

Title: PSI 2019/2020 - Quantum Theory (Dupuis) - Lecture 13

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TODAY: Correlation function and generating functional

$$\int dx(t) x(t_1) \dots x(t_n) e^{\frac{i}{\hbar} S[x(t)]} \longleftarrow \textcircled{1} \langle 0 | T[\hat{x}(t_1) \dots \hat{x}(t_n)] | 0 \rangle \text{ as a PI}$$

$$S[x(t)] + \int dt x(t) J(t) \longleftarrow \textcircled{2} \text{ Generating functional } Z[J] \text{ as a PI}$$

$\longleftarrow \textcircled{3}$ Know how to compute $Z[J]$ exactly?

$\textcircled{4}$ Functional derivatives on $Z[J]$

$$\downarrow \\ \langle 0 | T[\hat{x}(t_1) \dots \hat{x}(t_n)] | 0 \rangle$$

Perturbation theory $\longleftarrow$ No! $H = H_0 + \lambda H_I$ yes!

① Path Integral representation for correlation function

• Heisenberg picture

$$\hat{\alpha}(t) |x, t\rangle = x |x, t\rangle.$$

↑ instantaneous eigenstate.

⚠ $\neq$ dynamical time dependence of wave functions

$$\begin{aligned} \cdot \underbrace{K(x', T, x, 0)} &= \langle x' | e^{-iHT} | x \rangle \\ &= \underbrace{\langle x', T | x, 0 \rangle} \end{aligned}$$

$$\hat{a}(t) |x, t\rangle = x |x, t\rangle.$$

instantaneous eigenstate.

$\triangle \neq$ dynamical time dependence
of wave functions. (Schrödinger)

$$\begin{aligned} \langle x', 0 | &= \langle x' | e^{-iHT} | x \rangle \\ &= \langle x', T | x, 0 \rangle \end{aligned}$$

$$K = \langle x', T | x, 0 \rangle = \int dx[t] e^{iS/\hbar} \quad (\text{real time}).$$

$$\bullet \langle x', T | T [\hat{x}(t_1) \hat{x}(t_2)] | x, 0 \rangle$$

Suppose $t_1 > t_2$

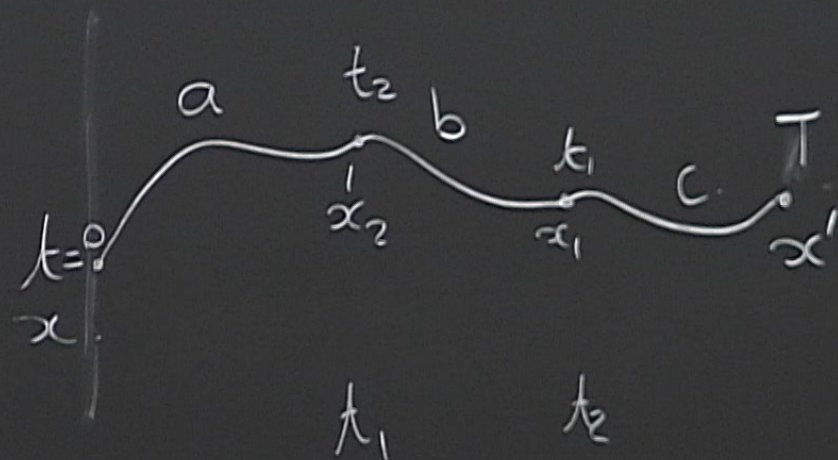
$$\langle x', T | \hat{x}(t_1) \hat{x}(t_2) | x, 0 \rangle = \int dx_1 dx_2 \langle x', T | x_1$$

$$\begin{aligned}
 & \overbrace{x_1 \langle x_1, t_1 |} \quad \overbrace{x_2 | x_2, t_2 \rangle} \\
 & \rangle = \int dx_1 dx_2 \langle x', T | x_1, t_1 \rangle \langle x_1, t_1 | \hat{x}(t_1) \hat{x}(t_2) | x_2, t_2 \rangle \langle x_2, t_2 \\
 & = \int dx_1 dx_2 x_1 x_2 \underbrace{\langle x', T | x_1, t_1 \rangle}_{\int_{x(T)=x'}^{x(t)=x_1} dx[t] e^{\frac{i}{\hbar} S}} \underbrace{\langle x_1, t_1 | x_2, t_2 \rangle}_{\int_{x(t_2)=x_2}^{x(t_1)=x_1} dx[t] e^{\frac{i}{\hbar} S}} \underbrace{\langle x_2, t_2 | x_1, 0 \rangle}_{\int_{x(0)=x}^{x(t_2)=x_2} dx[t] e^{\frac{i}{\hbar} S}} \\
 & = \int dx_1 dx_2 x_1 x_2 \int_{x(T)=x'}^{x(t)=x_1} dx[t] e^{\frac{i}{\hbar} S} \int_{x(t_2)=x_2}^{x(t_1)=x_1} dx[t] e^{\frac{i}{\hbar} S} \int_{x(0)=x}^{x(t_2)=x_2} dx[t] e^{\frac{i}{\hbar} S}
 \end{aligned}$$

$$\begin{aligned}
 &= \int dx_1 dx_2 \langle x', T | x_1, t_1 \rangle \langle x_1, t_1 | \hat{x}(t_1) \hat{x}(t_2) | x_2, t_2 \rangle \langle x_2, t_2 | x_1, 0 \rangle \\
 &= \int dx_1 dx_2 x_1 x_2 \langle x', T | x_1, t_1 \rangle \langle x_1, t_1 | x_2, t_2 \rangle \langle x_2, t_2 | x_1, 0 \rangle \\
 &= \int dx_1 dx_2 x_1 x_2 \int_{\substack{x(T)=x' \\ x(t_1)=x_1}} dx[t] e^{\frac{i}{\hbar} S} \int_{\substack{x(t_2)=x_2 \\ x(t_1)=x_1}} dx[t] e^{\frac{i}{\hbar} S} \int_{\substack{x(0)=x \\ x(t_2)=x_2}} dx[t] e^{\frac{i}{\hbar} S}
 \end{aligned}$$

check it! same result!

$t_2 |x, 0\rangle.$



$$|0\rangle = \int_{\substack{x(0)=x \\ x(T)=x'}} dx[t] \, x(t_1) x(t_2) e^{\frac{i}{\hbar} S} \quad (t_1 > t_2)$$

check it! $\rightarrow$ same result!

$$\Rightarrow \langle x', T | T[\hat{x}(t_1) \dots \hat{x}(t_n)] | x, 0 \rangle = \int_{\substack{x(T)=x' \\ x(0)=x}} dx[t] \, x(t_1) \dots x(t_n) e^{\frac{i}{\hbar} S}$$

$$② \quad S[x(t), J(t)] = \int_0^T dt \left[\frac{1}{2} m \dot{x}^2 - V(x) + x J \right] = S^J$$

$$K^J = \langle x', T | x, 0 \rangle_J = \int dx(t) e^{\frac{i}{\hbar} S^J}$$

Functional derivative?

$$\frac{\delta}{\delta J(t)} K^J = \int dx(t) \frac{i}{\hbar} x(t) e^{\frac{i}{\hbar} S^J}$$

$$\langle x', T | T[\hat{x}(t_1) \dots \hat{x}(t_n)] | x, 0 \rangle = (-i\hbar)^n \frac{\delta^n}{\delta J(t_1) \dots \delta J(t_n)}$$

$$\left. \int_{\substack{x(0)=x \\ x(T)=x'}} dx(t) e^{\frac{i}{\hbar} S^J} \right|_{J=0} K^J$$

Go back to point ① → how to extract vacuum contribution?

Euclidean n-point correlator:
(Wick rotation)

$$\int dx[z] x(z_1) \dots x(z_n) e^{-S_E[x(z)]/\hbar}$$

$$\left. \begin{array}{l} x(-\beta/2) = x' \\ x(\beta/2) = x \end{array} \right\}$$

=

$$\begin{aligned}
 |x, t\rangle &= e^{iHt/\hbar} |x\rangle, & \langle x, t| &= \langle x| e^{-iHt/\hbar} \\
 |x, \tau\rangle &= e^{iH\tau/\hbar} |x\rangle, & \langle x, \tau| &= \langle x| e^{-iH\tau/\hbar}
 \end{aligned}$$

of vacuum contribution:

$$dx[z] x(z_1) \dots x(z_n) e^{-S_E[x(z)]/\hbar}$$

$$\left\{ \begin{array}{l} x(-\beta/2) = x' \\ x(\beta/2) = x \end{array} \right.$$

$$= \langle x'', \beta/2 | e^{-\beta H/2\hbar} T[\hat{x}(z_1) \dots \hat{x}(z_n)] e^{-\beta H/2\hbar} | x' \rangle$$

$|x, t\rangle =$
 $|x', -\beta/2\rangle \sim |x, \tau\rangle =$

back to point ①

- Euclidean n-point correlator:
(Wick rotation)

$$\int dx[z] x(z_1) \dots x(z_n) e^{-\beta H[x]} \Big|_{\substack{x(-\beta/2)=x' \\ x(\beta/2)=x}} = \langle x'', \beta/2 | e^{-\beta H/2\hbar} | x' \rangle$$

• Reminder yesterday $\beta \rightarrow \infty, E_0$

$$K(0, \beta, 0, 0) = \langle x''=0 | e^{-\beta H/\hbar} | x'=0 \rangle$$

$$|x, t\rangle = e^{iHt/\hbar} |x\rangle, \quad \langle x, t|$$

$$|x', -\beta/2\rangle, |x, \tau\rangle = e^{H\tau/\hbar} |x\rangle, \quad \langle x, \tau|$$

$$e^{-\beta H/2\hbar} T[\hat{x}(\tau_1) \dots \hat{x}(\tau_n)] e^{-\beta H/2\hbar} |x'\rangle$$

$$|x=0\rangle_{\beta \rightarrow \infty} = \Psi_0(0) \Psi_0^*(0) e^{-\beta E_0/\hbar} \left[1 + O\left(e^{-\frac{\beta(E_1 - E_0)}{\hbar}}\right) \right]$$

$$(\tau_n) e^{-\beta H/2\hbar} |x'\rangle$$

$$\sum_k e^{-\beta E_k/2\hbar} \underbrace{\langle k|x'\rangle}_{\phi_k^*(x')}$$

$$(\tau_n)|0\rangle \phi_0(x'') \phi_0^*(x') e^{-\beta E_0/\hbar} \left[1 + O\left(e^{-\beta(E_1-E_0)/\hbar} \right) \right]$$

$$\begin{aligned}
& \lim_{\beta \rightarrow \infty} \underbrace{\langle x'' | e^{-\beta H / 2\hbar} T[\hat{x}_E(\tau_1) \dots \hat{x}_E(\tau_n)] e^{-\beta H / 2\hbar} | x' \rangle}_{\sum_n e^{-\beta E_n / 2\hbar} \underbrace{\langle x'' | n \rangle \langle n |}_{\phi_n(x'')}} \underbrace{\sum_k e^{-\beta E_k / 2\hbar} \underbrace{\langle k | x' \rangle}_{\phi_k^*(x')}} \\
& \stackrel{\text{vacuum}}{=} \underbrace{\langle 0 |}_{\substack{\uparrow \\ \text{dominant} \\ \text{term. } \beta \rightarrow \infty}} T[\hat{x}_E(\tau_1) \dots \hat{x}_E(\tau_n) | 0 \rangle \phi_0(x'') \phi_0^*(x') e^{-\beta E_0 / \hbar} [1 + \dots]
\end{aligned}$$

$$\Rightarrow \langle 0 | T[\hat{x}_E(\tau_1) \cdots \hat{x}_E(\tau_n)] | 0 \rangle = \lim_{\beta \rightarrow \infty} \frac{\langle x'' | e^{-\beta H / 2\hbar} T[x_E(\tau_1) \cdots x_E(\tau_n)] | x' \rangle}{\langle x'' | e^{-\beta H / 2\hbar} | x' \rangle}$$

Feynman correlator.

$$\begin{aligned} \langle x', 0 | &= \langle x' | e^{-iHT} | x \rangle \\ &= \langle x', T | x, 0 \rangle \end{aligned}$$

$$K = \langle x', T | x, 0 \rangle = \int \mathcal{D}x[t] e^{iS/\hbar} \quad (\text{real time})$$

$$|0\rangle = \lim_{\beta \rightarrow \infty} \frac{\langle x'' | e^{-\beta H / 2\hbar} T[x_\epsilon(\tau_1) \dots x_\epsilon(\tau_n)] e^{-\beta H / 2\hbar} | x' \rangle}{\langle x'' | e^{-\beta H / \hbar} | x' \rangle}$$

$$= \lim_{\beta \rightarrow \infty} \frac{\int dx(\tau) x(\tau_1) \dots x(\tau_n) e^{-S_\epsilon / \hbar}}{\int dx[\tau] e^{-S_\epsilon / \hbar}}$$

$$-x(\tau_n) \Big] e^{-\beta H/2\hbar} |x'\rangle$$

$$e^{-\beta H/\hbar} |x'\rangle$$

$$\tau_n) e^{-S_E/\hbar}$$

$$/\hbar$$

→ indpt of x', x''

→ choice of boundary conditions

• $x' = x'' = 0$ Dirichlet b.c.

• $x(\beta/2) = x(-\beta/2)$ Periodic b.c.

$$Z[J] = \frac{\int dx(t) e^{i(S + \int_{-\infty}^{+\infty} dt J(t)x(t))}}{\int dx(t) e^{iS}} \quad (Z[J=0]=1)$$

$$= \frac{\langle 0|0 \rangle_J}{\langle 0|0 \rangle_{J=0}}$$

② Generating functional

$$Z[J] = \frac{\int dx(t) e^{i(S + \int_{-\infty}^{\infty} dt J(t))}}{\int dx(t) e^{iS}}$$

$$= \frac{\langle 0|0 \rangle_J}{\langle 0|0 \rangle_{J=0}}$$

$$\textcircled{4} \left(\frac{1}{i} \frac{\delta}{\delta J(t_1)} \dots \frac{1}{i} \frac{\delta}{\delta J(t_n)} Z[J] \right) \Big|_{J=0} = \int dx(t) x(t_1) \dots x(t_n)$$

$$= \frac{\int d\alpha(t) e^{i(S + \int_{-\infty}^{+\infty} dt J(t) \alpha(t))}}{\int d\alpha(t) e^{iS}} \quad \left(Z[J=0] = 1 \right)$$

$$= \frac{\langle 0|0 \rangle_J}{\langle 0|0 \rangle_{J=0}}$$

$$= \frac{\int d\alpha(t) \alpha(t_1) \cdots \alpha(t_n) e^{iS}}{\int d\alpha(t) e^{iS}} = \langle 0 | T [\hat{\alpha}(t_1) \cdots \hat{\alpha}(t_n)] | 0 \rangle$$

For condensates? $\lim_{\alpha \rightarrow 0} \langle \alpha'' | e^{-p''/2h} | \alpha_E(\tau_1) \dots \alpha_E(\tau_n) \rangle$

$Z[J]?$

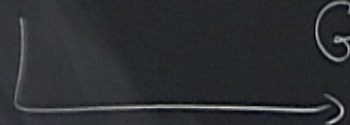
$\rightarrow$ HO ($\rightarrow$ very similar to $K[J]$)

$$Z[J] = e^{-i/m\hbar \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' J(t) G_0(t, t')}$$

$$(\hat{x}_1(\tau_1) \dots \hat{x}_n(\tau_n)) e^{-iH\tau/2\hbar} |\alpha\rangle$$

Do it explicitly!

$$J(t) G_0(t, t') J(t')$$



Green function

$$\left[\begin{aligned} (-\partial_t^2 - \omega(t)^2) G_0(t, t_1) &= \delta(t - t_1) \\ G(t'', t_1) &= 0 = G(t, t') \end{aligned} \right.$$

$$\Rightarrow G_0(t, t') = \frac{1}{2i\omega} e^{-i\omega|t-t'|}$$

HO (very similar to $R[J]$)

$$Z[J] = e^{-i/m\hbar \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' J(t) G_0(t, t') J(t')}$$

Green function

function

$$\begin{aligned} \langle 0 | T[\hat{x}(t) \hat{x}(t')] | 0 \rangle &= \left[\frac{\hbar}{i} \frac{\delta}{\delta J(t)} \frac{\hbar}{i} \frac{\delta}{\delta J(t')} Z[J] \right]_{J=0} \\ &= \frac{i\hbar}{m} \underline{\underline{G_0(t, t')}} \end{aligned}$$

TODAY: Correlation function and generating functional

$$\int dx(t) x(t_1) \dots x(t_n) e^{\frac{i}{\hbar} S[x(t)]} \longleftarrow \textcircled{1} \begin{array}{l} \downarrow \\ \langle 0 | T [\hat{x}(t_1) \dots \hat{x}(t_n)] | 0 \rangle \text{ as a PI} \\ \equiv G^{(n)}(t_1, \dots, t_n) \end{array}$$

$$S[x(t)] + \int dt x(t) J(t) \longleftarrow \textcircled{2} \text{ Generating functional } Z[J] \text{ as a PI}$$

$\longleftarrow \textcircled{3}$ Know how to compute $Z[J]$ exactly?

$\textcircled{4}$ Functional derivatives on $Z[J]$

$$\downarrow \\ \langle 0 | T [\hat{x}(t_1) \dots \hat{x}(t_n)] | 0 \rangle$$

Perturbation theory $\longleftarrow$ No! $H = H_0 + \lambda H_I$ yes!

function $\rightarrow$ n -point function?
 $G^{(n)}(t_1 \dots t_n)$

Wick's theorem

• n odd, $G^{(n)}(t_1 \dots t_n) = 0$

• n even, $G^{(n)}(t_1 \dots t_n) = \sum$

$$(t_1 \dots t_n) = 0$$

$$(t_1 \dots t_n) = \sum_{p \in \sigma_n} G^{(2)}(t_{p(1)}, t_{p(2)}) \dots G^{(2)}(t_{p(n-1)}, t_{p(n)})$$

$$\hookrightarrow (n-1)!! = (n-1)(n-3) \dots 3 \cdot 1 \text{ elements.}$$

• n even $G^{(n)}(t_1, \dots)$

Graphical notation.

$$A) \int G(\tau_1, \tau) J(\tau)$$

$$B) G(\tau_1, \tau_2) \equiv$$

$\prod$
dominant
term. $B \rightarrow \infty$

even $G^{(n)}(t_1, \dots, t_n) = \sum_{p \in \sigma_n} G(t_{p(1), p(2)} /$

$\hookrightarrow (n-1)!! = (n-1)(n-3) \dots 1$

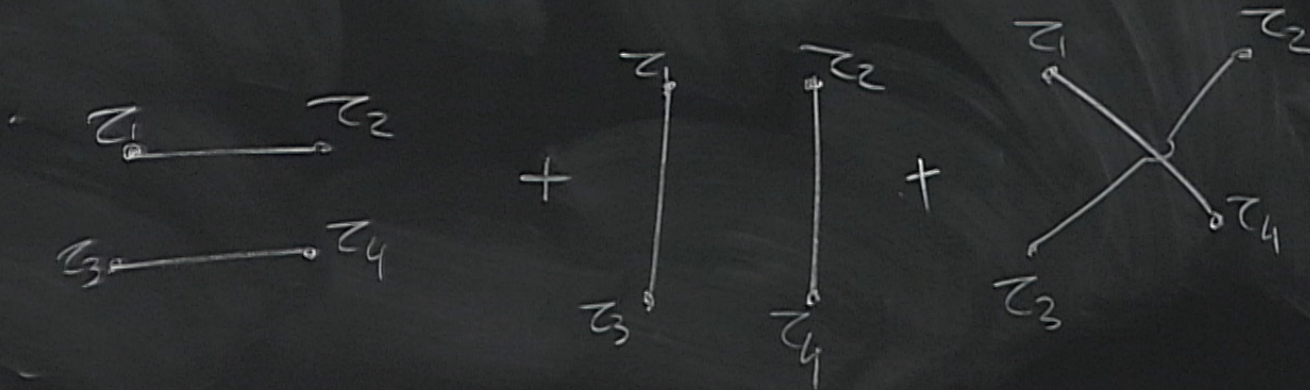
$\int G(\tau_1, \tau) J(\tau) d\tau \equiv \text{---} \circ$

$G(\tau_1, \tau_2) \equiv \tau_1 \text{---} \tau_2$

4-point
correlator

$$G^{(4)}(\tau_1, \tau_2, \tau_3, \tau_4) = G(\tau_1, \tau_2) G(\tau_3, \tau_4)$$

$$G(\tau_3, \tau_4) = G(\tau_1, \tau_2) G(\tau_3, \tau_4) + G(\tau_1, \tau_3) G(\tau_2, \tau_4) + G(\tau_1, \tau_4) G(\tau_2, \tau_3)$$



$$V(x) = V_0(x) + \lambda W(x) \quad \lambda \int d\tau W(x)$$

$\uparrow$
ex H.O.

$$\hookrightarrow S[x] = S_0 + S_I[x]$$

$$Z[J] = \mathcal{N} \left(e^{-\frac{1}{\hbar} S_I\left[\frac{\delta}{\delta J}\right]} \right) Z_0[J]$$

$\downarrow$
 $Z[0] = 1$

$\hookrightarrow$ power series expansion in λ