

Title: PSI 2019/2020 - Quantum Theory (Dupuis) - Lecture 9

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TODAY

- * Canonical quantization

↳ some explicit representations and wave functions.

- * Path Integral

The propagator as path integral on the configuration space

$$\hat{H} = \frac{1}{2m} \left(\hat{\vec{p}} + \frac{e}{c} \vec{A}(\hat{\vec{x}}, t) \right)^2 - e \phi(\hat{\vec{x}}, t)$$

↳ explicit time dependence

⇒ $U(t)$ "complicated" $[H(t), H(t')] \neq 0$
 → time ordered exponential.

$$\begin{aligned} H_{\pm}(t) &= U(t) H(\hat{\vec{x}}(t), \hat{\vec{p}}(t), t) U^{\dagger}(t) \\ &= H(\hat{\vec{x}}(t), \hat{\vec{p}}(t), t) \end{aligned}$$

Some explicit representations and wave functions

* Coordinate representation

- $\{|x\rangle\}$ eigenvectors of the position operator
 $\langle x|x'\rangle = \delta(x-x')$.

- $|\psi\rangle = \int_{\mathbb{R}} |x\rangle \underbrace{\langle x|\psi\rangle}_{\psi(x)} dx$.

representations and wave functions

position representation

• $\{|x\rangle\}$ eigenvectors of the position operator

$$\langle x|x'\rangle = \delta(x-x').$$

$$|\psi\rangle = \int_{\mathbb{R}} |x\rangle \underbrace{\langle x|\psi\rangle}_{\psi(x)} dx.$$

$$\hat{x}_\alpha |x\rangle = x_\alpha |x\rangle.$$

$$1 = \int_{\mathbb{R}} |x\rangle \langle x| dx$$

and wave functions

notation

$|x\rangle$ } eigenvectors of the position operator

$$\langle x|x'\rangle = \delta(x-x').$$

$$|\psi\rangle = \int_{\mathbb{R}} |x\rangle \underbrace{\langle x|\psi\rangle}_{\psi(x)} dx.$$

position
wave function

$$1 = \int_{\mathbb{R}} |x\rangle \langle x| dx$$

$$\hat{x}_d |x\rangle = x_d |x\rangle, \quad \text{, spectrum } \hat{x}_d =$$

$$\hat{x}_\alpha |x\rangle = x_\alpha |x\rangle, \quad \text{spectrum } \hat{x}_\alpha = \mathbb{R} \quad \langle x | \hat{f} | x' \rangle.$$

$$\langle x | dx \quad \left| \quad \hat{f} \text{ position basis} \right. \quad \hat{f} = \text{id} \hat{f} \text{id} = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dx' \overbrace{f(x, x')}^{\boxed{}} |x\rangle \langle x'|$$

* Included action

$$|\phi\rangle = \hat{x}|\psi\rangle$$

$$\hookrightarrow \underline{|\phi(x) \equiv \langle x|\phi\rangle} = \langle x|\hat{x}|\psi\rangle = x\langle x|\psi\rangle = \underline{x \psi(x)}$$

For an abstract operator A

$$A \cdot \psi(x) = \langle x|A|\psi\rangle$$

$$\begin{aligned}
 \langle \hat{x} | \Psi \rangle &= x \langle x | \Psi \rangle = \underline{x \Psi(x)} \quad \xrightarrow[\text{action}]{\text{induced}} \hat{x} \cdot \Psi(x) = x \Psi(x). \\
 A \cdot \Psi(x) &= \langle x | A | \Psi \rangle \quad \Psi(x) = \langle x | \Psi \rangle
 \end{aligned}$$

momentum operator $\hat{p}$? $\hat{p} \cdot \psi(x)$?

$U(a) = e^{-ia\hat{p}/\hbar}$ satisfying $U(a)|x\rangle = |x+a\rangle$

$$[\hat{x}, \hat{p}] = i\hbar \quad \rightarrow \quad [\hat{x}, \hat{p}^n] = i\hbar n \hat{p}^{n-1}$$

$$[\hat{x}, U(a)] = [\hat{x}, \sum \frac{(-ia/\hbar)^n}{n!} \hat{p}^n] = a U(a).$$

$$\hat{x} U(a)|x\rangle = U(a)(\hat{x} + a)|x\rangle = (x+a) U(a)|x\rangle$$

$$\Rightarrow U(a)|x\rangle \propto |x+a\rangle \quad \downarrow U(a) \text{ unitary}$$
$$U(a)|x\rangle = |x+a\rangle$$

Action of $\hat{p} \rightarrow$ infinitesimal level.

$$U(\varepsilon) |x\rangle = |x+\varepsilon\rangle \sim \left(1 - \frac{i\varepsilon}{\hbar} \hat{p}\right) |x\rangle \Rightarrow \left[\frac{\hat{p}}{i\hbar} \right]$$

↑
"small"

$\Rightarrow \langle x |$

$$\Rightarrow \boxed{\hat{p}|x\rangle = \hbar \frac{|x+\varepsilon\rangle - |x\rangle}{-i\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} i\hbar \frac{d}{dx}|x\rangle}$$

$$\Rightarrow \langle x|\hat{p}|\Psi\rangle = -i\hbar \frac{d}{dx}\langle x|\Psi\rangle = -i\hbar \frac{d}{dx}\Psi(x)$$

$$\boxed{\hat{p}\Psi(x) = -i\hbar \frac{d}{dx}\Psi(x)}$$

Momentum representation

$$\hat{p}_\alpha |p\rangle = p_\alpha |p\rangle$$

in the position basis

$$|p\rangle = \int dx |x\rangle \langle x|p\rangle$$

$$\hat{p} \cdot \langle x|p\rangle = -i\hbar \frac{d}{dx} \langle x|p\rangle = p \langle x|p\rangle.$$

solve differential eq $\rightarrow \langle x|p\rangle = C e^{i\frac{x p}{\hbar}}$

$$\begin{aligned}
 \delta(x-y) = \langle x|y \rangle &= \langle x | \int dp |p\rangle \langle p|y \rangle \\
 &= \underbrace{C^* C}_{2\pi\hbar} \underbrace{\int_R dp e^{\frac{i(x-y)p}{\hbar}}}_{\delta(x-y)}
 \end{aligned}$$

$$\int_R e^{iab} da = 2\pi\delta(b)$$

$$C \text{ is real} \Rightarrow C = \frac{1}{\sqrt{2\pi\hbar}}$$

continuum normalized
position wave function
of the momentum eigenvectors

$$\langle x|p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{+i\frac{x p}{\hbar}}$$

Time-evolution operator

Schrödinger
picture

$$i\hbar \frac{d}{dt} U(t) = H(t) U(t), \quad U(t_0) = \text{id}.$$

$$|\psi(t)\rangle = U(t) |\psi_0\rangle.$$

- For any basis $\{|\lambda\rangle\}$

- Position basis

$$K(\lambda, \lambda') := \langle \lambda | U(t) | \lambda' \rangle$$

$$K(\tilde{x}, x') := \langle x | U(t) | x' \rangle$$

Path Integral · propagator as path integral on the

H : non-relativistic, one dimensional, independent of time t .

Schrödinger
picture

$$\begin{aligned}\Psi(x'', t'') &= \langle x'' | \Psi(t'') \rangle \\ &= \langle x'' | U(t'', t') | \Psi(t') \rangle\end{aligned}$$

configuration space.

$$\Rightarrow \psi(x'', t'') = \int K(x'', t'', x', t') \psi(x', t') dx'$$

$\psi(x'')$

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}}$$

ation space.

$$\Psi(x'', t'') = \int K(x'', t'', x', t') \Psi(x', t') dx'$$

Wick notation

$$t \rightarrow -i\tau$$
$$f(t) \rightarrow g(z) = f(-i\tau)$$

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{+i\frac{xp}{\hbar}}$$

1. propagator as path integral on the configuration space
relativistic, one dimensional, independent of time t .

$$\begin{aligned}\Psi(x'', t'') &= \langle x'' | \Psi(t'') \rangle \\ &= \langle x'' | U(t'', t') | \Psi(t') \rangle \\ &= \int dx' \langle x'' | U(t'', t') | x' \rangle \langle x' | \Psi(t') \rangle\end{aligned} \quad \left. \vphantom{\int dx'} \right] \Rightarrow \Psi(x'', t'')$$

Wick

$$U(x'', t') \rightarrow U(\tau'', \tau') = e^{-\frac{i}{\hbar}(\tau'' - \tau')H}.$$

$$H = \frac{\vec{p}^2}{2m} + \alpha V(x) \leftarrow \text{typical Hamiltonian.}$$

Divide time interval into N equal subintervals.

$$U(\tau'', \tau') = \prod_{n=1}^N U(\tau_n, \tau_{n-1}) \quad \left\{ \begin{array}{l} \tau_0 = \tau' \\ \tau_n = \tau' + n\varepsilon \\ \tau_N = \tau'' \end{array} \right.$$

$$[H, H] = 0 \rightarrow$$

$$K = \langle x'' | e^{-i(\tau'' - \tau)H/\hbar} | x' \rangle$$

$$= \langle x'' | e^{-i(\tau'' - \tau)H/\hbar} e^{-i(\tau - \tau')H/\hbar} | x' \rangle$$

dividing
time
interval

$$= \langle x'' | e^{-i(\tau'' - \tau)H/\hbar} \int dx | x \rangle \langle x | e^{-i(\tau - \tau')H/\hbar} | x' \rangle$$

$$K = \int dx K(x'', \tau'', x, \tau) K(x, \tau, x', \tau')$$

$$E = \frac{\tau'' - \tau'}{N}$$

$$K = \int \prod_{i=1}^N dx_i \left(\langle x_i | U(\tau_i, \tau_{i-1}) | x_{i-1} \rangle \right)$$

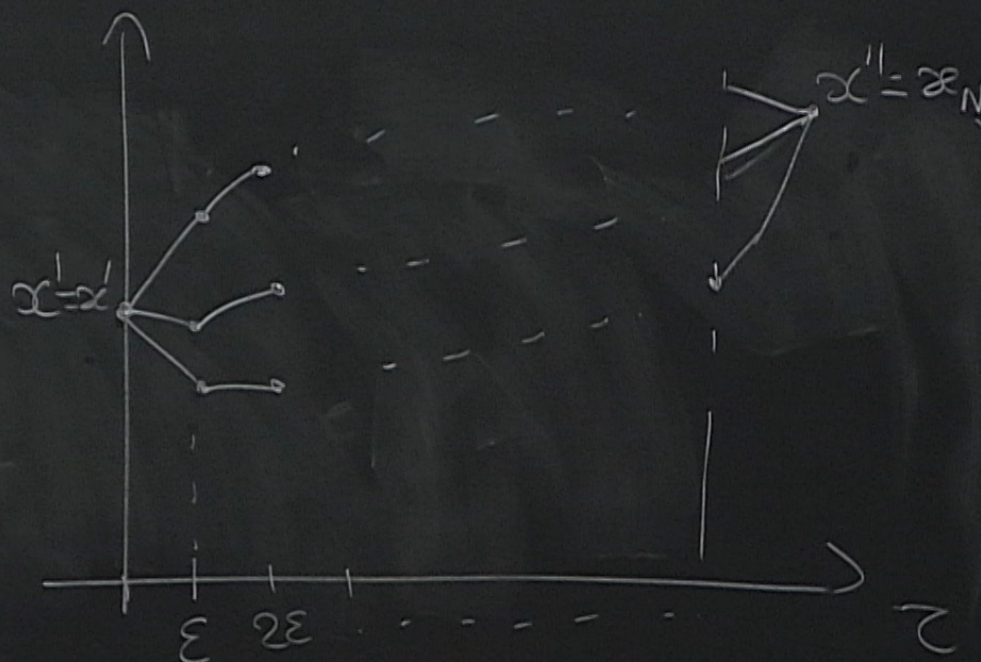
$$x_0 = x'$$

$$x_N = x''$$

$$x_0 = x'$$

$$x_N = x''$$

τ_i are fixed



$$N \rightarrow \infty$$

Sum over all possible path $x(\tau)$

$$\begin{aligned}
 \overbrace{dp |p\rangle \langle p| x_{n-1}} &= \int dp \, e^{\frac{-\varepsilon}{\hbar} \frac{p^2}{2m}} e^{\frac{i p (x_n - x_{n-1})}{\hbar}} \left(\frac{1}{2\pi\hbar} \right) \\
 &= \left(\frac{m}{2\pi\hbar\varepsilon} \right)^{1/2} e^{\frac{-(x_n - x_{n-1})^2 m}{2\hbar\varepsilon}}
 \end{aligned}$$

Subinterval.

$$K_{n,n-1} = \langle x_n | \cup(\tau_n, \tau_{n-1}) | x_{n-1} \rangle$$

$$\boxed{V=0}$$

exact computation

$$K_{n,n-1} = \langle x_n | e^{-\frac{\varepsilon}{\hbar} \frac{p^2}{2m}} | x_{n-1} \rangle = \langle x_n | e^{-\frac{\varepsilon}{\hbar} \frac{\hat{p}^2}{2m}} | x_{n-1} \rangle$$