Title: PSI 2019/2020 - Quantum Theory (Dupuis) - Lecture 11

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Collection: PSI 2019/2020 - Quantum Theory (Branczyk/Dupuis)

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Result from yesterday

$$K(x'',\tau'';x',\tau') = \langle x'' | U(\tau'',\tau') | x' \rangle$$

$$= \int_{\substack{x(\tau'')=x'' \\ x(\tau')=x'}} dx[\tau]\, e^{-\frac{1}{\hbar} S_E(x)}$$

Semi-classical expansion

$$S_E(x) = \int_{\tau'}^{\tau''} d\tau_1 \left[\frac{m}{2} \dot{x}^2(\tau_1) + V(x(\tau_1)) \right]$$

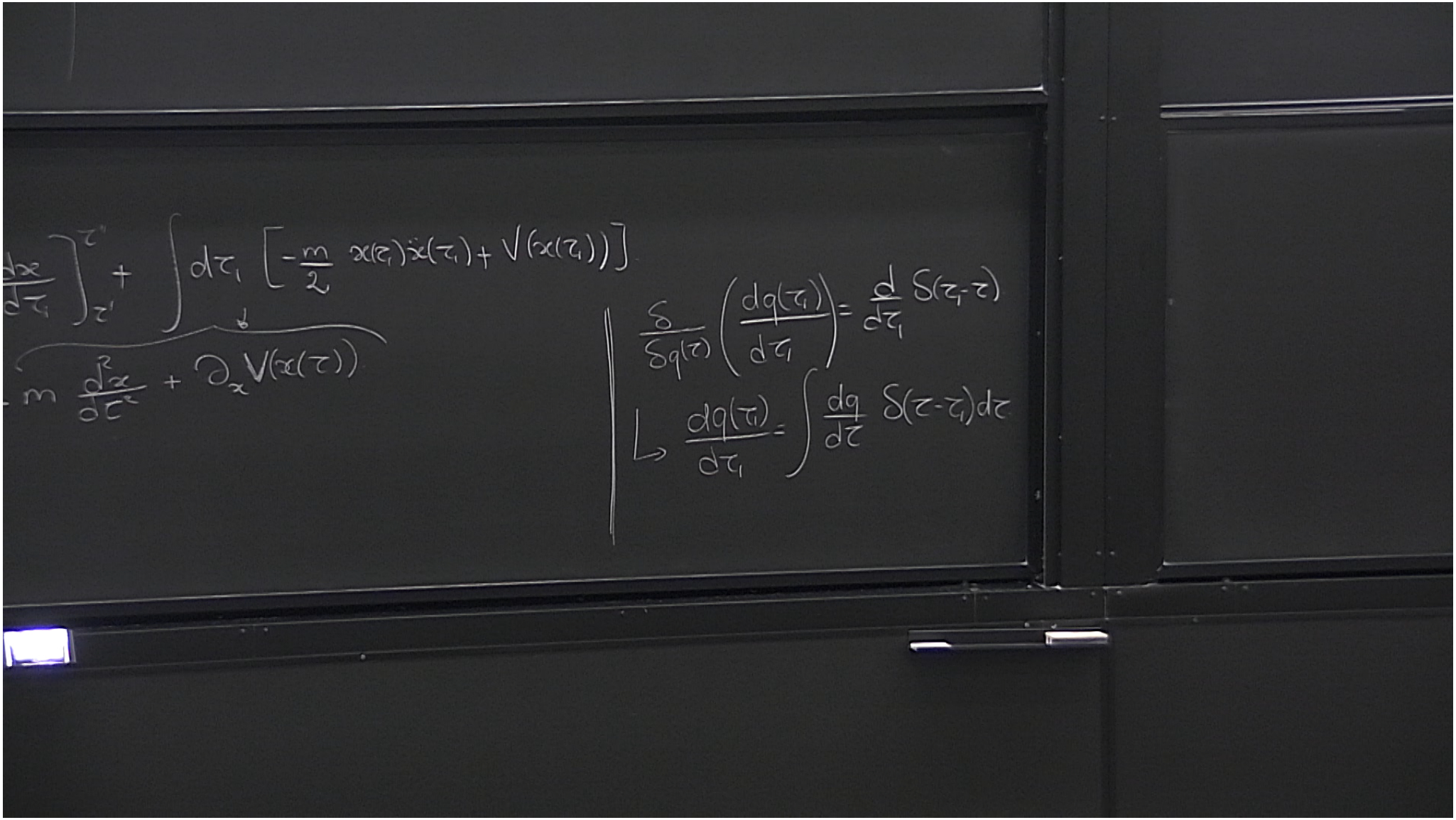
↳ $x(\tau) = x_c(\tau) + \sqrt{\hbar}\, U(\tau)$

$x_c(\tau') = x'$, $x_c(\tau'') = x''$, $U(\tau') = U(\tau'') = 0$

$$\hbar^{-1} S_E(x) = \hbar^{-1} S_E(x_c) + \frac{1}{\sqrt{\hbar}} \int_{\tau'}^{\tau''} \mu(\tau) \frac{\delta S_E}{\delta x(\tau)}\bigg|_{x=x_c} d\tau$$

$$+ \frac{1}{2} \int d\tau_1 d\tau_2 \, \mu(\tau_1)\mu(\tau_2) \frac{\delta^2 S_E}{\delta x(\tau_1)\delta x(\tau_2)}\bigg|_{x=x_c}$$

$$+ \frac{\sqrt{\hbar}}{3!} \int d\tau_1 d\tau_2 d\tau_3 \, \mu(\tau_1)\mu(\tau_2)\mu(\tau_3) \frac{\delta^3 S_E}{\delta x(\tau_1)\delta x(\tau_2)\delta x(\tau_3)}\bigg|_{x=x_c} + O(\hbar)$$


$\frac{\delta p}{\delta q(\tau)}\left(\frac{dp}{d\tau}\right) = \frac{d}{d\tau}\delta(\tau-\tau)$
$\hookrightarrow \frac{dp(\tau)}{d\tau} = \int \frac{\delta p}{\delta b p}\, \delta(\tau-\tau)\, d\tau$
$\int d\tau \left[-\frac{m}{2}\dot{x}(\tau)\dot{x}(\tau) + V(x(\tau))\right]$
$+ \partial_x V(x(\tau))$

$x_c(\tau) = x$, $x_c(\tau') = x'$, $v(\tau) = v(\tau') = 0$

- $\left.\frac{\delta S_E}{\delta x}\right|_{x=x_c} = 0$

- $S_E(x) = \int_{\tau'}^{\tau''} d\tau_1 \left[\frac{m}{2}\dot{x}^2 + V(x(\tau_1))\right] = \left[\frac{1}{2} m x(\tau_1) \frac{dx}{d\tau_1}\right]_{\tau'}^{\tau''} + \int d\tau_1 \left[-\frac{m}{2} x(\tau_1)\ddot{x}(\tau_1) + V(x(\tau_1))\right]$

↳ $\frac{\delta S}{\delta x(\tau)} = m \frac{dx(\tau)}{d\tau}\left(\delta(\tau''-\tau) - \delta(\tau'-\tau)\right) - m\frac{d^2x}{d\tau^2} + \partial_x V(x(\tau))$

$\frac{\delta S}{\delta x(\tau_1)\delta x(\tau_2)} = m \frac{d}{d\tau_1}\delta(\tau_1-\tau_2)\left[\delta(\tau_1-\tau'') - \delta(\tau_1-\tau')\right] + A(\tau_1,\tau_2)$

η

$\int d\tau_1 d\tau_2\, u(\tau_1)\, u(\tau_2)\, \eta(\tau_1,\tau_2) =$

η

$\neq \eta \rightarrow \int d\tau_1 d\tau_2 \, u(\tau_1) u(\tau_2) \eta(\tau_1, \tau_2) = 0$

$\underline{u(\tau') = u(\tau'') = 0.}$

$\rightarrow$ disappear at the next order. (undpt of x).

$$* \boxed{A(\tau_1,\tau_2) = \left(-m\frac{d^2}{d\tau_1^2} + \partial^2_{x(\tau_1)x(\tau_1)} V(x(\tau_1))\right)\delta(\tau_1,\tau_2)}$$

$$\frac{\delta^3 S}{\delta x(\tau_1)\delta x(\tau_2)\delta x(\tau_3)} = \frac{\partial^3 V}{\partial x^3}\,\delta(\tau_1-\tau_2)\,\delta(\tau_2-\tau_3)$$

$$K(x'',\tau'',x',\tau') = e^{-\frac{1}{\hbar}S_E[x_c]} \int_{u(\tau')=u(\tau'')=0} \mathcal{D}u(\tau)\; e^{-\frac{1}{2}\iint d\tau_1 d\tau_2\, u(\tau_1) A(u_1,u_2) u(\tau_2)} \left[1 + \frac{\sqrt{\hbar}}{3!}\int d\tau\, u(\tau)^3 \left.\frac{\partial^3 V}{\partial x^3}\right|_{x=x_c} + O(\hbar)\right]$$

2 next steps → determine x_c

→ compute quadratic integral in u that is evolving A ⟶ Functional determinant

ex A is $n \times n$ Hermitian matrix with positive-definite eigenvalues λ_k

$$\int d^n x \exp\left(-\sum \frac{1}{2} x_i A_{ij} x_j\right) = (2\pi)^{n/2} (\det A)^{-1/2} = \frac{(2\pi)^{n/2}}{\sqrt{\prod_k \lambda_k}}$$

→ Solve an eigenvalue pb for $\det A$

in the case of a quadratic lagrangian

$$K = \text{cst} \cdot e^{-\frac{S_E[x_c]}{\hbar}} \cdot (\det A)^{-1/2}.$$

Harmonic Oscillator $V(x) = \frac{1}{2} m\omega^2 x^2$

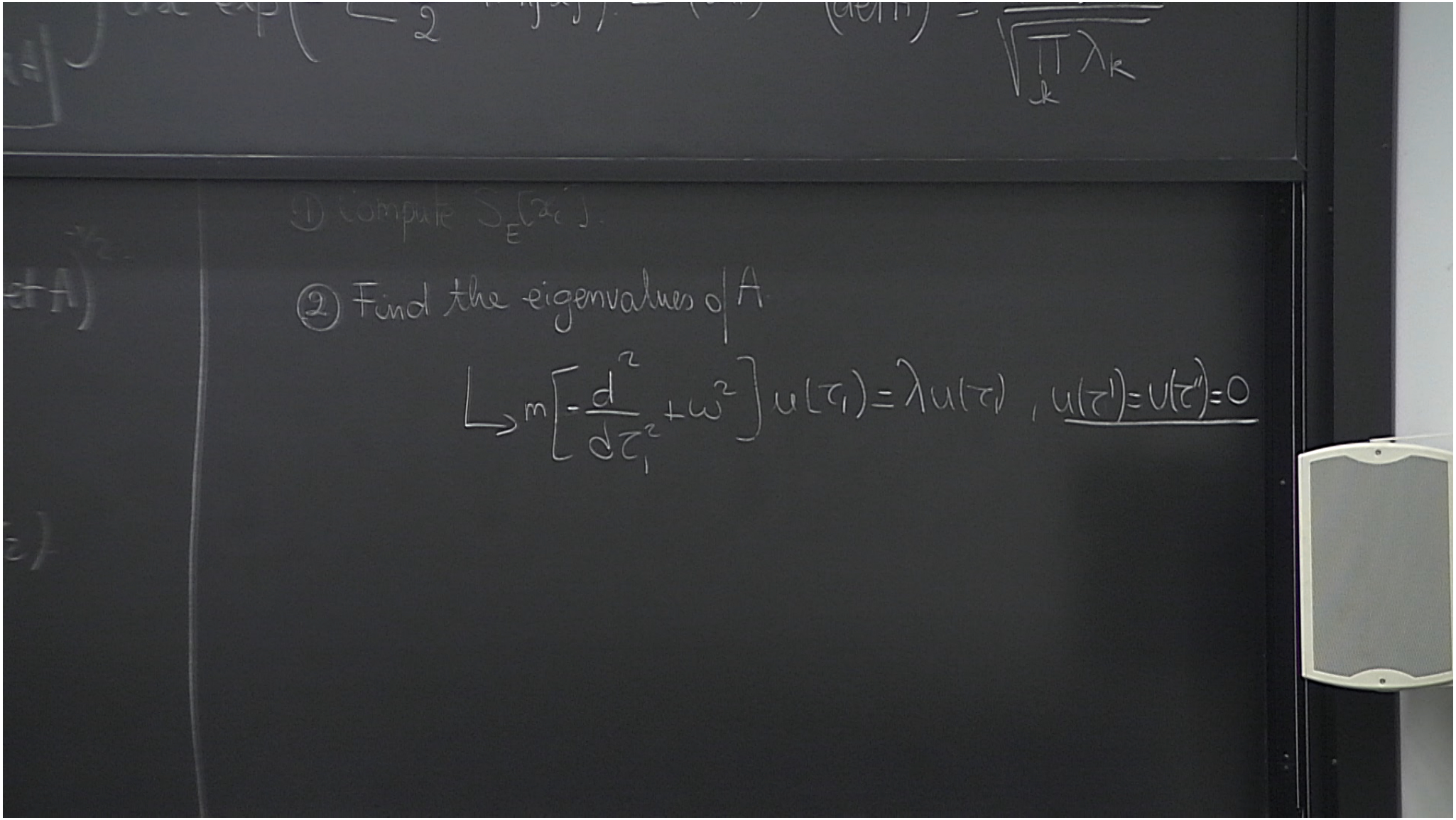
$A(\tau_1, \tau_2) =$

in the case of a quadratic lagrangian

$$K = \text{constant} \cdot e^{-\frac{S_E[x_c]}{\hbar}} \cdot (\det A)^{-1/2}.$$

Harmonic Oscillator $V(x) = \frac{1}{2} m\omega^2 x^2$

$$A(\tau_1, \tau_2) = m\left[-\frac{d^2}{d\tau_1^2} + \omega^2\right] \delta(\tau_1, \tau_2)$$


① compute $S_E[x_c]$.
② Find the eigenvalues of A
$\hookrightarrow m\left[-\frac{d^2}{d\tau_1^2}+\omega^2\right]u(\tau_1)=\lambda u(\tau_1)$, $u(\tau')=u(\tau'')=0$

$A e^{\omega\tau} + B e^{-\omega\tau} = x_c(\tau)$

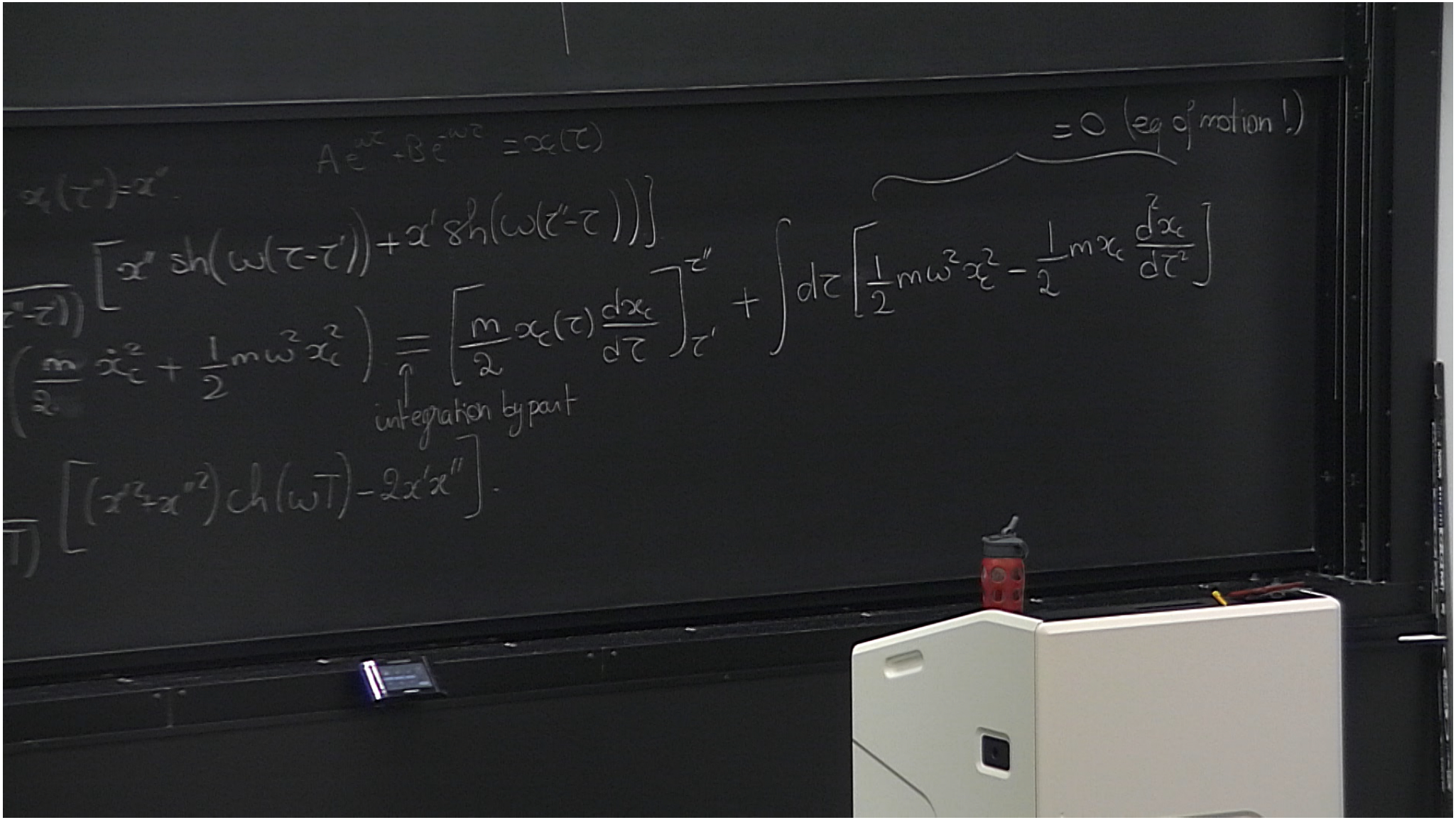
① $\frac{d^2 x_c}{d\tau^2} = \omega^2 x_c$, $x_c(\tau') = x'$, $x_c(\tau'') = x''$

$$x_c(\tau) = \frac{1}{\mathrm{sh}(\omega(\tau''-\tau'))}\left[x'' \,\mathrm{sh}(\omega(\tau-\tau')) + x' \,\mathrm{sh}(\omega(\tau''-\tau))\right]$$

$$S_E(x_c) = \int d\tau \left(\frac{m}{2}\dot{x}_c^2 + \frac{1}{2} m\omega^2 x_c^2\right) = \left[\frac{m}{2} x_c(\tau)\frac{dx_c}{d\tau}\right]_{\tau'}^{\tau''} + \int d\tau \left[\right.$$

integration by part

$T = \tau'' - \tau'$ $\quad S_E(x_c) = \frac{m\omega}{2\,\mathrm{sh}(\omega T)}\left[(x'^2 + x''^2)\,\mathrm{ch}(\omega T) - 2x'x''\right]$.


$A e^{\omega\tau} + B e^{-\omega\tau} = x_c(\tau)$
$x_c(\tau') = x'$
$\left[x' \sinh(\omega(\tau - \tau')) + x' \sinh(\omega(\tau' - \tau))\right]$
$\left(\frac{m}{2}\dot{x}_c^2 + \frac{1}{2}m\omega^2 x_c^2\right) = \left[\frac{m}{2} x_c(\tau) \frac{dx_c}{d\tau}\right]_{\tau'}^{\tau''} + \int d\tau \left[\frac{1}{2}m\omega^2 x_c^2 - \frac{1}{2} m x_c \frac{d^2 x_c}{d\tau^2}\right]$
$= 0$ (eq of motion !)
integration by part
$\left[(x'^2 + x''^2)\cosh(\omega T) - 2x'x''\right]$.

$$\frac{\delta S}{\delta x(\tau_1)\delta x(\tau_2)} = m \frac{d}{d\tau_1} \delta(\tau_1-\tau_2)\left[\delta(\tau_1-\tau')-\delta(\tau_1-\tau'')\right] + A(\tau_1,\tau_2)$$

② $\left(-\frac{d}{d\tau_1^2}+\omega^2\right)u(\tau_1)=\lambda u(\tau_1)$ $\quad u(\tau')=u(\tau'')=0$

$$u_n(\tau)=\sqrt{\frac{2}{T}}\sin\left(\frac{\pi n}{T}(\tau-\tau')\right)$$

with $\lambda_n=\frac{n^2\pi^2}{T^2}+\omega^2$ $\quad n\in\mathbb{N}^*$

$$+ \int d\tau \left[-\frac{m}{2} x(\tau) \ddot{x}(\tau) + V(x(\tau)) \right]$$

$$\frac{K(x'', \tau'', x', \tau')}{K^{(0)}(x'', \tau'', x', \tau')} = e^{-\frac{1}{\hbar} S_E(x_c) + \frac{1}{\hbar} S_E^{(0)}(x_c^0)} \prod_n \left(\frac{\lambda_n^{(0)}}{\lambda_n} \right)^{1/2}$$

$V = 0$

$$\lambda_n^{(0)} = \frac{n^2 \pi^2}{T^2}$$

$n \in \mathbb{N}^*$

$$K(x'', \tau'', x', \tau') = K^{(0)}(x'', \tau'', x', \tau') \, e^{\frac{1}{\hbar}\left(-S_E(x_c) + S_E^{(0)}(x_c^0)\right)} \left(\prod_n \frac{1}{1 + \frac{\omega^2 T^2}{\pi^2 n^2}} \right)^{1/2}$$

$$\prod_{n>0} \left(1 + \frac{\omega^2 T^2}{\pi^2 n^2} \right) = \frac{\sinh(\omega T)}{\omega T}$$

$$\frac{\delta^3 S}{\delta x(\tau_1) \delta x(\tau_2) \delta x(\tau_3)} = \frac{\partial^3 V}{\partial x^3} \delta(\tau_1 - \tau_2) \delta(\tau_2 - \tau_3)$$

H.O.

$$\Rightarrow K(x'',\tau'';x',\tau') = \left(\frac{m\omega}{2\pi\hbar\,\mathrm{sh}(\omega T)}\right)^{1/2} e^{-\frac{m}{2\hbar\,\mathrm{sh}(\omega T)}\left((x''^2+x'^2)\,\mathrm{ch}(\omega T)-2x'x''\right)}$$

$$U = e^{\frac{1}{i\hbar}\hat{H}} \qquad \langle x'|\psi(t)\rangle = U(t',t)|\psi\rangle$$

$$\langle x'|U|x\rangle \qquad \psi(x',t) = \int dx \underbrace{\langle x'|U|x\rangle}_{K}\langle x|\psi\rangle$$

TODAY

P I
* semi-classical expansion
* perturbative theory.

↓

$H = H_0 + \lambda H_p$

Ground state energy

Time ordered correlator on the ground state $\langle 0|T[\hat{x}(t_1)\cdots\hat{x}(t_n)]|0\rangle$

can be expressed as an expression involving propagator.

"interacting" part

tor can be written in terms of a free propagator with source term added.

$$S[x]=\int d\tau\,(L^{(0)}+J(\tau)x(\tau))$$

ex HO

integration by part

Harmonic Oscillator

Perturbation theory — use to determine → Ground state energy
general strategy → Time ordered correlator on the gro

① can be expressed as an expression

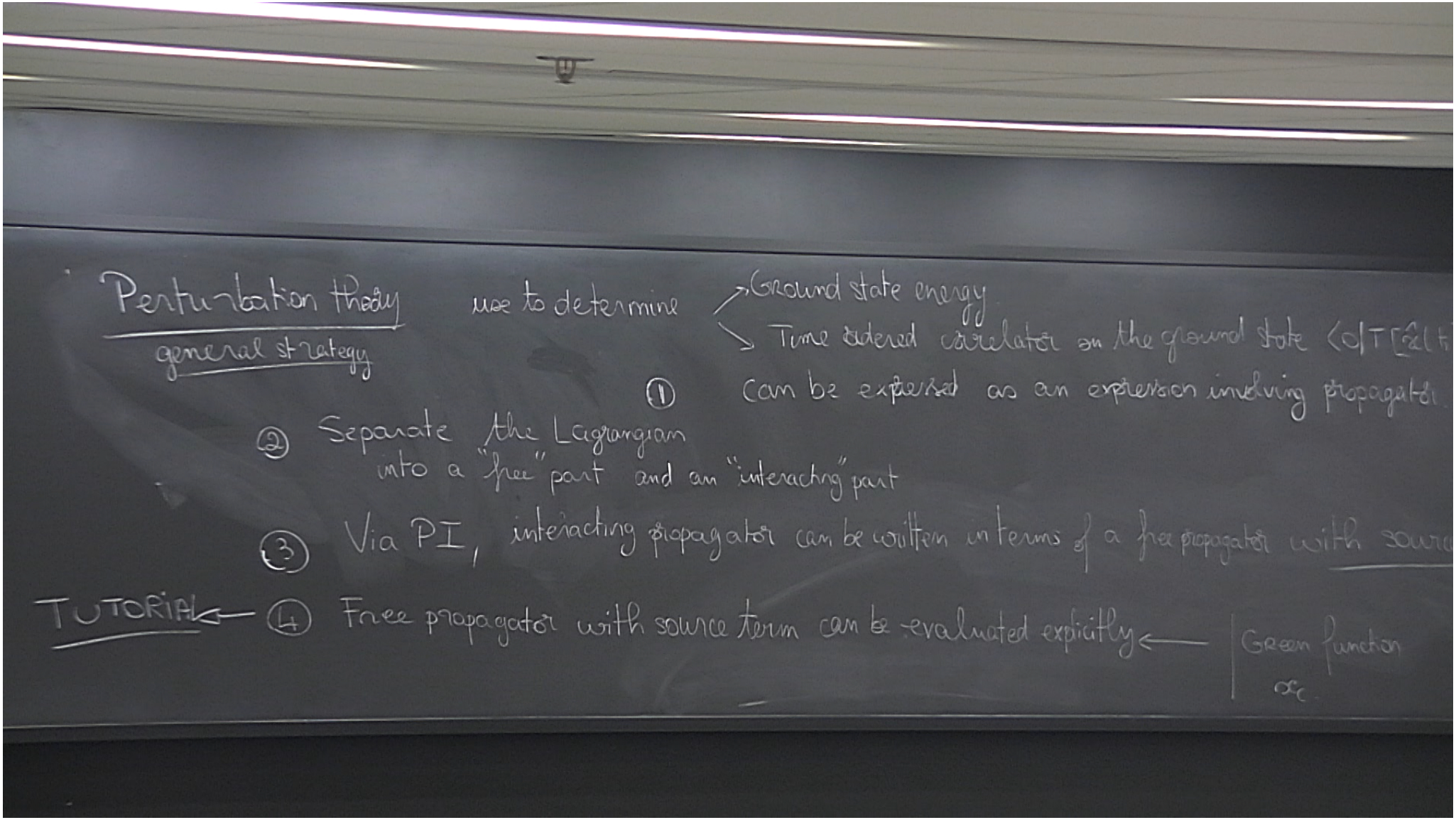
② Separate the Lagrangian into a "free" part and an "interacting" part

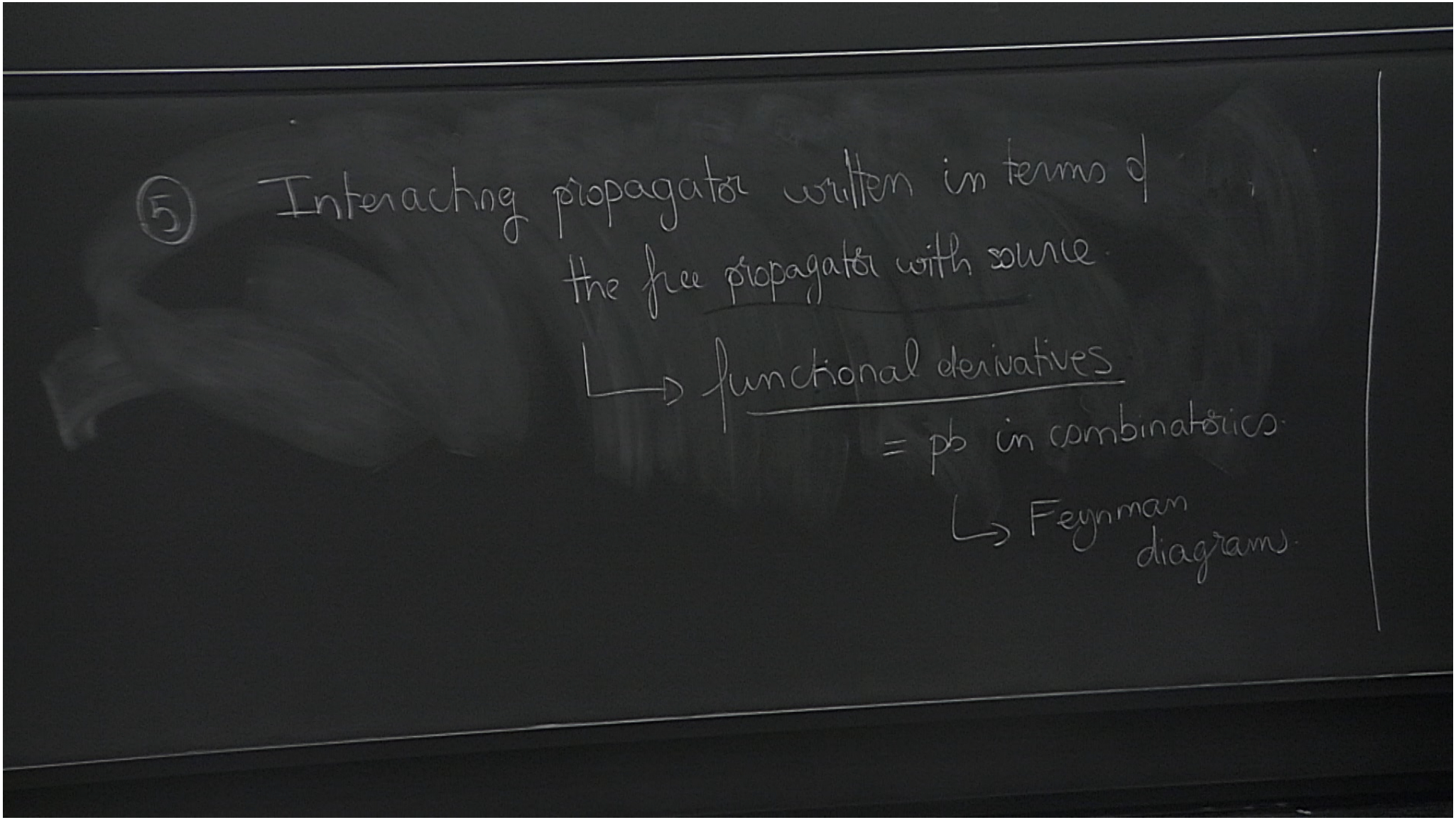
③ Via PI, interacting propagator can be written in terms of a free pro

TUTORIAL ← ④ Free propagator with source term can be evaluated explicitly.

integration by part

$T = t'' - t' \qquad S_E(x) = \frac{m\omega}{2\sinh(\omega T)} \left[(x'^2 + x''^2)\cosh(\omega T) - 2x'x'' \right]$


Perturbation theory
general strategy
use to determine
Ground state energy
Time ordered correlator on the ground state
can be expressed as an expression involving propagator
Separate the Lagrangian into a "free" part and an "interacting" part
Via PI, interacting propagator can be written in terms of a free propagator with sour
TUTORIAL
Free propagator with source term can be evaluated explicitly
Green function


⑤ Interacting propagator written in terms of
the free propagator with source
↳ functional derivatives
= pb in combinatorics
↳ Feynman
diagrams