omega^2 H_0 \frac{t^3}{3!} + \cdots =$
② Matrix elements $\langle\theta|U_0(t)|\theta'\rangle = 1 + \theta^*\theta' e^{-\omega t} = e^{(\theta^*\theta' e^{-\omega t})}$
③ $U_0(t) = [U_0(t/N)]^N$
Divide time interval into N-subinterval
ex: check explicitly $U_0(t-t')\,U_0(t') = U_0(t)$

$N \to \infty \quad \frac{t}{N} \to 0 \quad \langle \theta | U_0(\frac{t}{N}) | \theta' \rangle = 1 + \theta^* \theta' \left(1 -$

Introduce resolution of identity.

$$\int d\theta^* d\theta \, |\theta\rangle\langle\theta| \, e^{-\theta^*\theta} = \mathbb{1} \overset{\text{(prove)}}{=} \int d\theta^* d\theta \left(|0\rangle +$$

$$= 1 + \theta^* \theta' \left(1 - \frac{\omega t}{N}\right) + O\left(\frac{t}{N}\right)^2 = \exp\left(\theta^* \theta' \left(1 - \frac{\omega t}{N}\right)\right) + O\left(\frac{t^2}{N}\right)$$

(prove)

$$= \int d\theta^* d\theta \left(|0\rangle + |1\rangle\theta\right)\left(\langle 0| + \theta^* \langle 1|\right)\left(1 - \theta^* \theta\right)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}.$$

recall

$$\int d\theta_1 d\theta_2 f(\theta_1, \theta$$

$$= \frac{\partial}{\partial \theta_1} \frac{\partial}{\partial \theta_2}$$

$N \to \infty \qquad \frac{t}{N} \to 0 \qquad \langle \theta | U_0(\frac{t}{N}) | \theta' \rangle = 1 + \theta^* \theta'$

④ Introduce resolution of identity.

$$\int d\theta^* d\theta \, |\theta\rangle\langle\theta| \, e^{-\theta^*\theta} = \mathbb{1} \overset{\text{(prove)}}{=} \int d\theta^* d\theta \, (|$$

$$\langle \theta'' | U(t_0) | \theta \rangle = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} d\theta_k^* \, d\theta_k \qquad = |\rangle\langle| +$$

$N \to \infty \quad \frac{t}{N} \to 0 \qquad \langle \theta | U_0(\frac{t}{N}) | \theta' \rangle = 1 + \theta^* \theta' \left(1 - \right.$

Introduce resolution of identity.

$$\int d\theta^* d\theta \, |\theta\rangle\langle\theta| \, e^{-\theta^*\theta} = \mathbb{1} \quad \Big| \overset{\text{(prove)}}{=} \int d\theta^* d\theta \left(|0\rangle + | \right.$$

$$\langle \theta'' | U(t_0) | \theta \rangle = \lim_{N \to \infty} \int \prod_{k=1}^{N-1} d\theta_k^* \, d\theta_k \, \exp\left[-\theta_N^* \theta_N - S_\varepsilon(\theta, \theta^*) \right] \Big| = \quad |0\rangle\langle 0| + |1\rangle\langle$$

with $S_\varepsilon(\theta, \theta^*) = \sum_{k=1}^{N} \theta_k^* (\theta_k - \theta_{k-1}) + \omega \varepsilon \theta_k^* \theta_{k-1}$, $\varepsilon = t/N$

$$1 + \theta^* \theta' \left(1 - \frac{\omega t}{N}\right) + O\left(\frac{t}{N}\right)^2 = \exp\left(\theta^* \theta' \left(1 - \omega \frac{t}{N}\right)\right) + O\left(\left(\frac{t}{N}\right)^2\right)$$

$$\int d\theta^* d\theta \left(|0\rangle + |1\rangle\theta\right)\left(\langle 0| + \theta^* \langle 1|\right)\left(1 - \theta^* \theta\right)$$

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$$

recall

$$\int d\theta_1 d\theta_2 \, f(\theta_1, \theta_2) = \frac{\partial}{\partial \theta_1} \frac{\partial}{\partial \theta_2} f(\theta_1,$$

θ_{k-1}, $\varepsilon = t/N$, $\theta_0 = \theta'$, $\theta_N = \theta''$

④ Introduce resolution of identity.

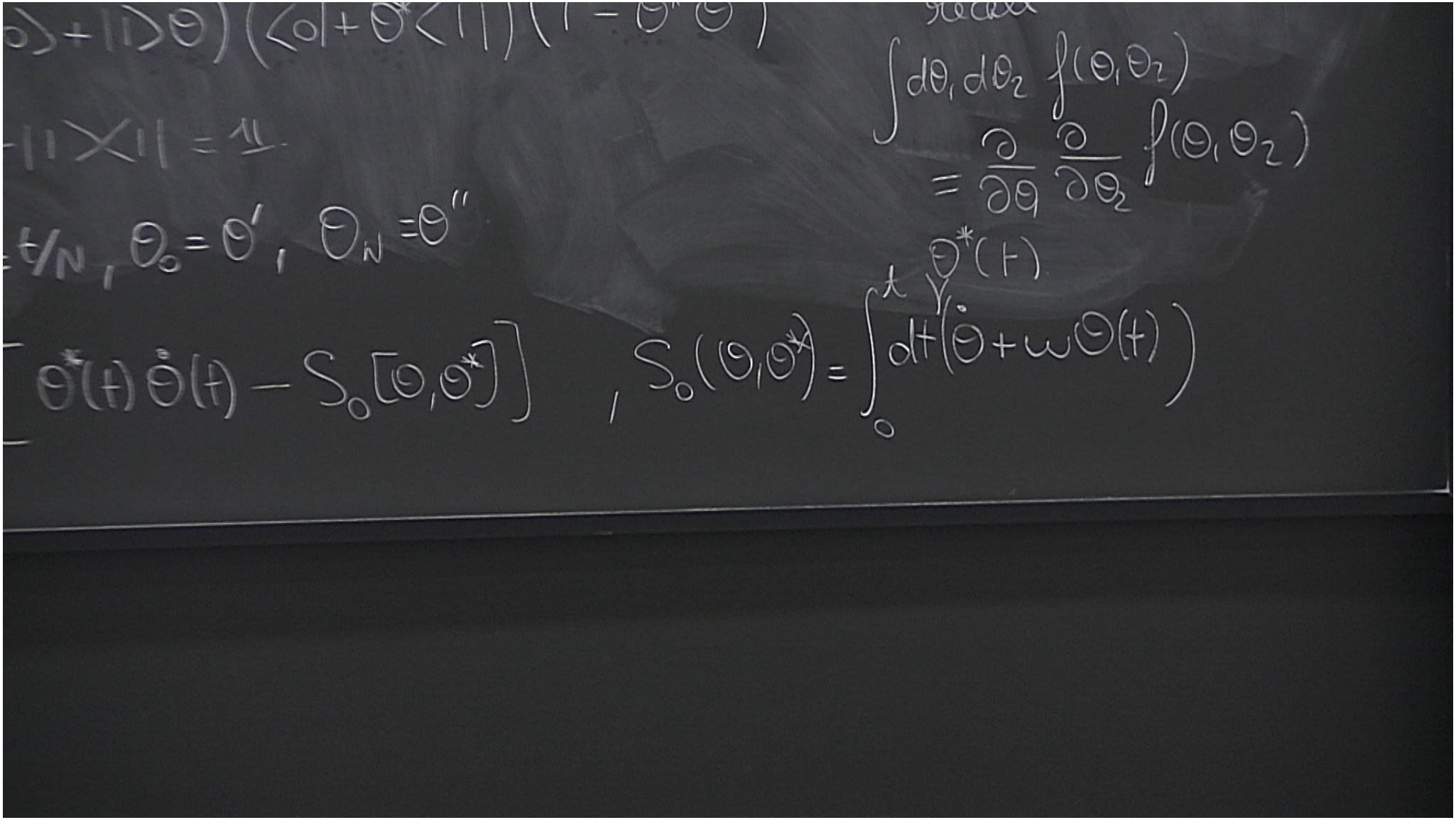
$$\int d\theta^* d\theta \, |\theta\rangle\langle\theta| \, e^{-\theta^*\theta} = \mathbb{1}$$

(prove) ↓ =

$$\langle\theta''|U(t_0)|\theta\rangle = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} d\theta_k^* \, d\theta_k \, \exp\left[-\theta_N^*\theta_N - S_\varepsilon(\theta,\theta^*)\right]$$

with $S_\varepsilon(\theta,\theta^*) = \sum_{k=1}^{N} \theta_k^* (\theta_k - \theta_{k-1}) + \omega\varepsilon\,\theta_k^*\,\theta_{k-1}$

⑤ $N\to\infty$ $\quad \langle\theta''|U_0(t)|\theta'\rangle = \int_{\substack{\theta(0)=\theta' \\ \theta(t)=\theta''}} [d\theta^*(t)\, d\theta(t)]$


$\int d\theta_1 d\theta_2 f(\theta_1, \theta_2)$
$= \frac{\partial}{\partial \theta_1} \frac{\partial}{\partial \theta_2} f(\theta_1, \theta_2)$
$\theta^*(t)$
$\theta^*(t)\dot{\theta}(t) - S_0[\theta, \theta^*]$
$S_0(\theta, \theta^*) = \int_0^t dt \left(\dot{\theta} + \omega\theta(t)\right)$

TODAY: . Summary
on bosonic path integral
. (Short!) introduction
on fermionic
path integral.