Title: PSI 2019/2020 - Quantum Theory (Dupuis) - Lecture 8

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Collection: PSI 2019/2020 - Quantum Theory (Branczyk/Dupuis)

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URL: http://pirsa.org/19090050

1900, Planck

$E = hf$

$\hookrightarrow h = 6.63 \times 10^{-34}\ m^2\,kg/s$

$F = ma + h$ X DIMENSION

$F = ma\left(1 + \frac{xp}{h}\right)$ X

$F = ma\left(1 + \frac{h}{xp}\right)$ X

Classical

- starting point: H, hamiltonian
  $H(p,q)$
- $\{x,p\} = 1$
- - antisymmetry
  - linearity
  - addition rule
  - product rule
  - Jacobi identity

Poisson algebra

$m(\ ,\ )$ ← pointwise product
$(g.h)(x,p) = g(x,p)$

$\{\ ,\ \}$ ← Lie bracket (non-assoc

Compatibility $\{f, m(a,b)\} =$

hamiltonian

(p, q)

Poisson algebra

$m(\,,\,)$ ← pointwise product

$(g \cdot h)(x, p) = g(x, p)\, h(x, p)$

$\{\,,\,\}$ ← Lie bracket (non-associative ← Jacobi)

Compatibility $\{f, m(a, b)\} = m(a, \{f, b\}) + m(\{f, a\}, b)$

↳ product rule

starting point: H, hamiltonian
$H(p, q)$

- $\{x, p\} = 1$

- antisymmetry
  linearity
  addition rule
  product rule
  Jacobi identity

Poisson algebra

$m(\ ,\ )$ ← pointwise product
associative
$(g\,h)(x,p) = g(x,p)\,h(x,p)$

$\{\ ,\ \}$ ← Lie bracket (non-associative)

Compatibility $\{f, m(a,b)\} = m(a,$
↳ product rule

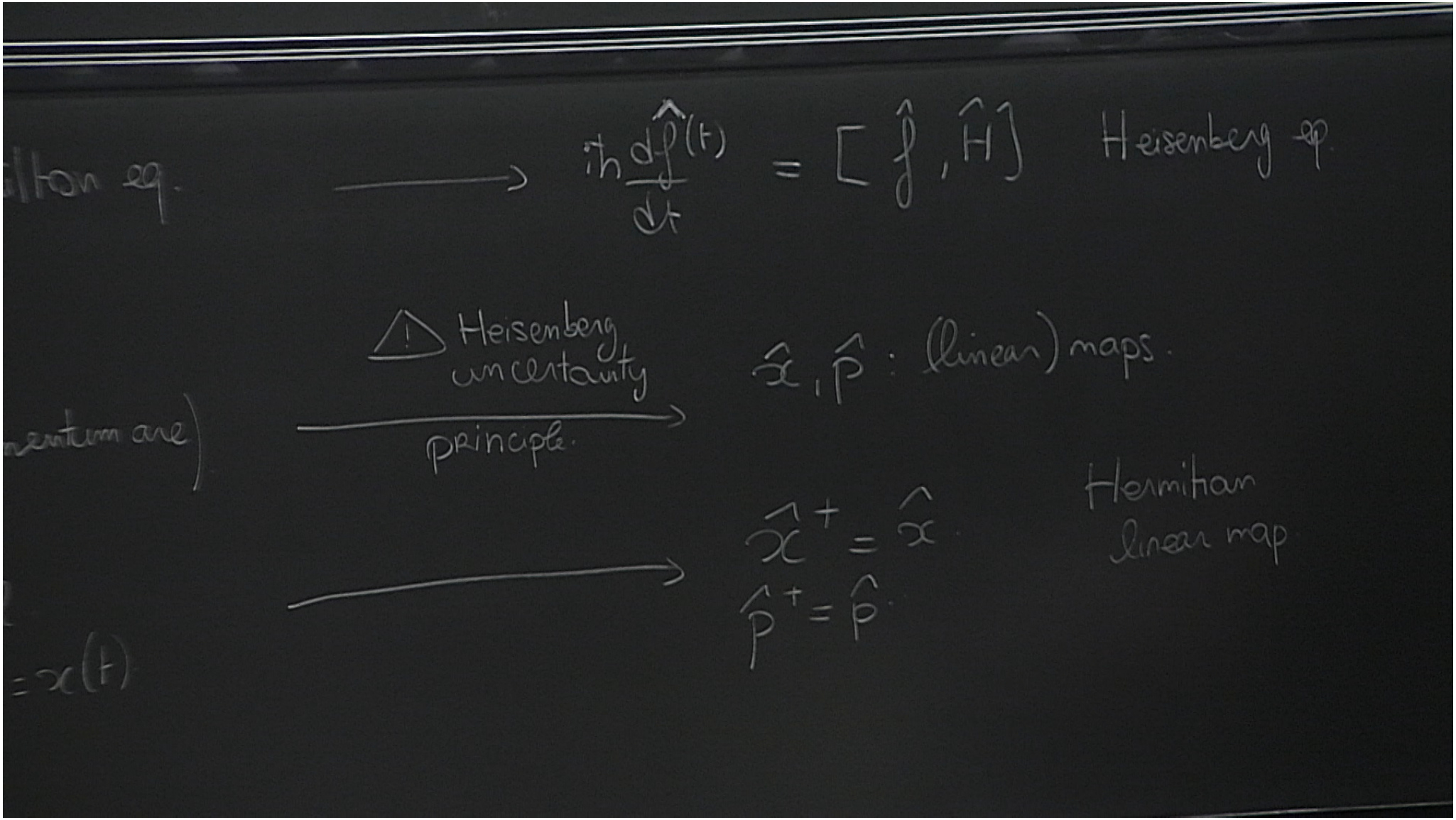
- $\frac{df}{dt} = \{f, H\}$ Hamilton eq. $\longrightarrow$ $i\hbar \frac{d\hat{\rho}(t)}{dt}$

- $x(t), p(t)$.
  - ↳ commutativity (measurem^t position / momentum are compatible)

⚠ Heisenberg uncertainty principle.

- Measurement values are real

$$x^*(t) = x(t)$$


ilton eq.
$i\hbar \frac{d\hat{f}(t)}{dt} = [\hat{f}, \hat{H}]$ Heisenberg eq.
Heisenberg uncertainty principle
$\hat{x}, \hat{p}$: (linear) maps.
entum are)
$\hat{x}^+ = \hat{x}$
$\hat{p}^+ = \hat{p}$
Hermitian linear map
$= x(t)$

$H(p,q)$

$\{x, p\} = 1$

$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}$

- antisymmetry
- linearity
- addition rule
- product rule
- Jacobi identity

Poisson al

$m(\ ,\ )$ ←

associative

$\{\ ,\ \}$ ←

Compatibility

• starting point: H, hamiltonian
H(p, q)

✓ 1. $\{x, p\} = 1$

$X_f\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}$

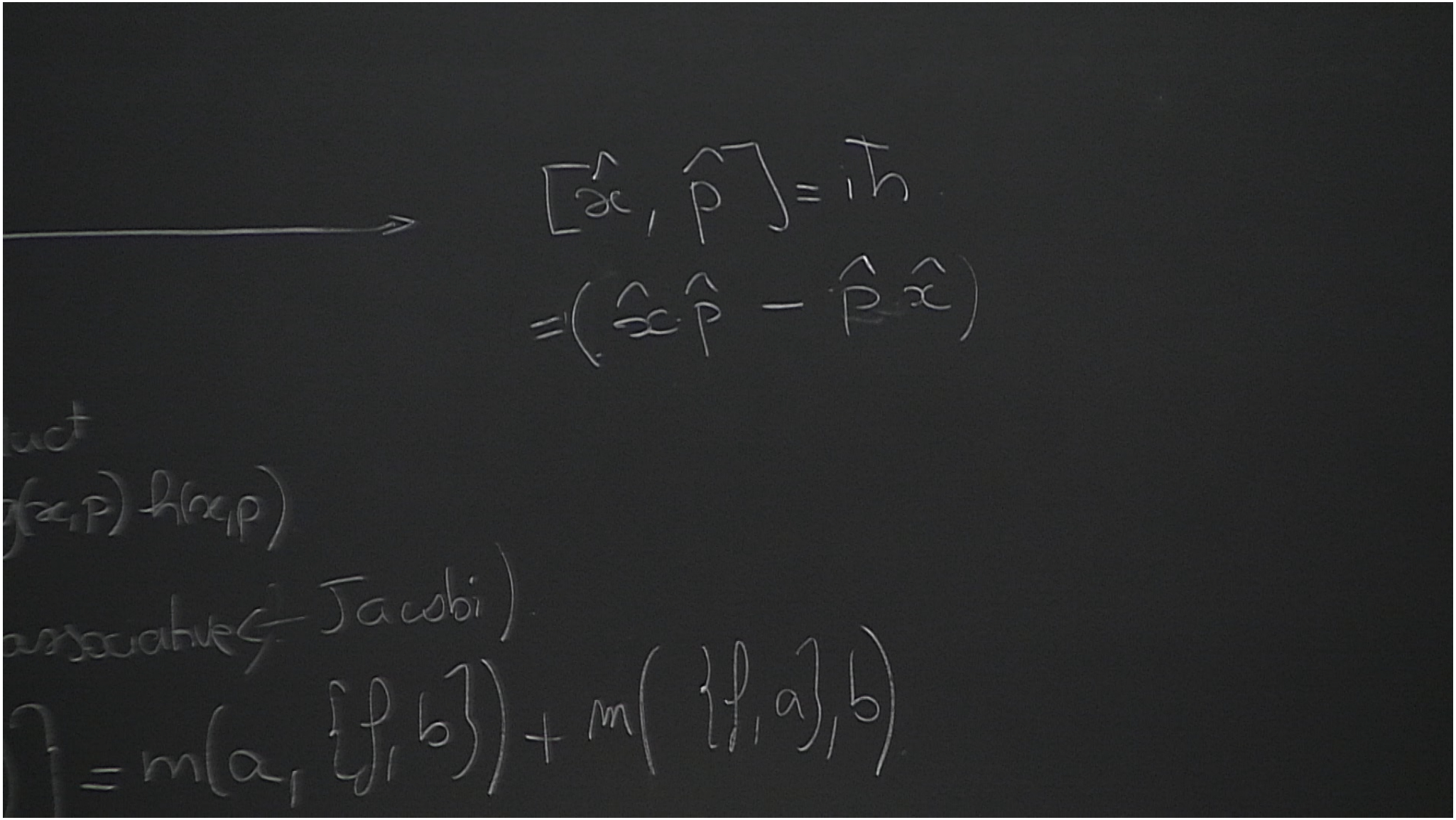
- antisymmetry
- linearity
- addition rule
- product rule
- Jacobi identity

Poisson a

m(,)

associative

{ , }


$[\hat{x}, \hat{p}] = i\hbar$
$= (\hat{x}\hat{p} - \hat{p}\hat{x})$
uct
$g(x,p)\,h(x,p)$
associative ⇐ Jacobi)
$= m(a, \{f, b\}) + m(\{f, a\}, b)$

ex: Free particle?

$\hat{x}(t), \hat{p}(t)$ ∞-matrices!

. number as outcome?

$\langle\psi|A|\psi\rangle$

only depends on abstract operator abstract vector

2.3. Hilbert space.

$[\hat{f}, \hat{H}]$ Heisenberg eq.

ex: Free particle

$\hat{x}(t), \hat{p}(t)$

$\hat{p}$: linear) maps.

Hermitian
linear map

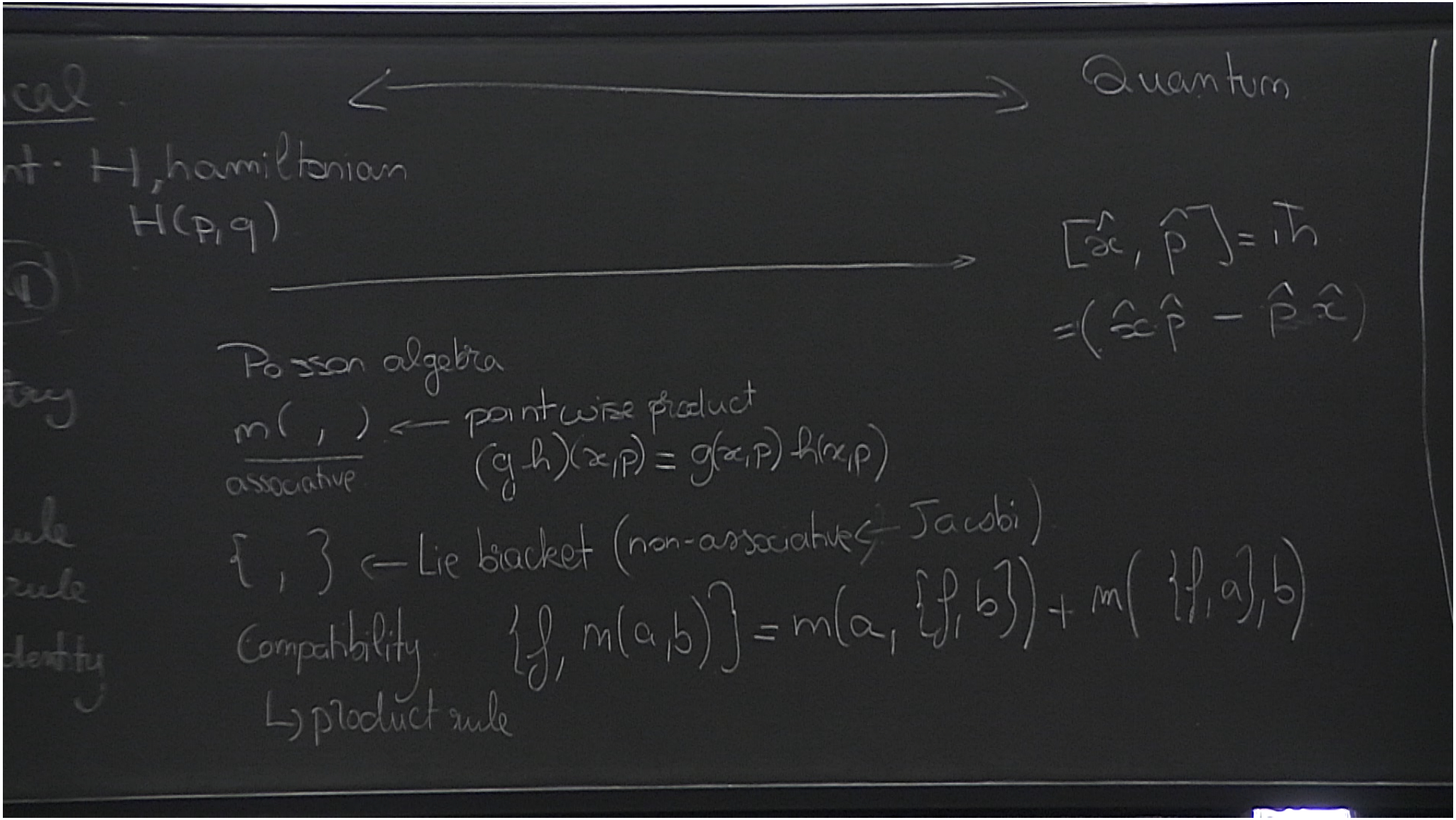
$= \hat{x}$

$\hat{p}$

• number as outcome?

$\langle \psi | \hat{A}(t) | \psi \rangle$

only depends
on abstra
abstrac


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Quantum
nt· H, hamiltonian
H(p, q)
$[\hat{x}, \hat{p}] = i\hbar$
$= (\hat{x}\hat{p} - \hat{p}\hat{x})$
Poisson algebra
m(,) ← pointwise product
associative
$(g \cdot h)(x,p) = g(x,p)\, h(x,p)$
{ , } ← Lie bracket (non-associative ⇐ Jacobi)
Compatibility $\{f, m(a,b)\} = m(a, \{f,b\}) + m(\{f,a\}, b)$
↳ product rule

senberg eq, time-evolution operator, Heisenberg picture, Schrödinger picture

- $\frac{d\hat{A}}{dt} = \frac{1}{i\hbar}[\hat{A}, \hat{H}]$

↳ formal solution $A(t) = \underbrace{e^{iHt/\hbar}}_{U(t)} A(0) \underbrace{e^{-iHt/\hbar}}_{U(t)}$

- Time evolution $U(t) = e^{-iHt/\hbar}$ valid only for H time independent

OP:

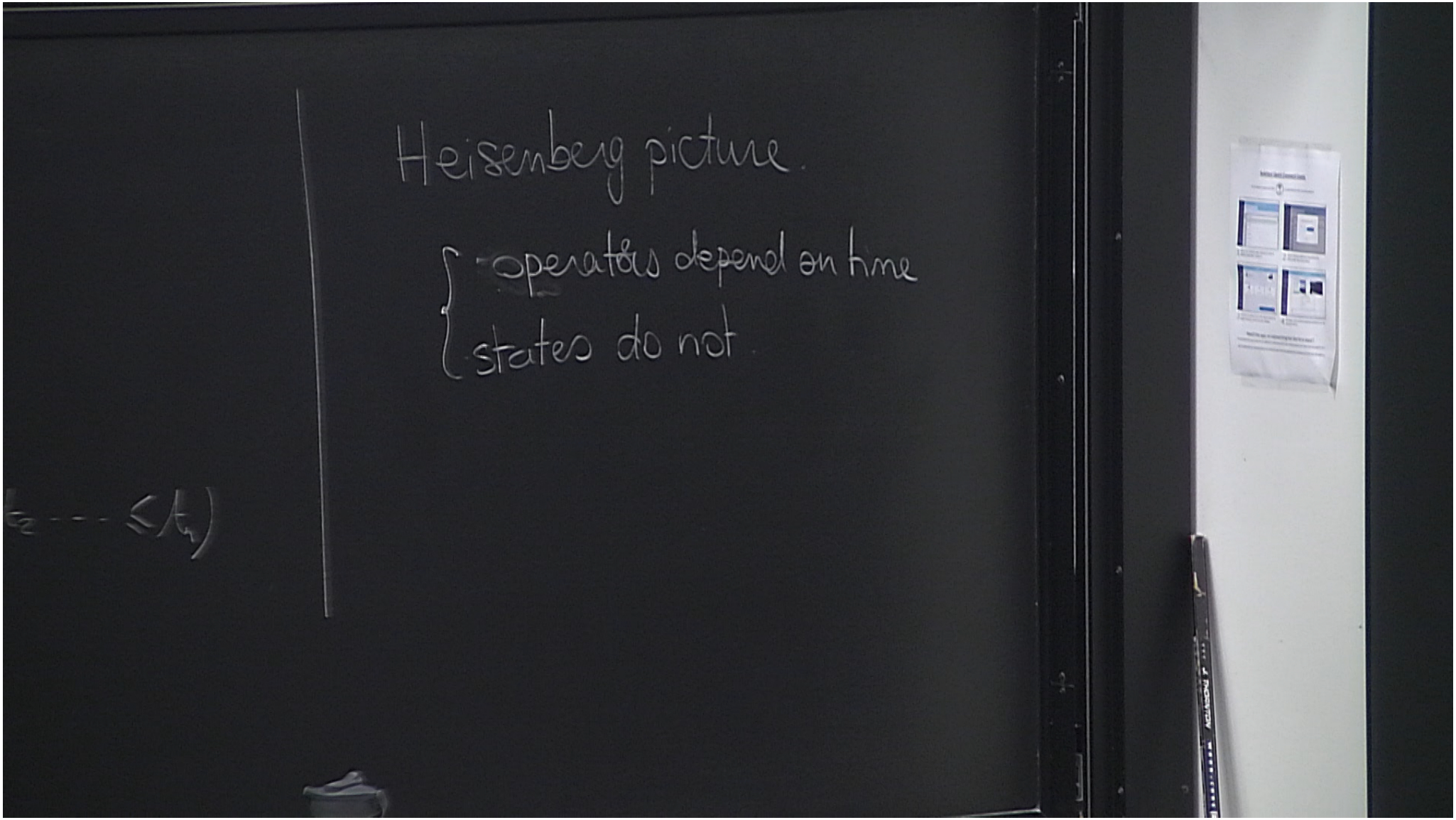
More generally, U is the solution of $\begin{cases} i\hbar \frac{dU}{dt} = U(t)H(t) \\ U(t_0) = id \end{cases}$

Solution: $\boxed{U(t) = \overrightarrow{T}\, e^{\frac{1}{i\hbar}\int_{t_0}^{t} H(t)\,dt}}$

↳ left-to-right chronological ordering of the

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{i\hbar}\right)^n \int_{t_0}^{t} \cdots \int_{t_0}^{t} dt_1 \cdots dt_n \; T\big(H(t_1) \cdots$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{i\hbar}\right)^n \int_{t_0}^{t} dt_n \cdots \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 \; H(t_1) H(t_2$$

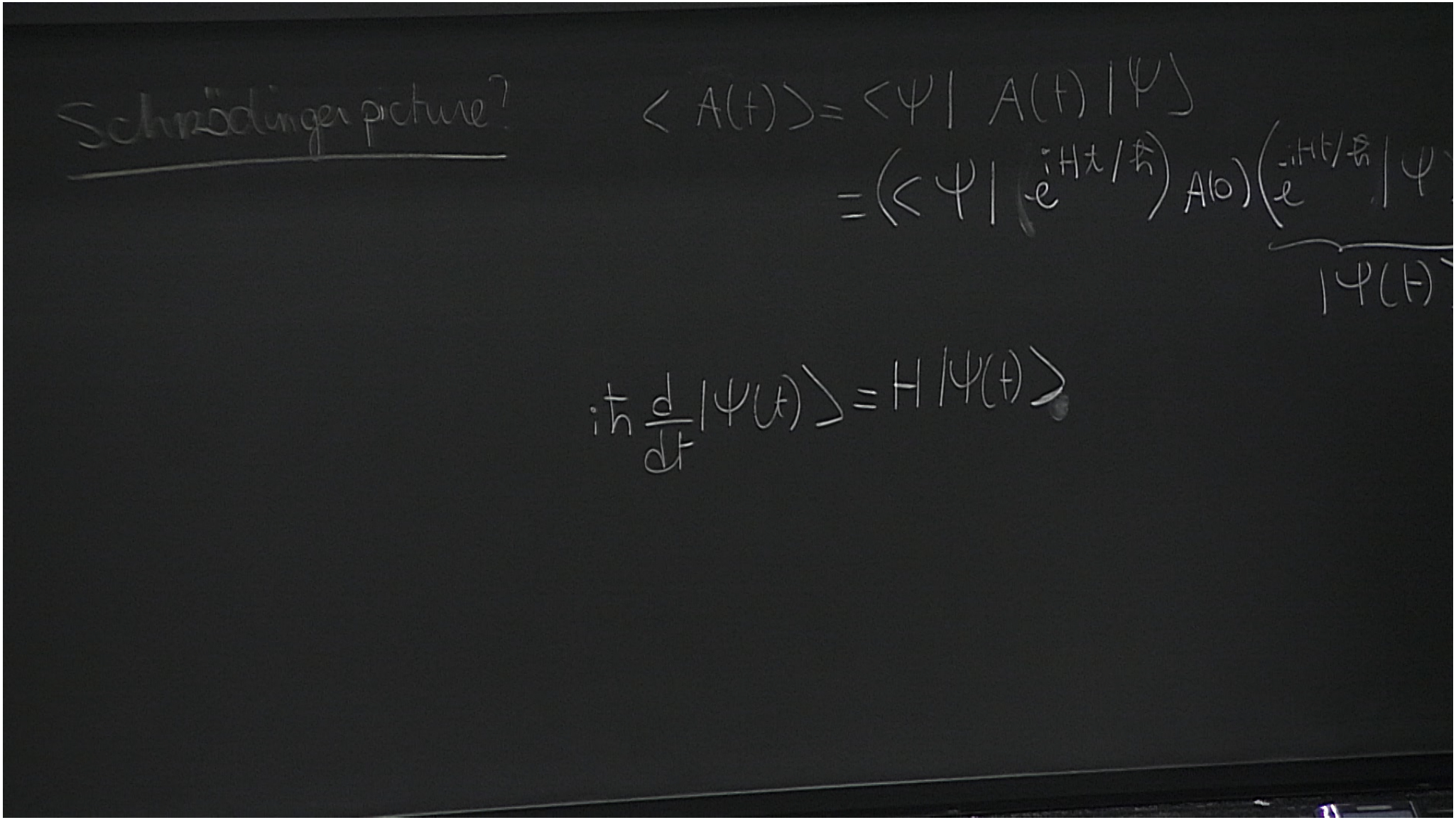

Heisenberg picture
operators depend on time
states do not

$$A(t) = U^{\dagger}(t)\, A(0)\, U(t)$$

Schrödinger picture?

$$\langle A(t) \rangle = \langle \psi |\, A(t)\, | \psi \rangle$$

$$= \langle \psi |$$


Schrödinger picture?
$\langle A(t) \rangle = \langle \psi | A(t) | \psi \rangle$
$= (\langle \psi | e^{iHt/\hbar}) A(0) (e^{-iHt/\hbar} | \psi$
$|\psi(t)\rangle$
$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$

$$\langle A(t) \rangle = \langle \psi | A(t) | \psi \rangle$$

$$= \left(\langle \psi | e^{iHt/\hbar} \right) A(0) \underbrace{\left(e^{-iHt/\hbar} | \psi \rangle \right)}_{|\psi(t)\rangle}$$

Schrodinger state

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \quad \text{(H time independent)}.$$

$\langle\psi| e^{iHt/\hbar} A(0) \left(e^{-iHt/\hbar}|\psi\rangle\right)$

$\underbrace{\qquad\qquad\qquad\qquad}$

$|\psi(t)\rangle$. Schrodinger state

(H time independent).

More generally

$|\psi(t)\rangle = U(t)|\psi\rangle$

$H|\psi(t)\rangle$

evolution operator, Heisenberg picture, Schrödinger picture

$H]$

olution $A(t) = e^{iHt/\hbar} A(0) e^{-iHt/\hbar}$ to.

$U(t)$

$t) = e^{-iHt/\hbar}$ valid only for H time independent.

U is the solution $i\hbar \frac{dU}{dt} = U(t) H(t)$

$U(t_0) = id$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \qquad (H \text{ time independent}).$$

$$i\hbar \frac{dU}{dt}|\psi\rangle_H = U(t)H(t)|\psi\rangle_H = U(t)H(t)U^\dagger(t)$$

$$i\hbar \frac{d}{dt}\left(U|\psi\rangle_H\right) = U(t)H(t)U^\dagger(t)\left(U|\psi\rangle_H\right)$$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \quad (H \text{ time independent}).$$

$$i\hbar \frac{dU}{dt}|\psi\rangle_H = U(t)H(t)|\psi\rangle_H = U(t)H(t)U^\dagger(t)$$

$$i\hbar \frac{d}{dt}\underbrace{\left(U|\psi\rangle_H\right)}_{|\psi(t)\rangle_S} = \underbrace{U(t)H(t)U^\dagger(t)}_{H_S(t)}\underbrace{\left(U|\psi\rangle_H\right)}_{|\psi(t)\rangle_S}$$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$ (H ti

Time-dependent H

$$i\hbar \frac{dU}{dt}|\psi\rangle_H = U(t)H(t)|\psi\rangle_H =$$

$$i\hbar \frac{d}{dt}\underbrace{(U|\psi\rangle_H)}_{|\psi(t)\rangle_S} = \underbrace{U(t)H(t)U^\dagger(t)}_{H_S(t)}$$

$= H|\psi(t)\rangle$ (H time independent).

$= U(t)H(t)|\psi\rangle_H = U(t)H(t)\overbrace{U^\dagger(t)U(t)}^{\mathbb{1}}|\psi\rangle_H$.

$|\psi\rangle_H$

$= \underbrace{U(t)H(t)U^\dagger(t)}_{H_S(t)} \underbrace{(U|\psi\rangle_H)}_{|\psi(t)\rangle_S}$

$|\psi(t)\rangle_S$

mberg eq, time-evolution operator, Heisenberg picture, Schrödinger picture

$$\frac{d\hat{A}}{dt} = \frac{1}{i\hbar}[\hat{A}, \hat{H}]$$

$$\hat{x}(t) = U^{+}(t)\,\hat{x}(t_0)\,U(t)$$

t_0

↳ formal solution $A(t) = e^{iHt/\hbar} A(0) \underbrace{e^{-iHt/\hbar}}_{U(t)}$

Time evolution $U(t) = e^{-iHt/\hbar}$ valid only for H time independent

QP

More generally, U is the solution

$$i\hbar \frac{dU}{dt} = U(t)H(t)$$

$$U(t_0) = id$$

Interaction picture (Dirac).

$$\hat{H}_S(t) = \underbrace{\frac{\hat{p}(t_0)^2}{2m} + \frac{m\omega^2}{2}\hat{x}(t_0)^2}_{H_{0,S}} + \underbrace{e^{-at^2}\hat{p}(t_0)\hat{x}(t_0)^3\hat{p}(t_0)}_{H_{1,S}}$$

The operators $A_I(t)$ evolve according to the "easy" Hamil

The states $|\psi_I(t)\rangle$ evolve with respect to the difficult Ham

$$A_I(t) = U_{0,S}^{+}(t) A_S U_{0,S}(t)$$

$$i\hbar \frac{\partial}{\partial t} |\psi_I(t)\rangle = H_{1,I} |\psi_I(t)\rangle$$

check tutorial 1.