Title: PSI 2019/2020 - Quantum Theory (Branczyk) - Lecture 6

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Collection: PSI 2019/2020 - Quantum Theory (Branczyk/Dupuis)

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Schmidt Decomposition

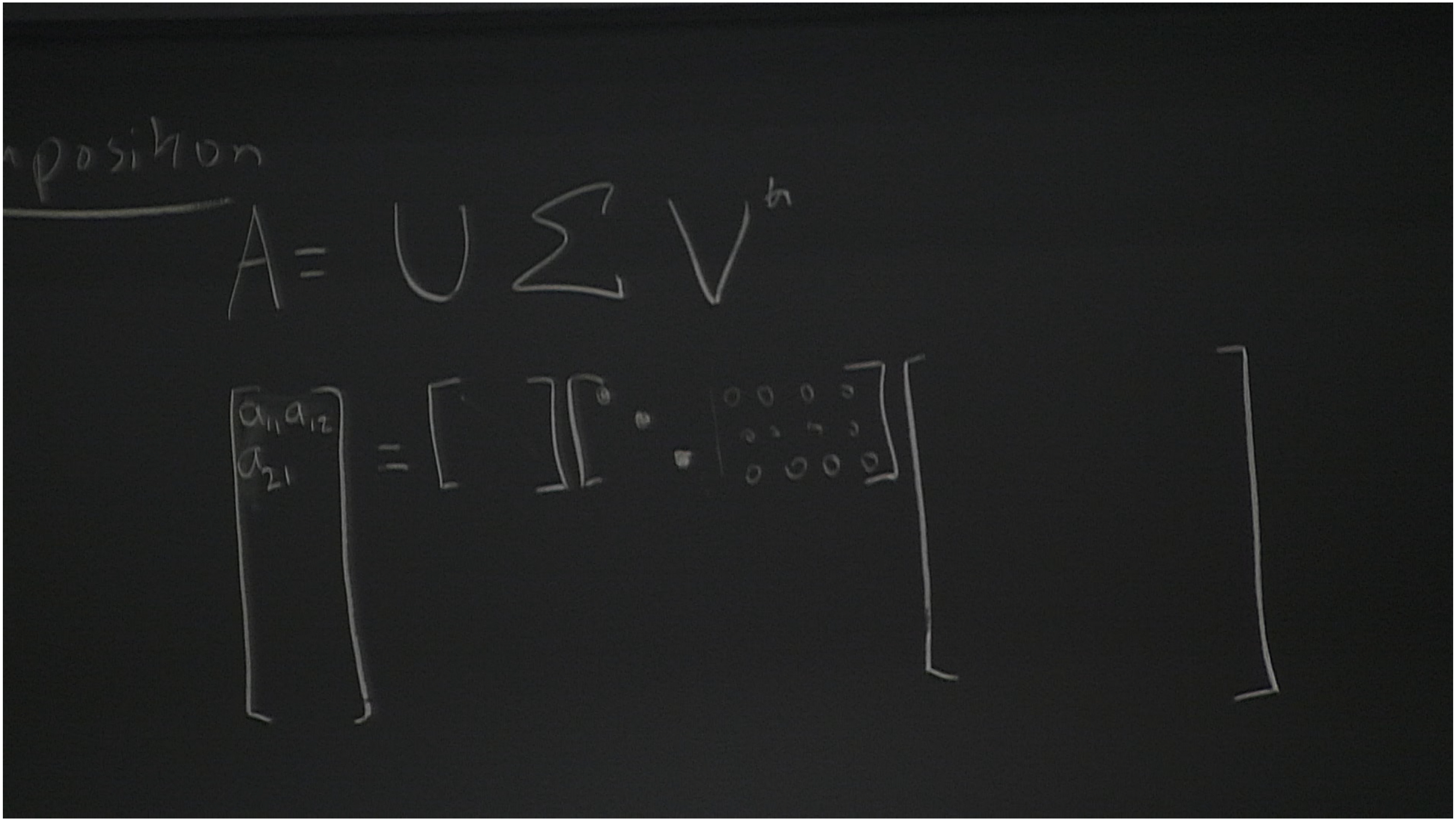
$$|\psi\rangle_{AB} = \sum_{ij} a_{ij} |e_i\rangle_A |f_j\rangle_B$$

$$= \sum_k \lambda_k |\phi_k\rangle_A |\varphi_k\rangle_B$$

$$A = U \Sigma V^{\dagger}$$

$$\rho_A = \mathrm{Tr}_B \left[|\psi\rangle\langle\psi|_{AB} \right] = \sum_k \lambda_k^2 |\phi_k\rangle\langle\phi_k|_A$$

$$\rho_B = \mathrm{Tr}_A \left[|\psi\rangle\langle\psi|_{AB} \right] = \sum_k \lambda_k^2 |\varphi_k\rangle\langle\varphi_k|_B$$


nposition
$A = U \Sigma V^{\dagger}$
$a_{11}\ a_{12}$
a_{21}

per mixture"

proper mixture"

prepare $|0\rangle$ with $p_0 = \frac{1}{2}$

$|1\rangle$ with $p_1 = \frac{1}{2}$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

Preparations.

First motivation: probabilistic $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Second motivation: reduced $\rho_A = \mathrm{Tr}_B\left[|\psi\rangle\langle\psi|_{AB}\right]$

prepare $|0\rangle$ with $p_0 = \frac{1}{2}$

$|1\rangle$ with $p_1 = \frac{1}{2}$

"proper mixture"

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$\langle\Psi|_{AB}]$

"improper mixture"

prepare $|\Psi\rangle_{AB} = \dfrac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}$

mixture"

prepare $|0\rangle$ with $p_0 = \frac{1}{2}$

$|1\rangle$ with $p_1 = \frac{1}{2}$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

mixture"

prepare $|\psi\rangle_{AB} = \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}$

$$\rho_A = \mathrm{Tr}_B\left[|\psi\rangle\langle\psi|_{AB}\right] = \frac{1}{2}|0\rangle\langle 0|_A + \frac{1}{2}|1\rangle\langle 1|_B$$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$ "proper mixture"

$$\rho_A = \mathrm{Tr}_B\left[|\psi\rangle\langle\psi|_{AB}\right]$$ "improper mixture"

prepare $|0\rangle$

$|1\rangle$

$\rho = \frac{1}{2}|0\rangle\langle 0|$

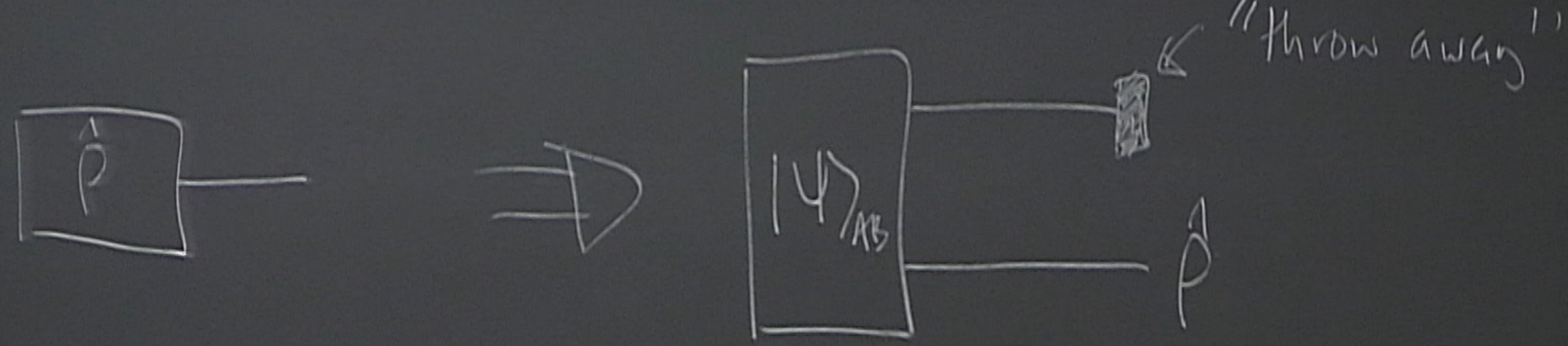
prepare $|\psi\rangle_{AB} =$

$\rho_A = \mathrm{Tr}_B\left[|\psi\rangle\langle\psi|_{AB}\right] =$

Purification

Any density operator ρ_A can be realized as the reduced state associated with a pure state on an extended Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

That pure state is known as a "purification".



$\hat{\rho}$
$|\psi\rangle_{AB}$
"throw away"
$\hat{\rho}$

Postulate 1

Preparations: $\hat{\rho}$

$$\mathrm{Tr}[\hat{\rho}] = 1$$

Postulate 2:

Transformations: $\mathcal{E}(\hat{\rho}) = \sum_k A_k \rho A_k^\dagger$

$$\sum_k A_k^\dagger A_k = \mathbb{1}$$

Postulate 3.

Measurements: $\{E_m\}$

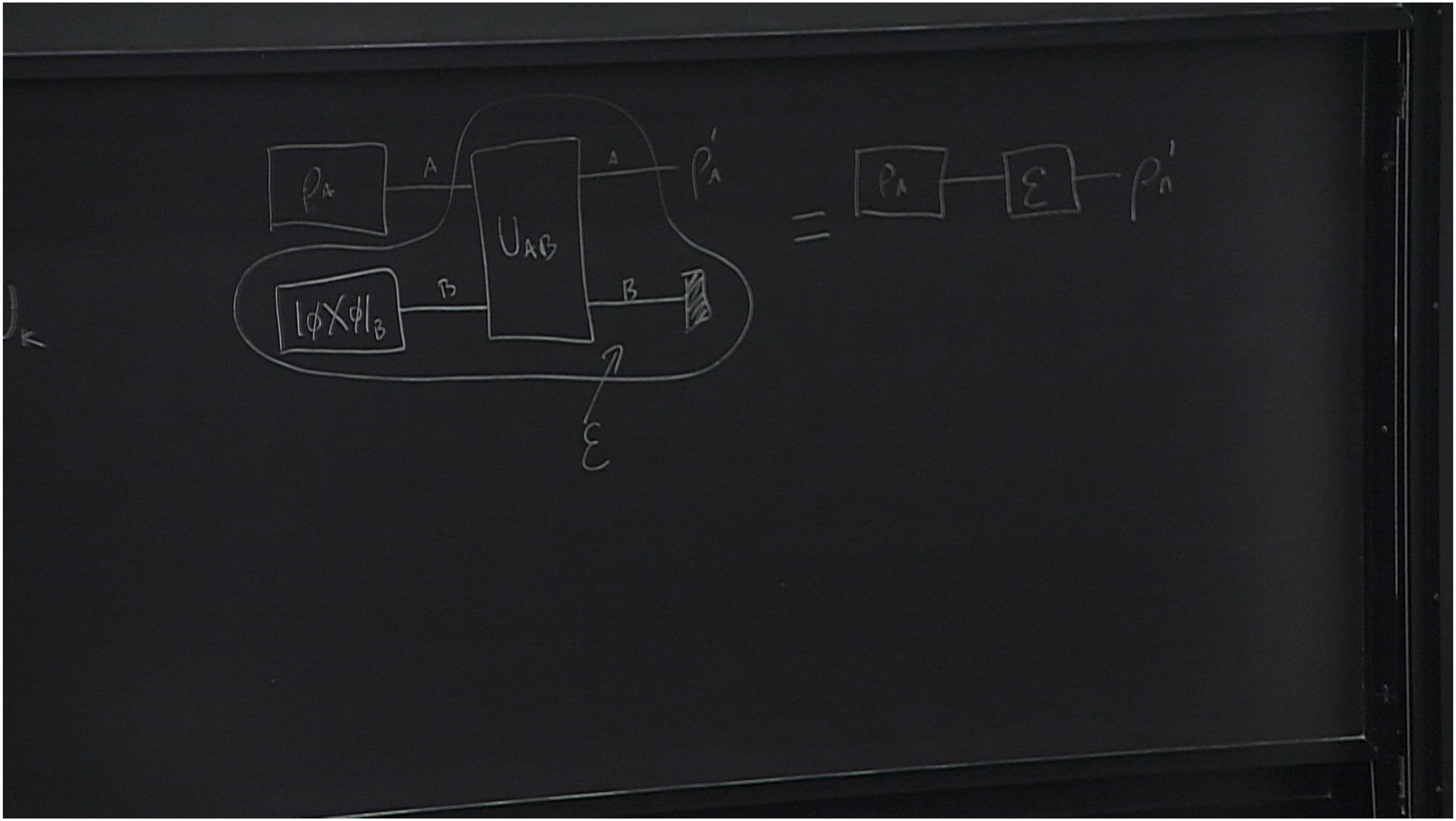
$$\sum_m E_m = \mathbb{1}$$

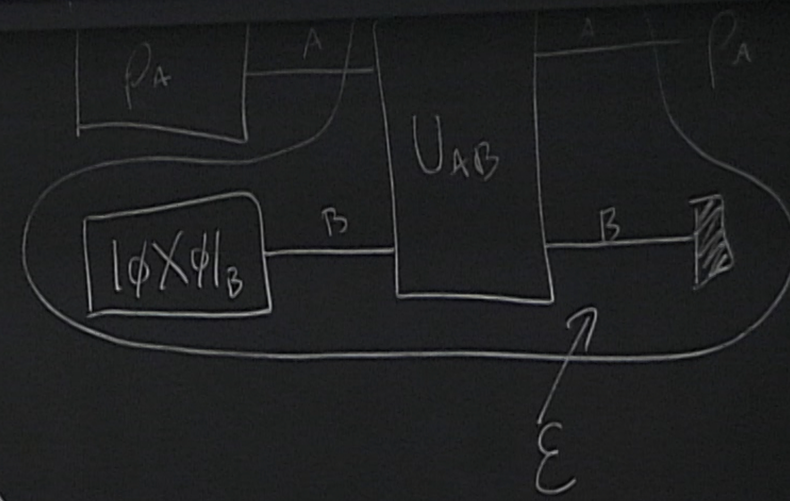
$$\Pr(m) = \mathrm{Tr}[\hat{\rho} E_m]$$

Transformations

1st Motivation: $\sum_i p_i U_i \rho U_i^\dagger = \mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger$ where $A_k := \sqrt{p_k}\, U_k$

2nd Motivation: $\mathrm{Tr}_B\left[U_{AB}\left(\rho_A \otimes |\phi\rangle\langle\phi|_B\right)U_{AB}^\dagger\right]$


ρ_A
A
U_{AB}
A
ρ'_A
$|\phi\rangle\langle\phi|_B$
B
B
$\mathcal{E}$
=
ρ_A
$\mathcal{E}$
ρ'_A



$$\sum_k A_k \rho A_k^\dagger \quad \text{where } A_k = \sqrt{p_k}\, U_k$$

$$U_{AB}^\dagger \Big] = \sum_k \underbrace{\langle k|_B U_{AB} |\phi\rangle_B}_{A_k} \rho_A \underbrace{\langle\phi|_B U_{AB}^\dagger |k\rangle}_{A_k^\dagger}$$

$$= \sum_k A_k \rho A_k^\dagger \quad \text{where } A_k = \langle k|_B U_{AB} |\phi\rangle_B$$

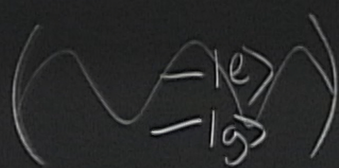
atom inside a cavity

coupling

$$|e\rangle_A|0\rangle_F \rightarrow \cos\left(\frac{gt}{2}\right)|e\rangle|0\rangle + i\sin\left(\frac{gt}{2}\right)|g\rangle|1\rangle$$

$$\rho_A = |e\rangle\langle e|_A \Rightarrow \cos\left(\frac{gt}{2}\right)^2|e\rangle\langle e| - \sin\left(\frac{gt}{2}\right)^2|g\rangle\langle g|$$

Example. atom inside a cavity



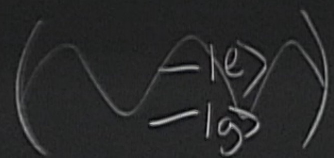
$$|e\rangle_A |0\rangle_F \rightarrow \cos\Big(\frac{\overset{\text{coupling}}{g}t}{2}\Big)|e\rangle|0\rangle + i\sin\Big(\frac{g}{}$$

$$\rho_A = |e\rangle\langle e|_A \rightarrow \cos\Big(\frac{gt}{2}\Big)^2 |e\rangle\langle e| + s$$

For atom coupled to free EM field,
Can have effective description:

$$A_0 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$$

Example: atom inside a cavity



$$|e\rangle_A |0\rangle_F \rightarrow \cos\left(\frac{\overset{\text{coupling}}{g}t}{2}\right)|e\rangle|0\rangle + i\sin\left(\frac{gt}{2}\right)|g$$

$$\rho_A = |e\rangle\langle e|_A \rightarrow \cos\left(\frac{gt}{2}\right)^2 |e\rangle\langle e| + \sin\left(\frac{gt}{2}\right.$$

Stinespring Dilation Theorem: Any $\mathcal{E}(\rho_A)$ can be expressed of U_{AB} on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. i.e. $\mathcal{E}(\rho_A) = \mathrm{Tr}_B\left[U_{AB}(\rho_A \otimes \sigma_B)U_{AB}^\dagger\right]$

For atom coupled to free EM field,

Can have effective description:

$$A_0 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$$

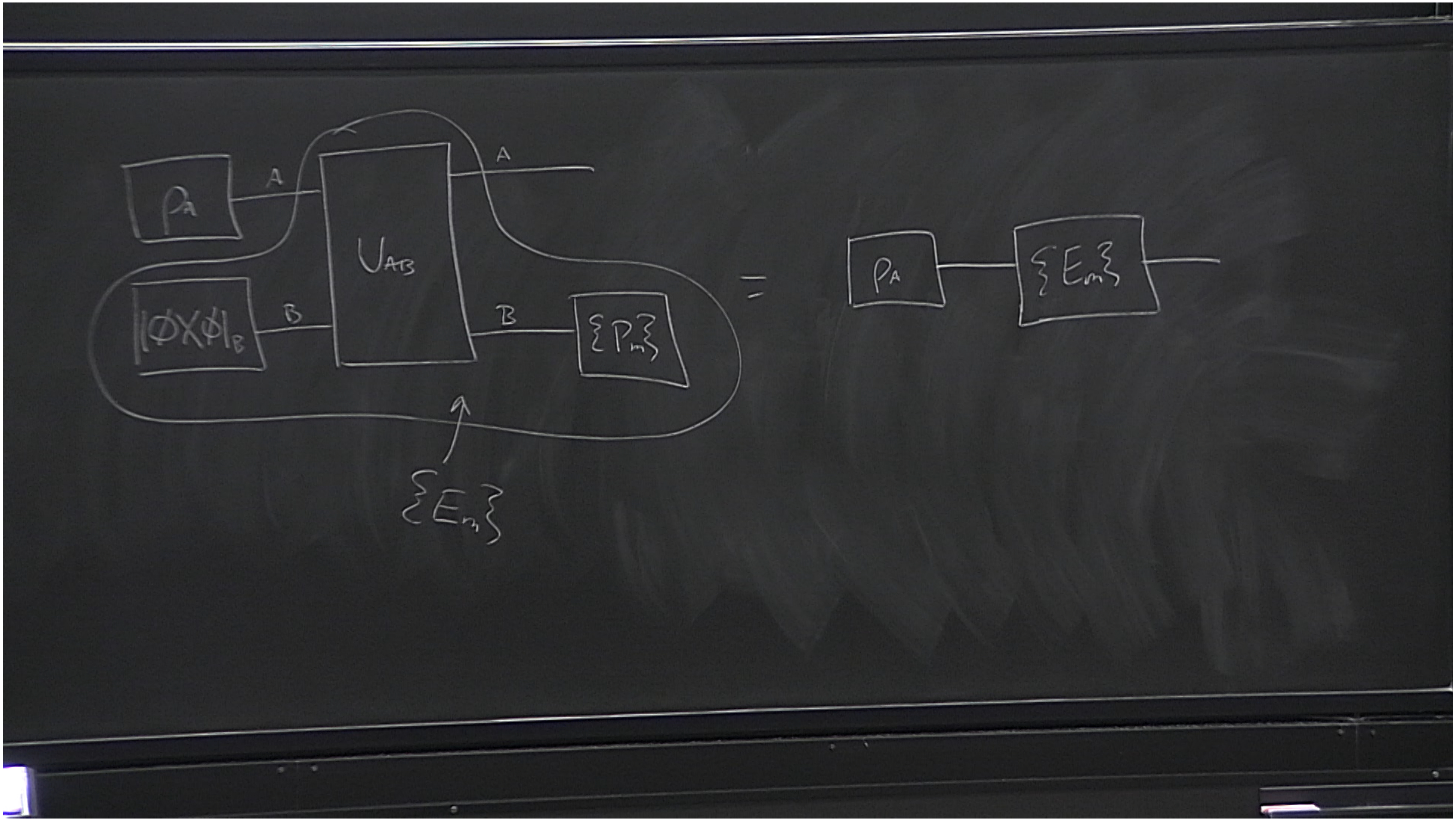
$i \sin\left(\frac{gt}{2}\right)|g\rangle|1\rangle$

$\rangle\langle e| + \sin\left(\frac{gt}{2}\right)^2 |g\rangle\langle g|$

be expressed as the reduced action

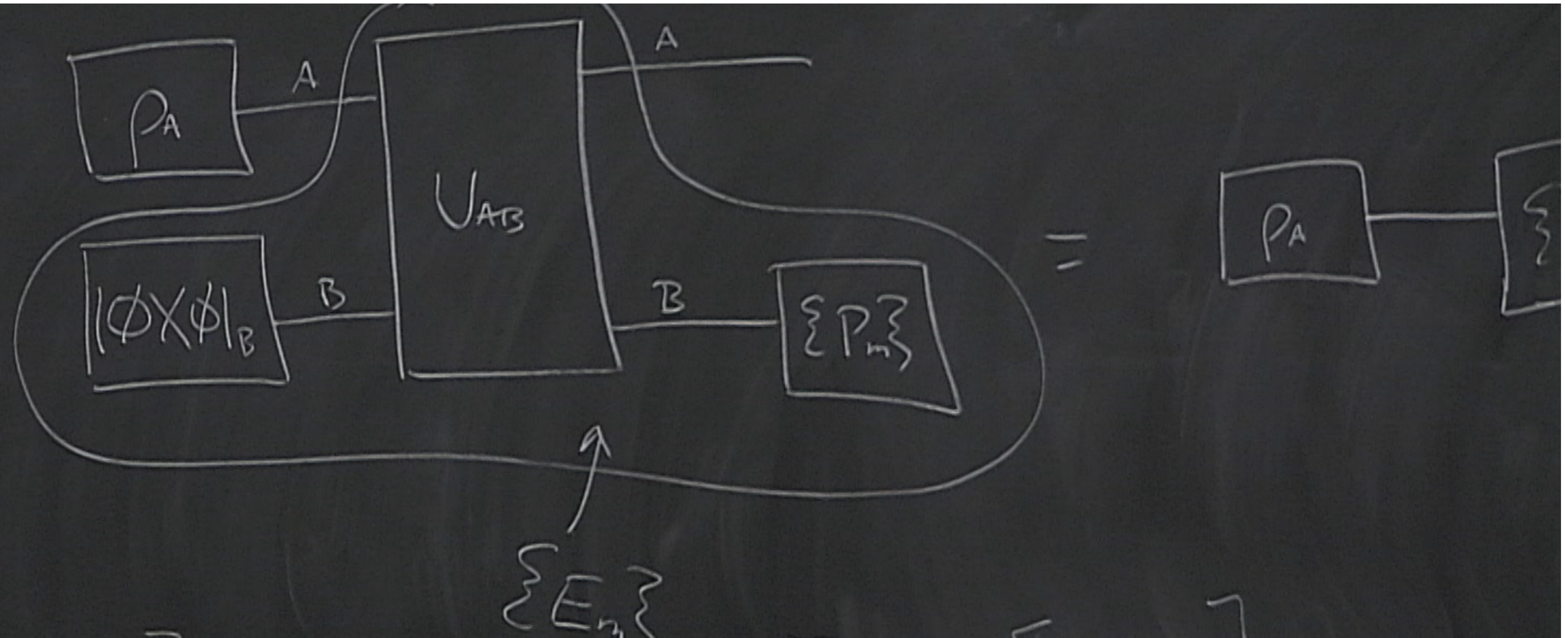
$\otimes \sigma_B) U_{AB}^\dagger]$

<u>Not unique</u>

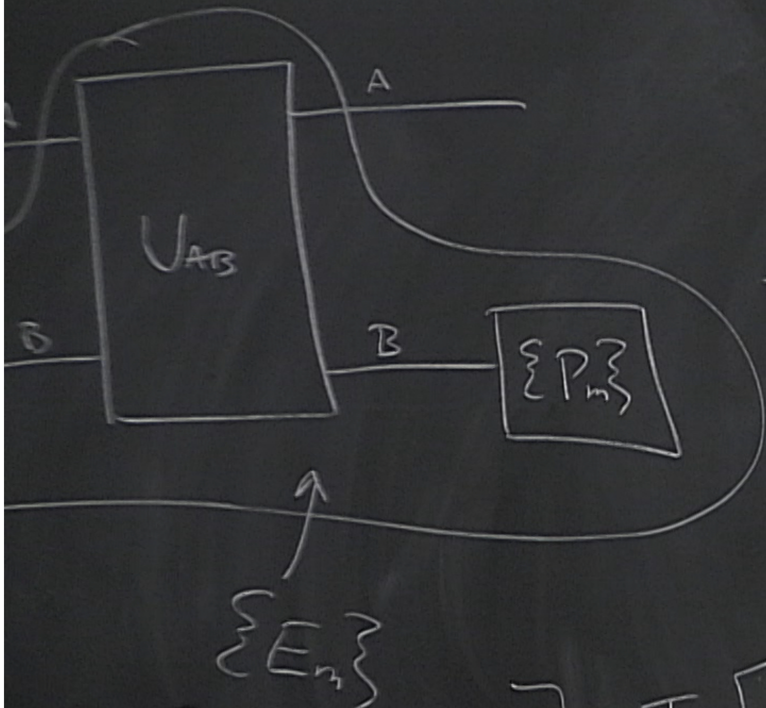


ρ_A
A
U_{AB}
A
$|\phi\rangle\langle\phi|_B$
B
B
$\{P_m\}$
=
ρ_A
$\{E_m\}$
$\{E_m\}$

t space.



$$\ldots{}_B|\phi\rangle_B\,\rho_A\,\underbrace{\langle\phi|_B\,U^\dagger_{AB}\,|m\rangle_B}_{M^\dagger}\Big] = \mathrm{Tr}\left[M_m^\dagger M_m\,\rho_A\right] = \mathrm{Tr}\left[E_m\,\rho_A\right]$$



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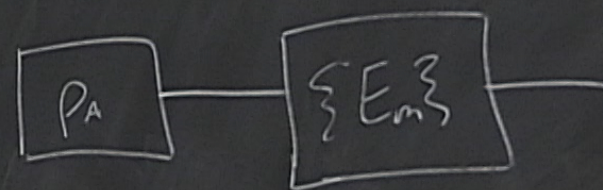


$$\mathrm{Tr}[M_m^\dagger M_m \rho_A] = \mathrm{Tr}[E_m \rho_A] \quad \text{where } E_m = M_m^\dagger M_m$$

$$M_m = {}_B\langle m | U_{AB} | \phi \rangle_B$$

Measurements

Motivation 1: probabilistic projective meas.

Motivation 2: Projective meas on larger Hilbert space.

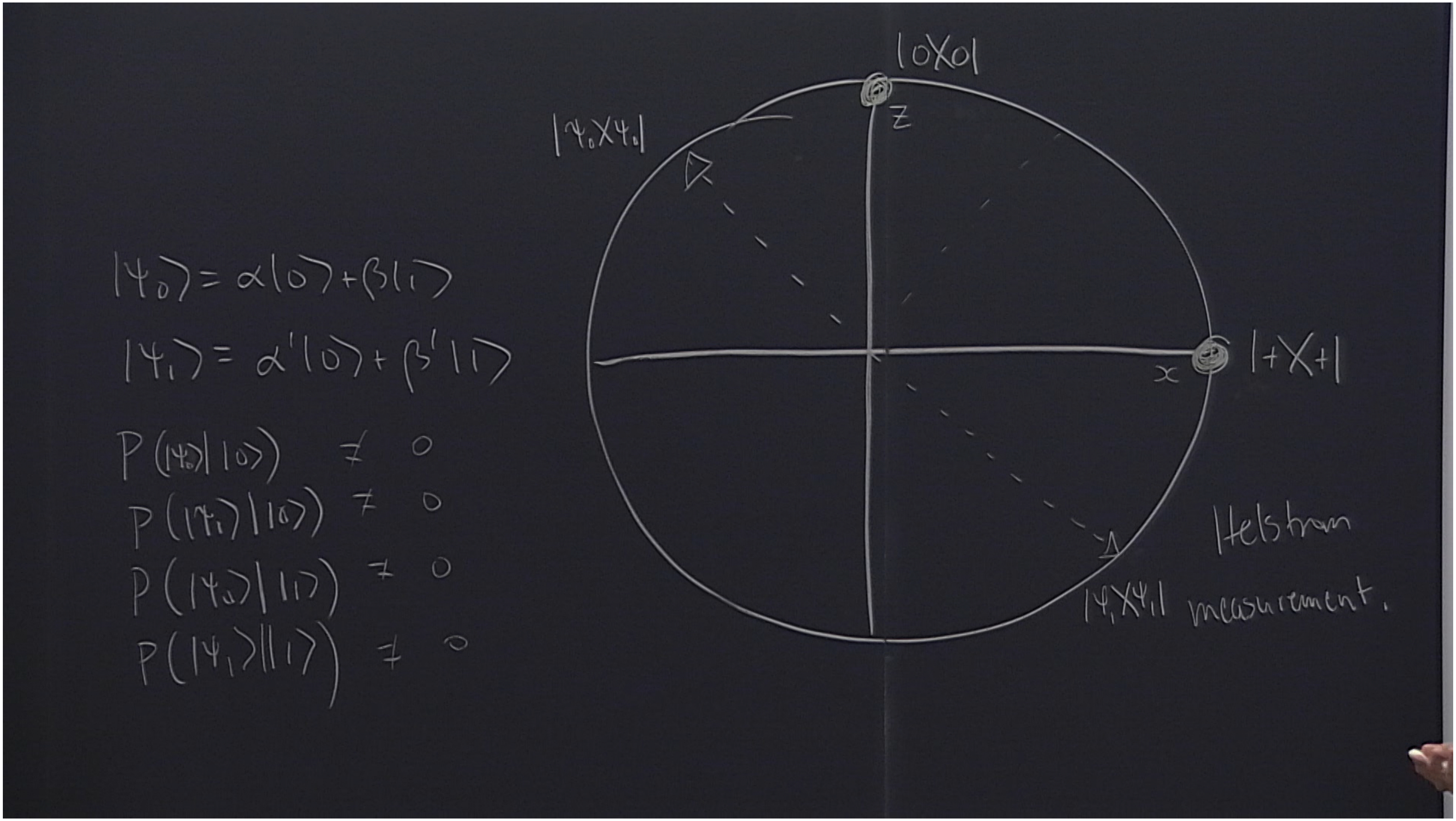
$$\Pr(m) = \mathrm{Tr}\left[U_{AB} \left(\rho_A \otimes |\phi\rangle\langle\phi|_B \right) U_{AB}^{\dagger} \left(\mathbb{1}_A \otimes |m\rangle\langle m|_B \right) \right] = \mathrm{Tr}\Big[\underbrace{\langle m|_B U_{AB} |\phi\rangle_B}_{M_m} \rho_A \ldots$$

Neumark's (Naimark's) Dilation Theorem

For any $\{E_m\}$ and any ρ_A, there exists a $\{P_m\}$ acting on $\mathcal{H}_{AB}$ and a state $|\phi\rangle\langle\phi|_B$ s.t.

$$\mathrm{Tr}\left[E_m \rho_A\right] = \mathrm{Tr}\left[(\rho_A \otimes |\phi\rangle\langle\phi|) P_m^{(AB)}\right]$$

$\{P_m^{(AB)}\}$ can always be realized as $U_{AB}(\mathbb{1}_A \otimes P_m^{(B)})$


$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$
$|\psi_1\rangle = \alpha'|0\rangle + \beta'|1\rangle$
$P(|\psi_0\rangle \,|\, |0\rangle) \neq 0$
$P(|\psi_1\rangle \,|\, |0\rangle) \neq 0$
$P(|\psi_0\rangle \,|\, |1\rangle) \neq 0$
$P(|\psi_1\rangle \,|\, |1\rangle) \neq 0$
$|0\rangle\langle 0|$
z
$|\psi_0\rangle\langle\psi_0|$
$|+\rangle\langle +|$
x
$|\psi_1\rangle\langle\psi_1|$
Helstrom measurement.

Generalized meas. are useful.

Consider $E_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle\langle 1|$

$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} |-\rangle\langle -|$

$E_{fail} = \mathbb{1} - E_0 - E_1$

$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$

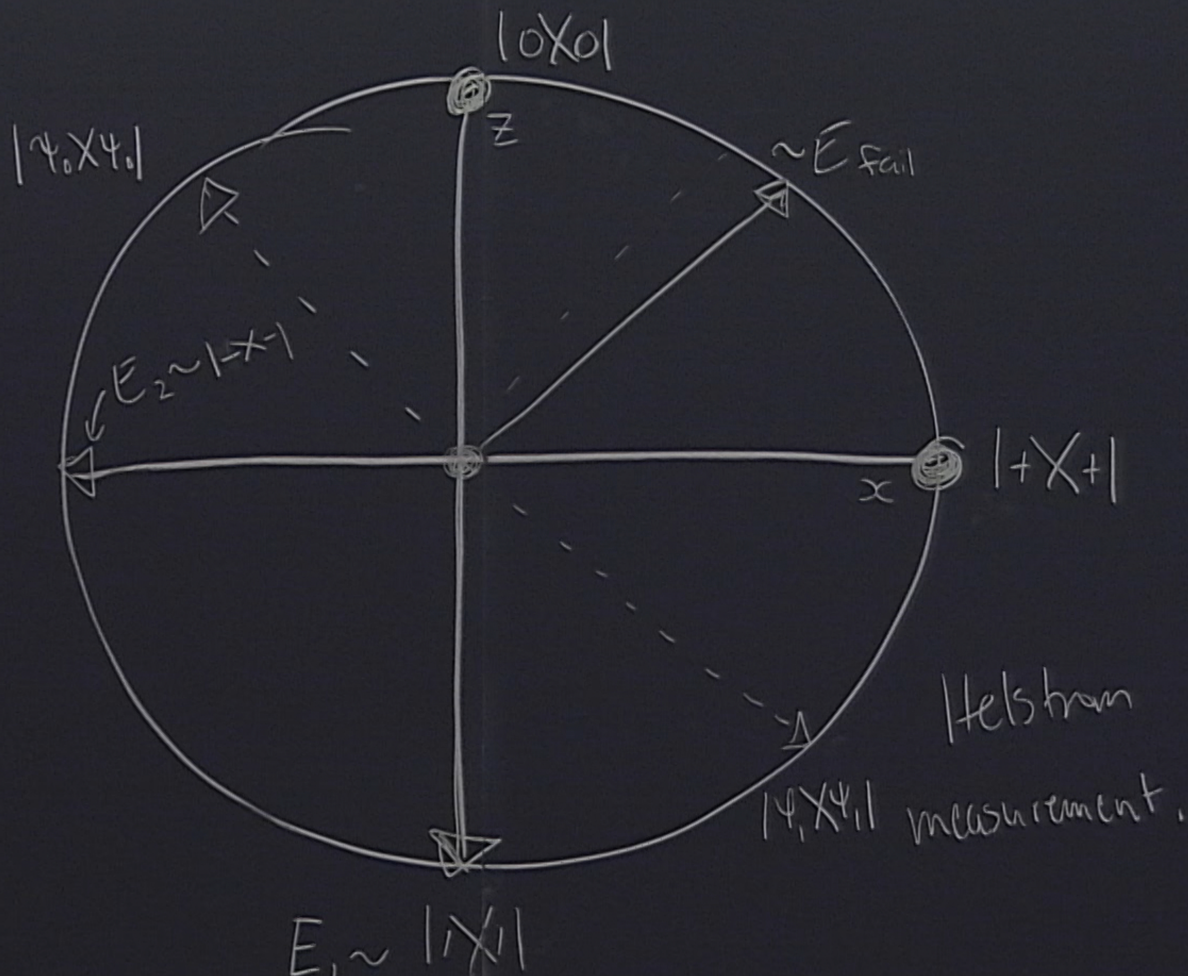
$|\psi_1\rangle = \alpha'|0\rangle + \beta'|1\rangle$

$P(|\psi_0\rangle \,|\, |0\rangle) \neq 0$

$P(|\psi_1\rangle \,|\, |0\rangle) \neq 0$

$P(|\psi_0\rangle \,|\, |1\rangle) \neq 0$

$P(|\psi_1\rangle \,|\, |1\rangle) \neq 0$

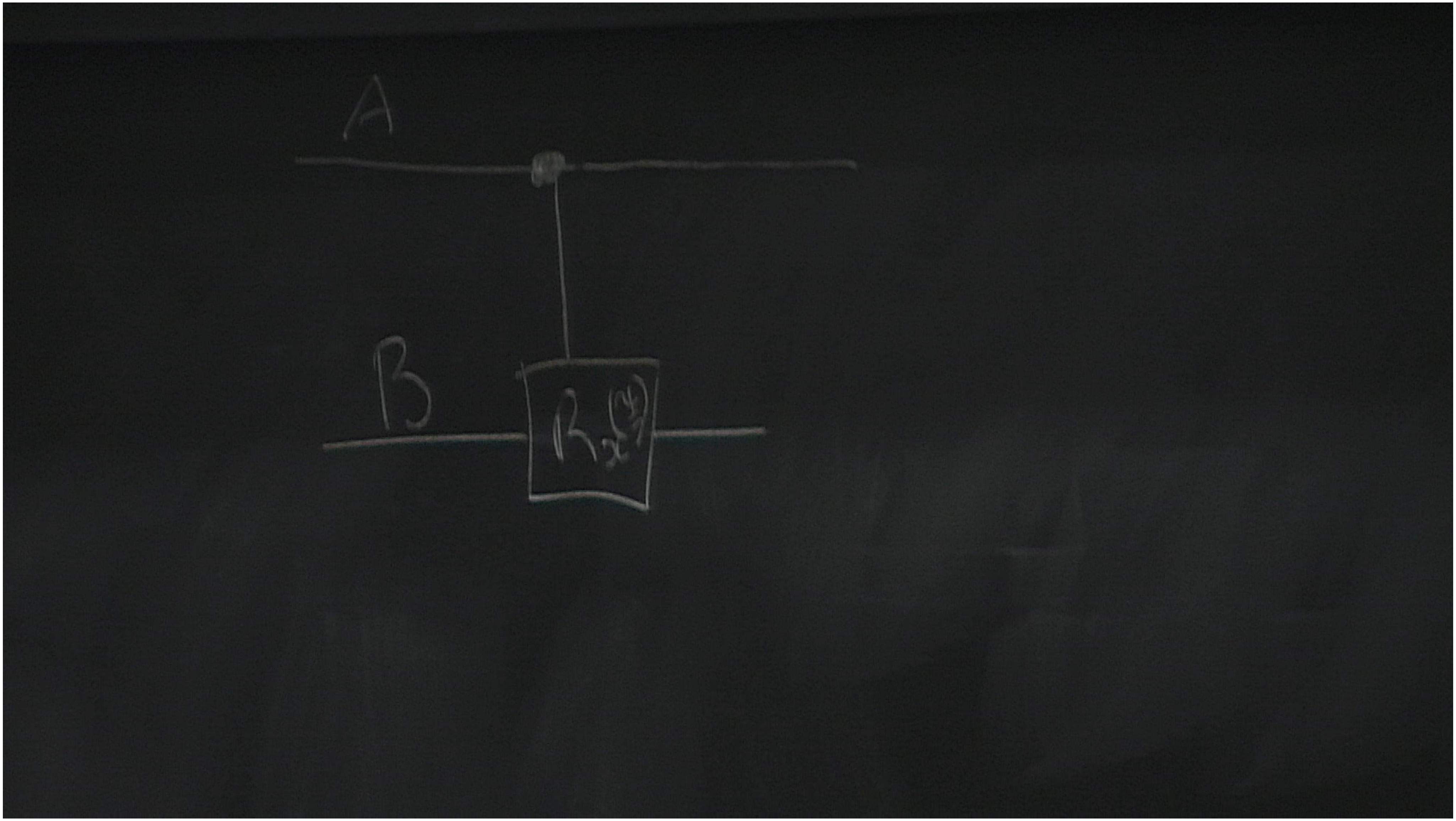


$U_{AB} \neq U_A \otimes U_B$

Weak measurement

Conider the following entangling unitary.

$$U_{AB} = |0\rangle\langle 0|_A \otimes \mathbb{1}_B + |1\rangle\langle 1|_A \otimes \underbrace{\left(\cos(\chi)\,\mathbb{1}_B + i\sin(\chi)(\sigma_x)_B\right)}_{R_x\left(\frac{\chi}{2}\right)}$$


A
B

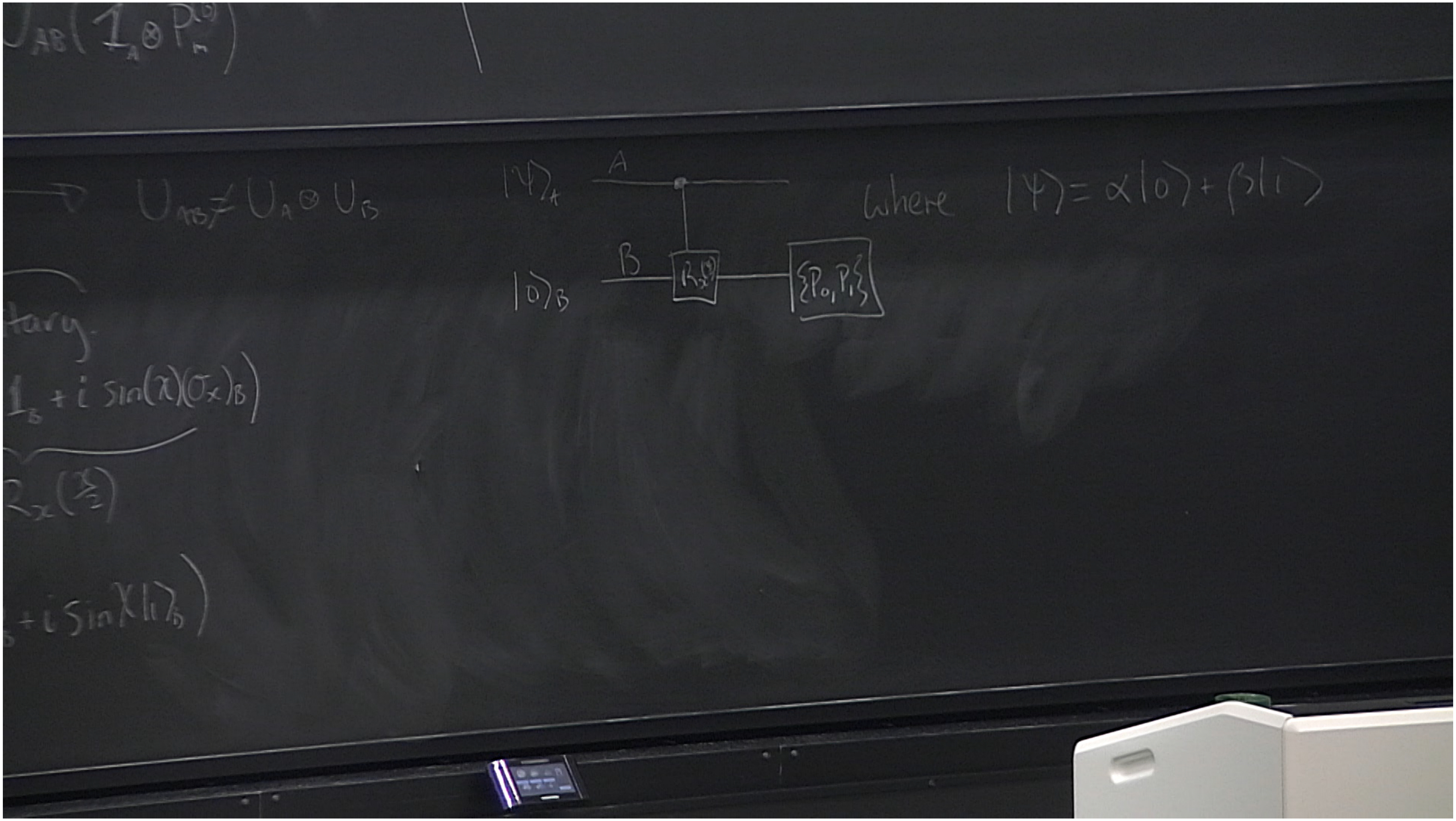
$U_{AB} \neq U_A \otimes U$

Weak measurement

Conider the following entangling unitary.

$$U_{AB} = |0\rangle\langle 0|_A \otimes \mathbb{1}_B + |1\rangle\langle 1|_A \otimes \underbrace{\left(\cos(\chi)\, \mathbb{1}_B + i \sin(\chi) (\sigma_x)_B \right)}_{R_x\left(\frac{\chi}{2}\right)}$$

$$U_{AB} |\psi\rangle_A |0\rangle_B = \alpha |0\rangle_A |0\rangle_B + \beta |1\rangle_A \left(\cos\chi |0\rangle_B + i \sin\chi |1\rangle_B \right)$$


$U_{AB}(\mathbb{1}_A \otimes P_m^{(B)})$
$\Rightarrow U_{AB} \neq U_A \otimes U_B$
$|\psi\rangle_A$
A
B
$|0\rangle_B$
$R_x(\theta)$
$\{P_0, P_1\}$
where $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
tary.
$\mathbb{1}_B + i\sin(\chi)(\sigma_x)_B)$
$R_x(\frac{\chi}{2})$
$+ i\sin\chi|1\rangle_B)$

no info no disturbance

max info max disturbance (equal to Proj meas on A) if $\chi = 0$ Output state $(\alpha|0\rangle_A + \beta|1\rangle_A)\otimes|0\rangle_B$

if $\chi = \pi/2$ " " $\alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B$