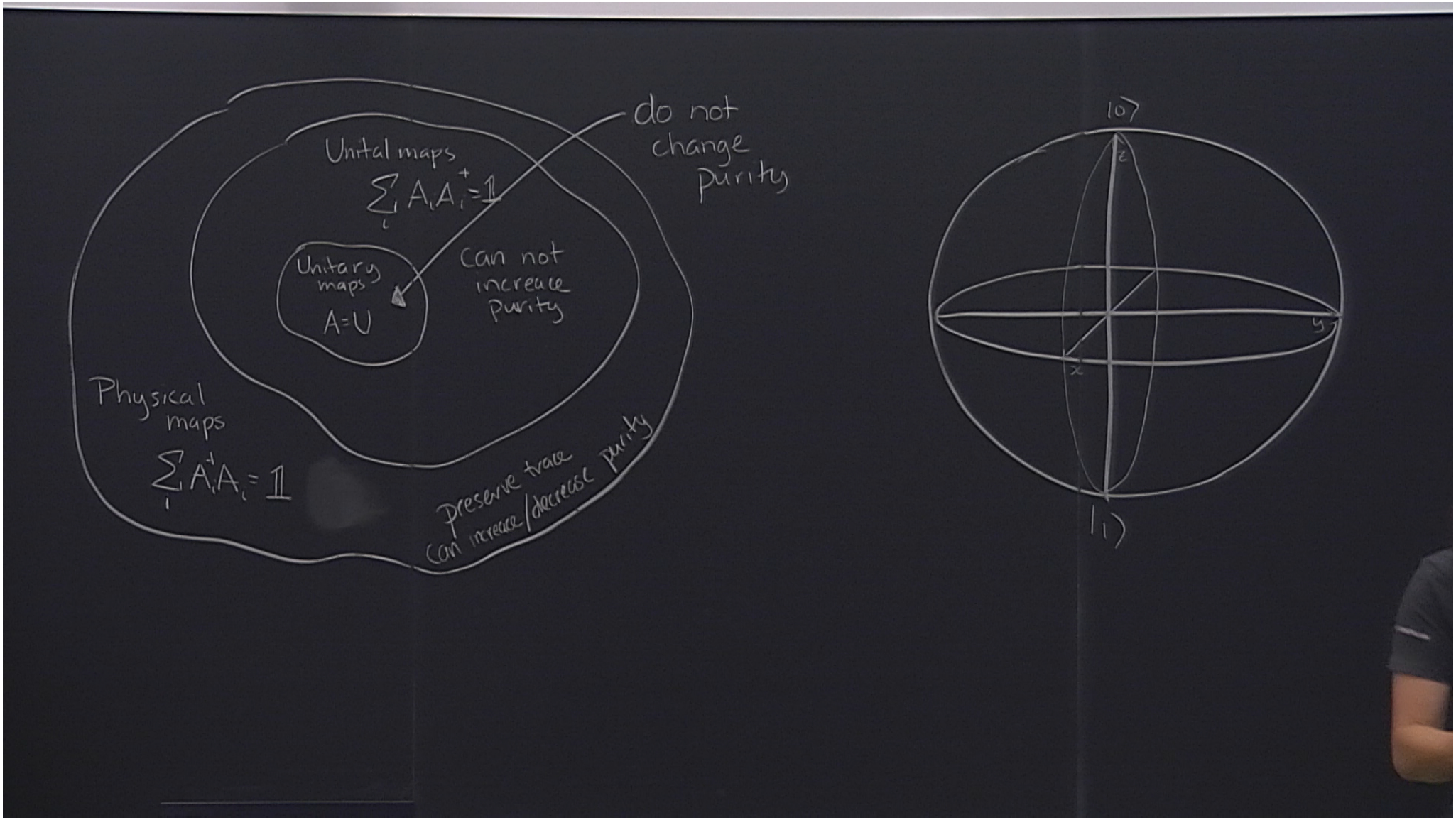
Title: PSI 2019/2020 - Quantum Theory (Branczyk) - Lecture 5

Speakers: Agata Branczyk

Collection: PSI 2019/2020 - Quantum Theory (Branczyk/Dupuis)

Date: September 09, 2019 - 9:00 AM

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Unital maps
$\sum_i A_i A_i^\dagger = \mathbb{1}$
Unitary maps
$A = U$
do not change purity
Can not increase purity
Physical maps
$\sum_i A_i^\dagger A_i = \mathbb{1}$
preserve trace
Can increase/decrease purity
$|0\rangle$
$|1\rangle$
z
x
y

Composite systems.

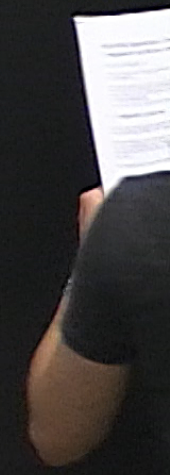
$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$

$\mathcal{H}_A$ $\quad$ $\mathcal{H}_B$

$\{|a_k\rangle\}$ $\quad$ $\{|b_\ell\rangle\}$

Want all possible combinations of basis states.

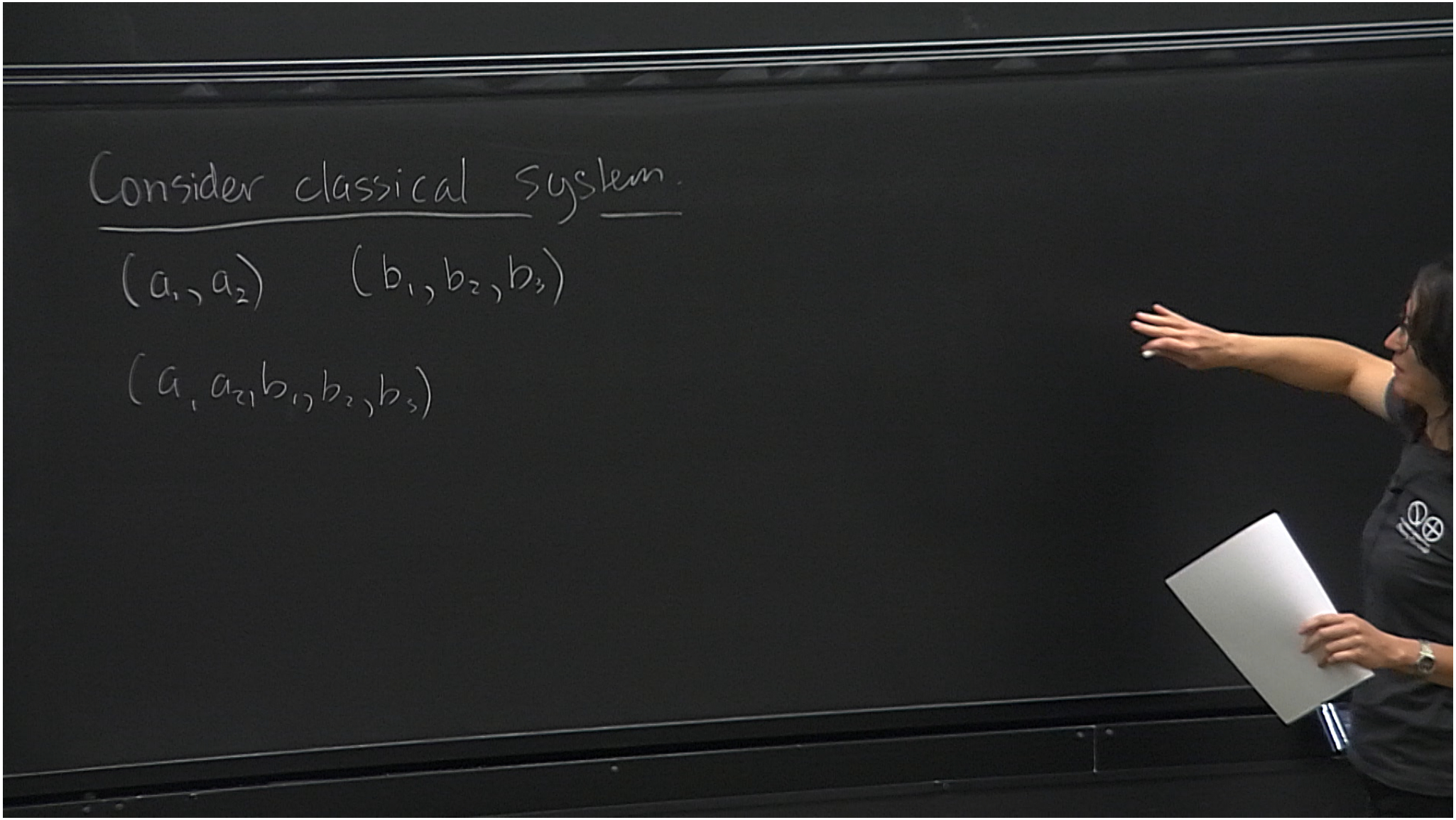
Right structure → tensor product.

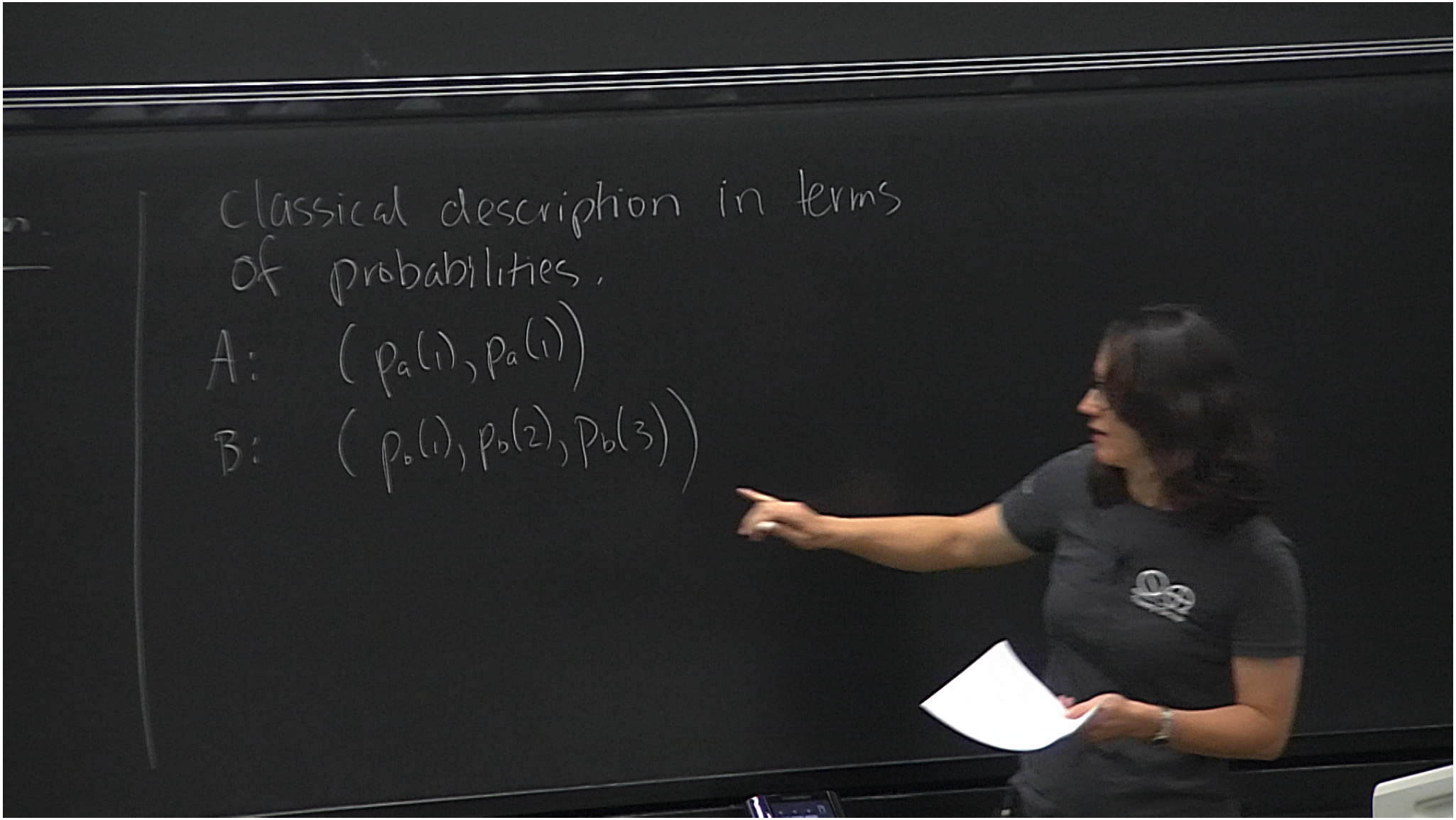


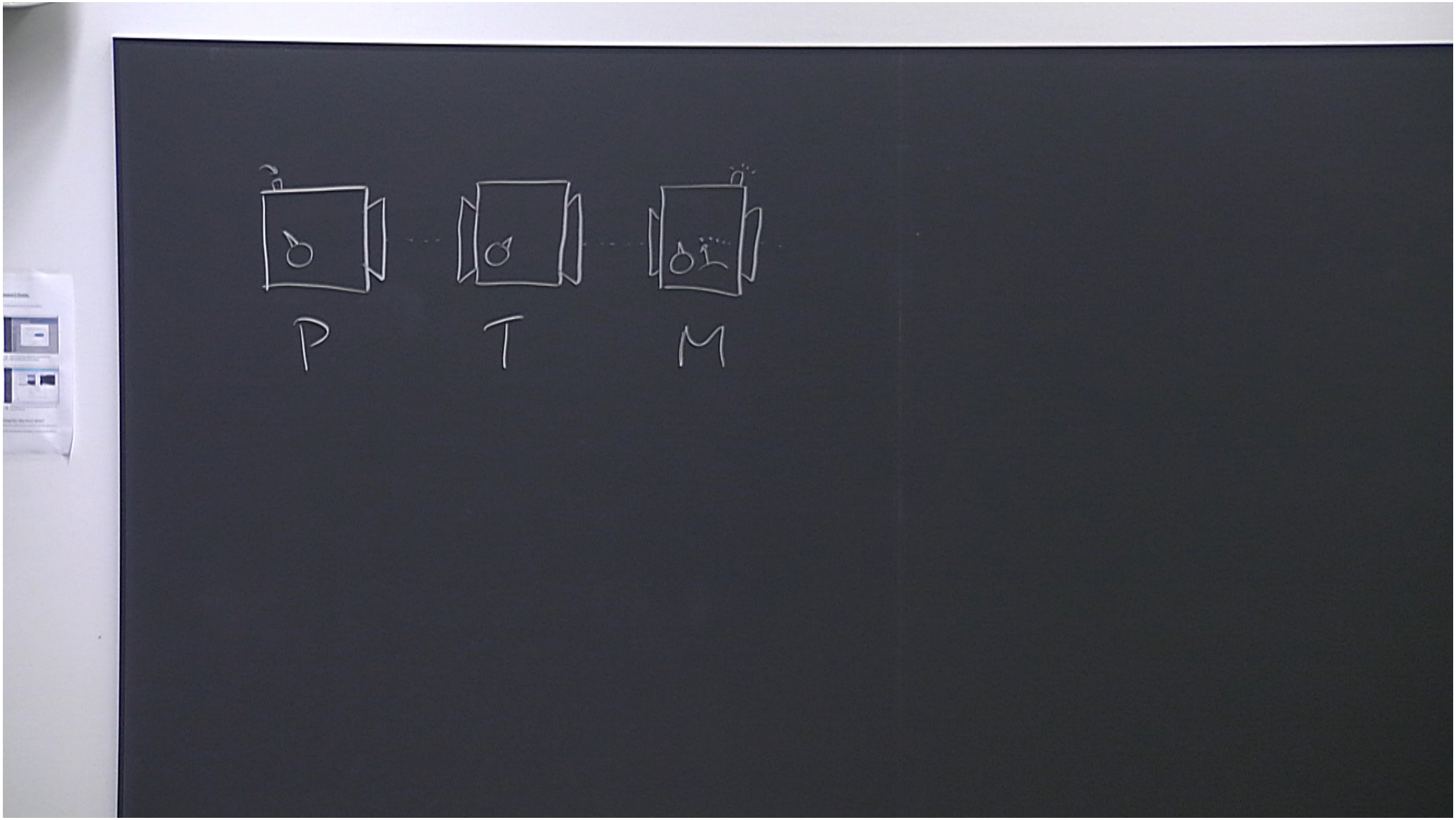
$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ The dimension $\dim(\mathcal{H}_{AB}) = \dim(\mathcal{H}_A)\dim(\mathcal{H}_B)$

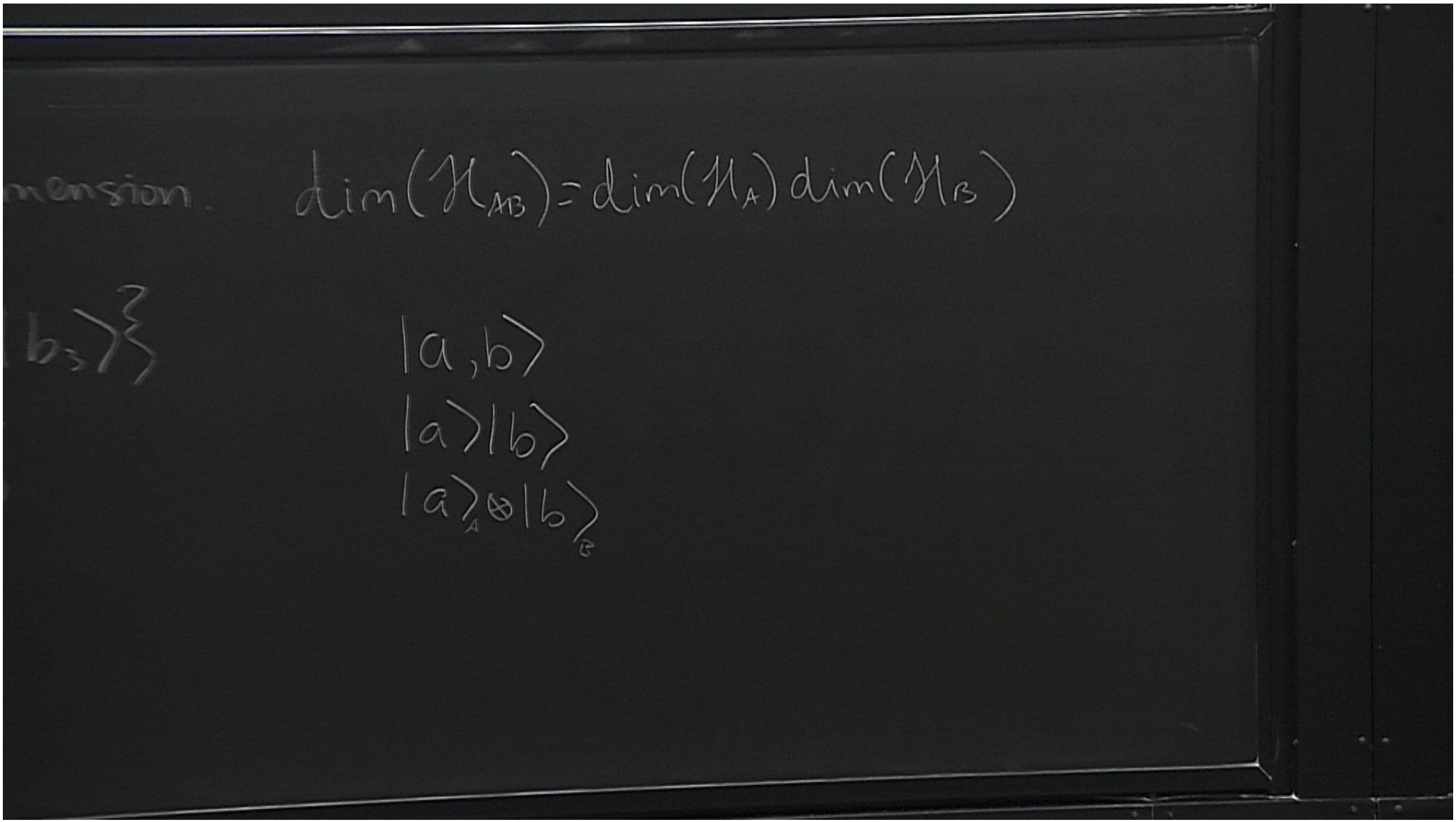
e.g. $\{|a_1\rangle, |a_2\rangle\}$ $\{|b_1\rangle, |b_2\rangle, |b_3\rangle\}$

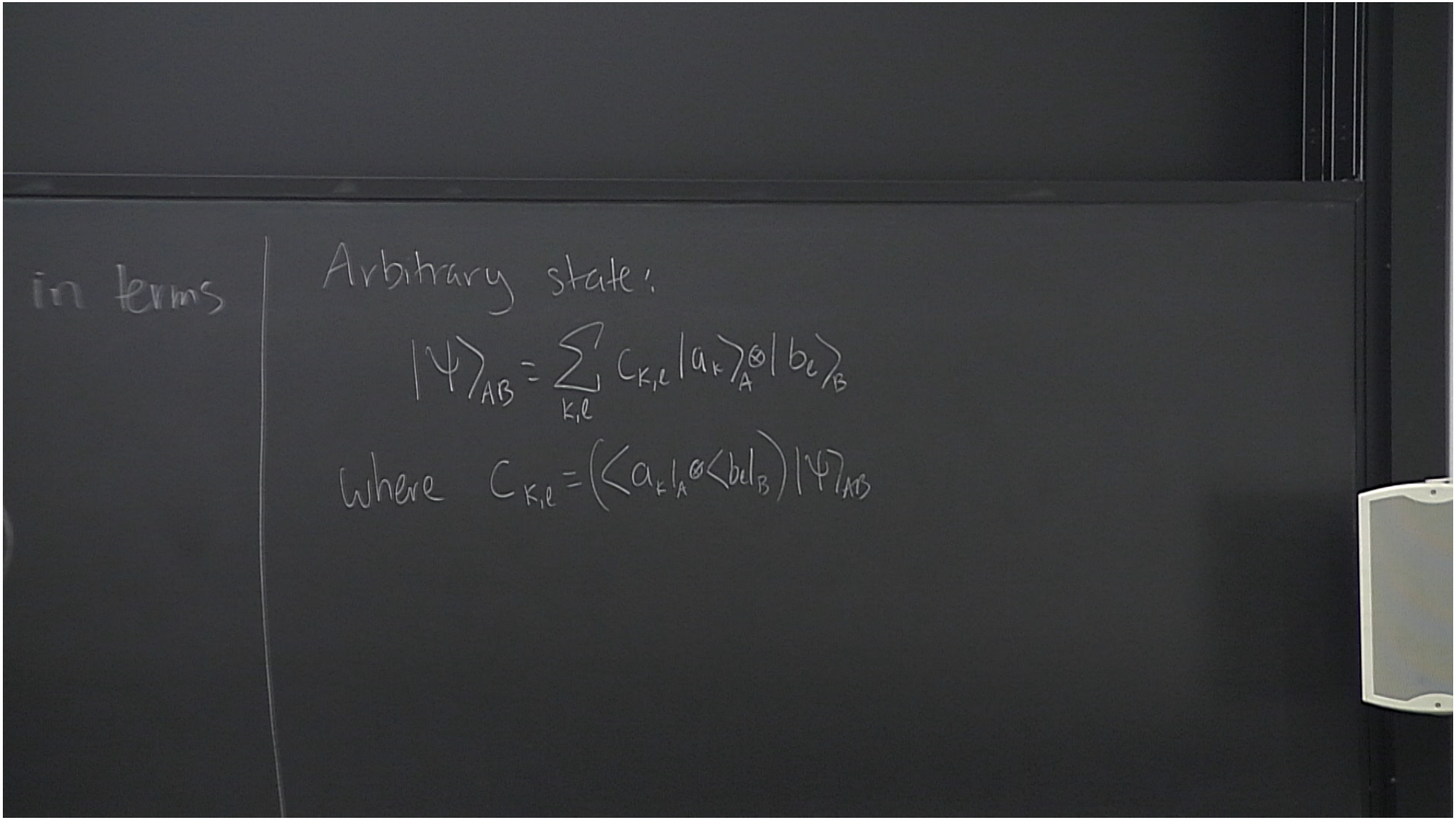
Composite: $\{\underbrace{|a_1,b_1\rangle, |a_1,b_2\rangle, \ldots |a_2,b_3\rangle}_{6 \text{ elements}}\}$


Consider classical system.
(a_1, a_2) (b_1, b_2, b_3)
$(a_1, a_2, b_1, b_2, b_3)$


Classical description in terms of probabilities.
A: $(p_a(1), p_a(1))$
B: $(p_b(1), p_b(2), p_b(3))$


P
T
M


mension.
$\dim(\mathcal{H}_{AB}) = \dim(\mathcal{H}_A)\dim(\mathcal{H}_B)$
$b_3\rangle\}$
$|a,b\rangle$
$|a\rangle|b\rangle$
$|a\rangle_A \otimes |b\rangle_B$


in terms
Arbitrary state:
$|\psi\rangle_{AB} = \sum_{k,l} c_{k,l} |a_k\rangle_A \otimes |b_l\rangle_B$
where $c_{k,l} = (\langle a_k|_A \otimes \langle b_l|_B)|\psi\rangle_{AB}$

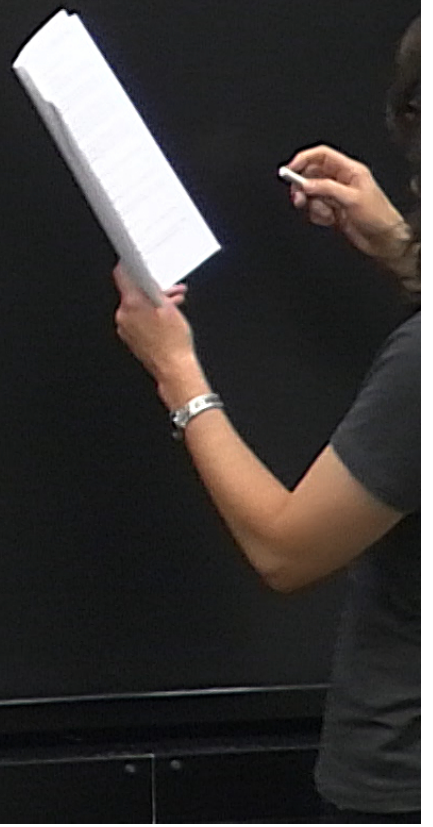
Schmidt decomposition.

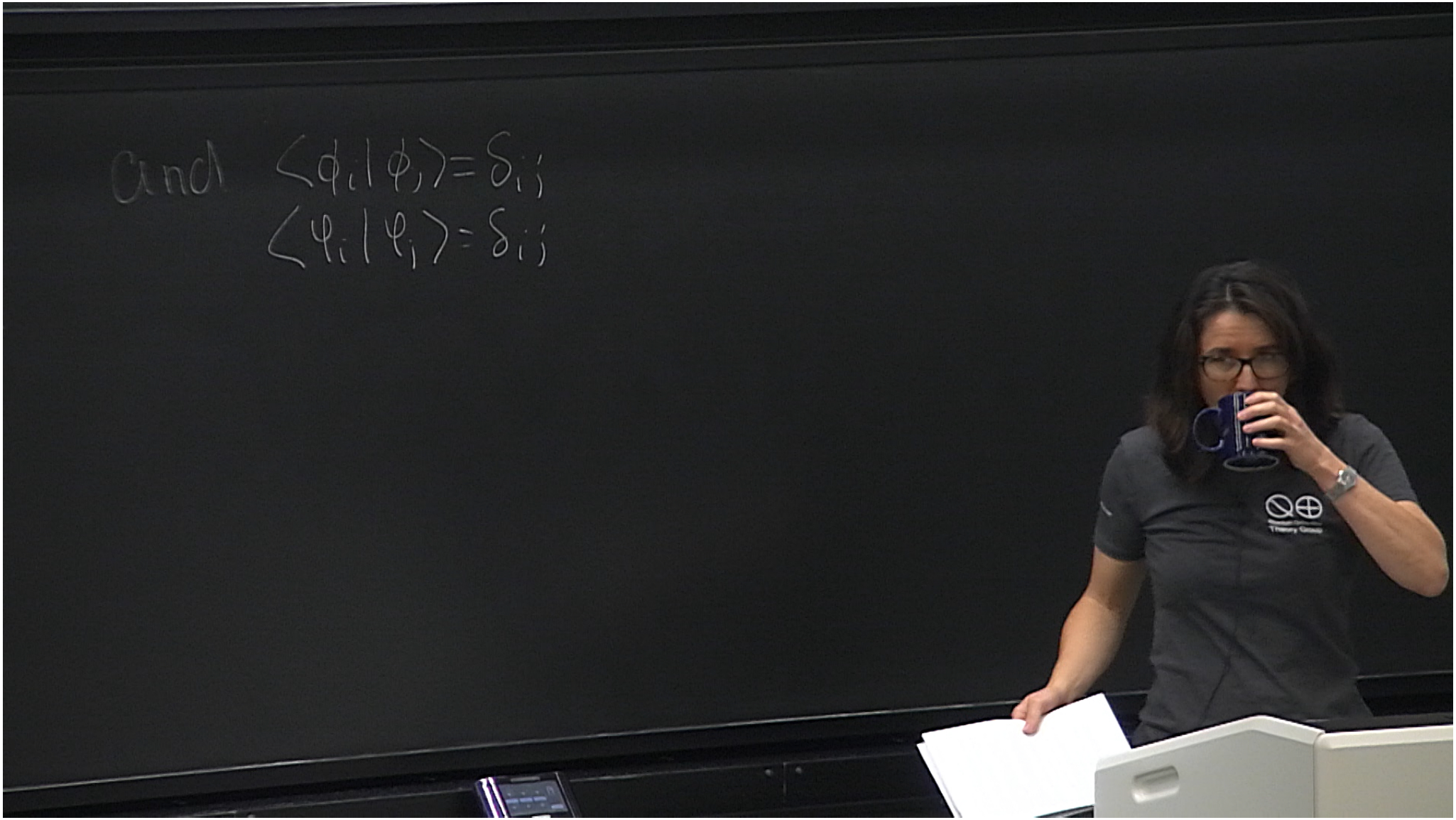
Any time you have (1), you can always find decomposition:

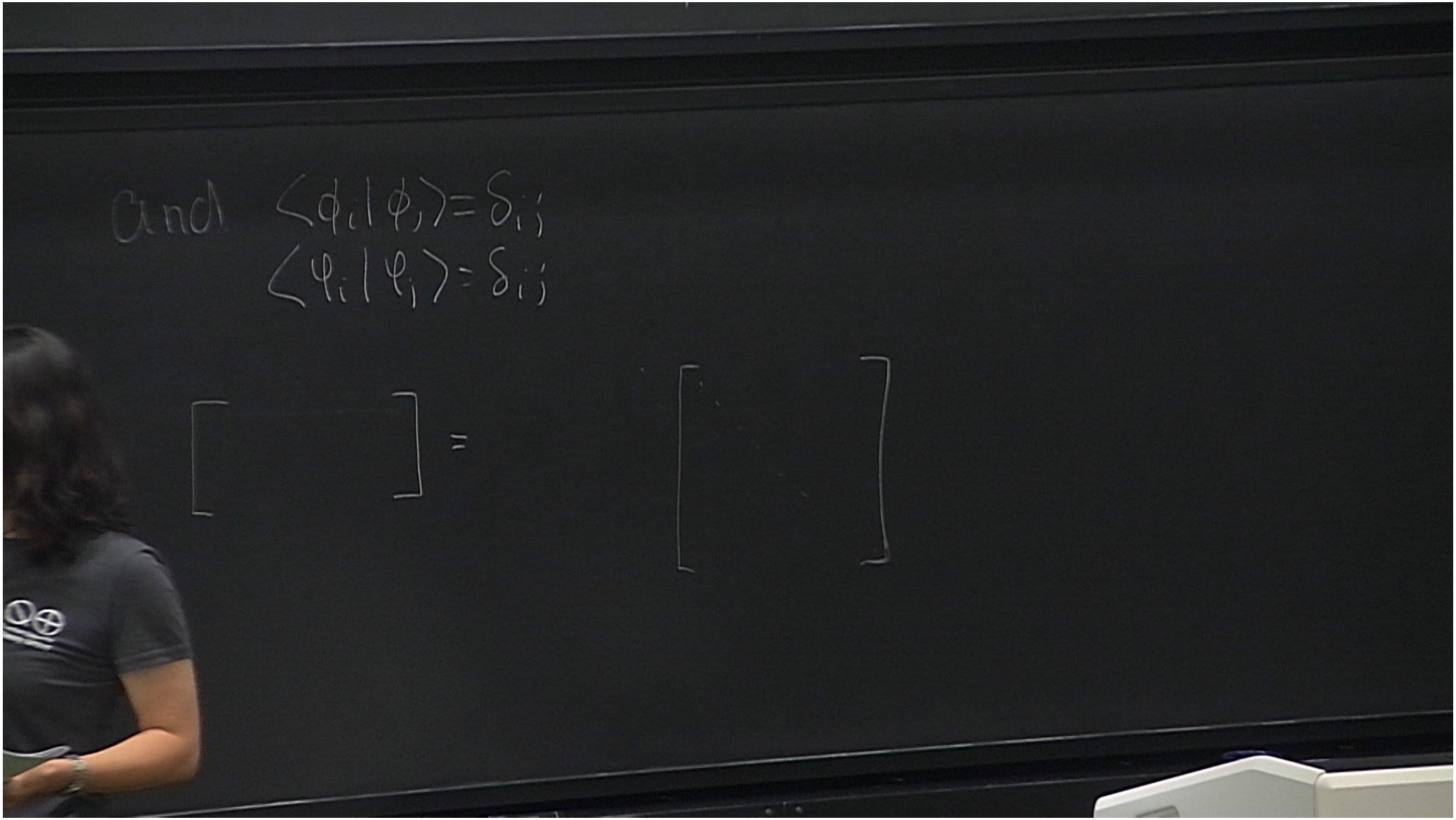
$$|\psi\rangle_{AB} = \sum_i \lambda_i |\phi_i\rangle_A |\varphi_i\rangle_B$$

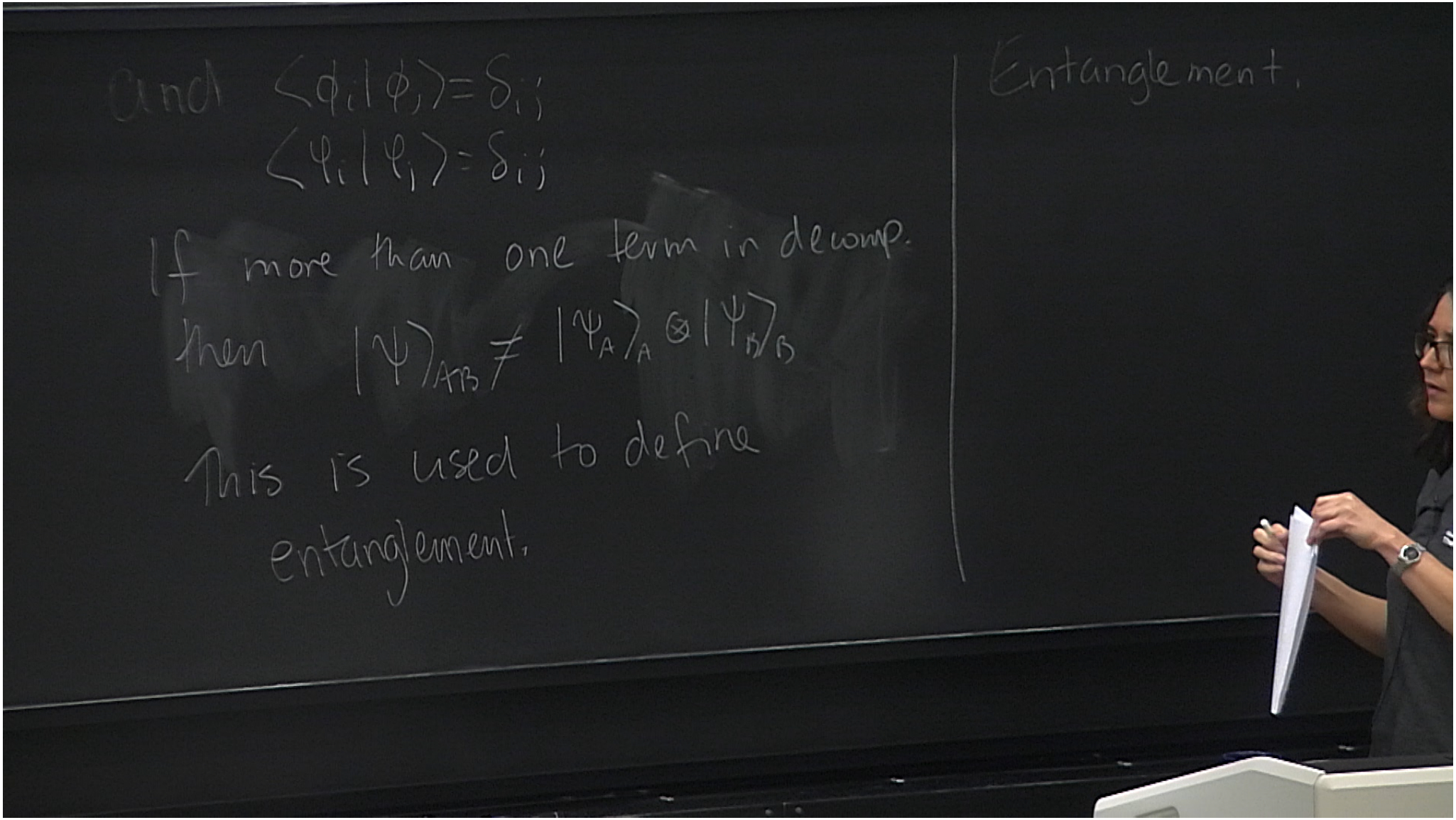
Where $\lambda_i \geq 0$ and $\sum_i \lambda_i^2 = 1$

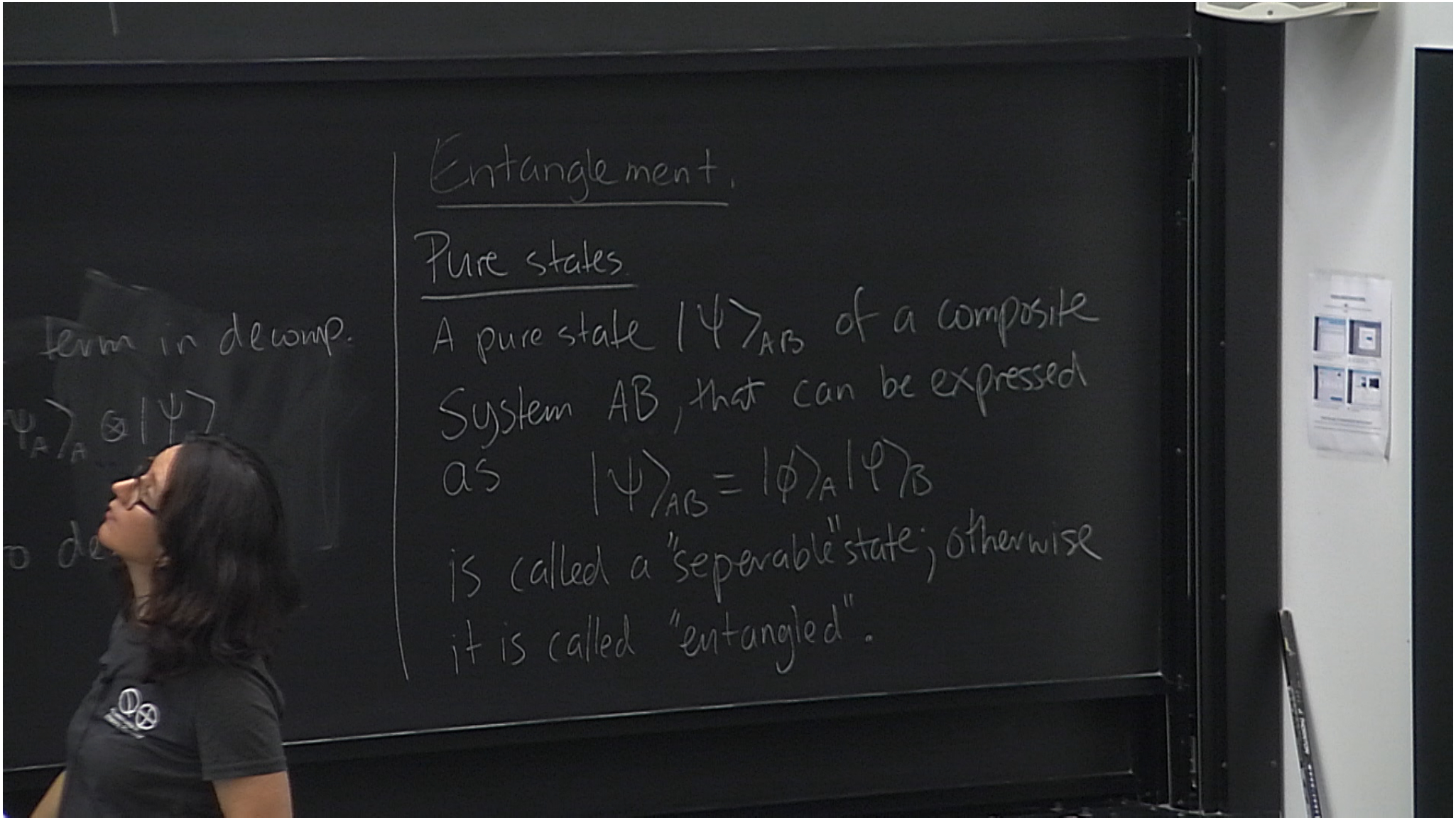
and




and $\langle \phi_i | \phi_j \rangle = \delta_{ij}$
$\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$


and $\langle \phi_i | \phi_j \rangle = \delta_{ij}$
$\langle \psi_i | \psi_j \rangle = \delta_{ij}$
=


and $\langle \phi_i | \phi_j \rangle = \delta_{ij}$
$\langle \psi_i | \psi_j \rangle = \delta_{ij}$
If more than one term in decomp.
then $|\psi\rangle_{AB} \neq |\psi_A\rangle_A \otimes |\psi_B\rangle_B$
This is used to define
entanglement.
Entanglement.

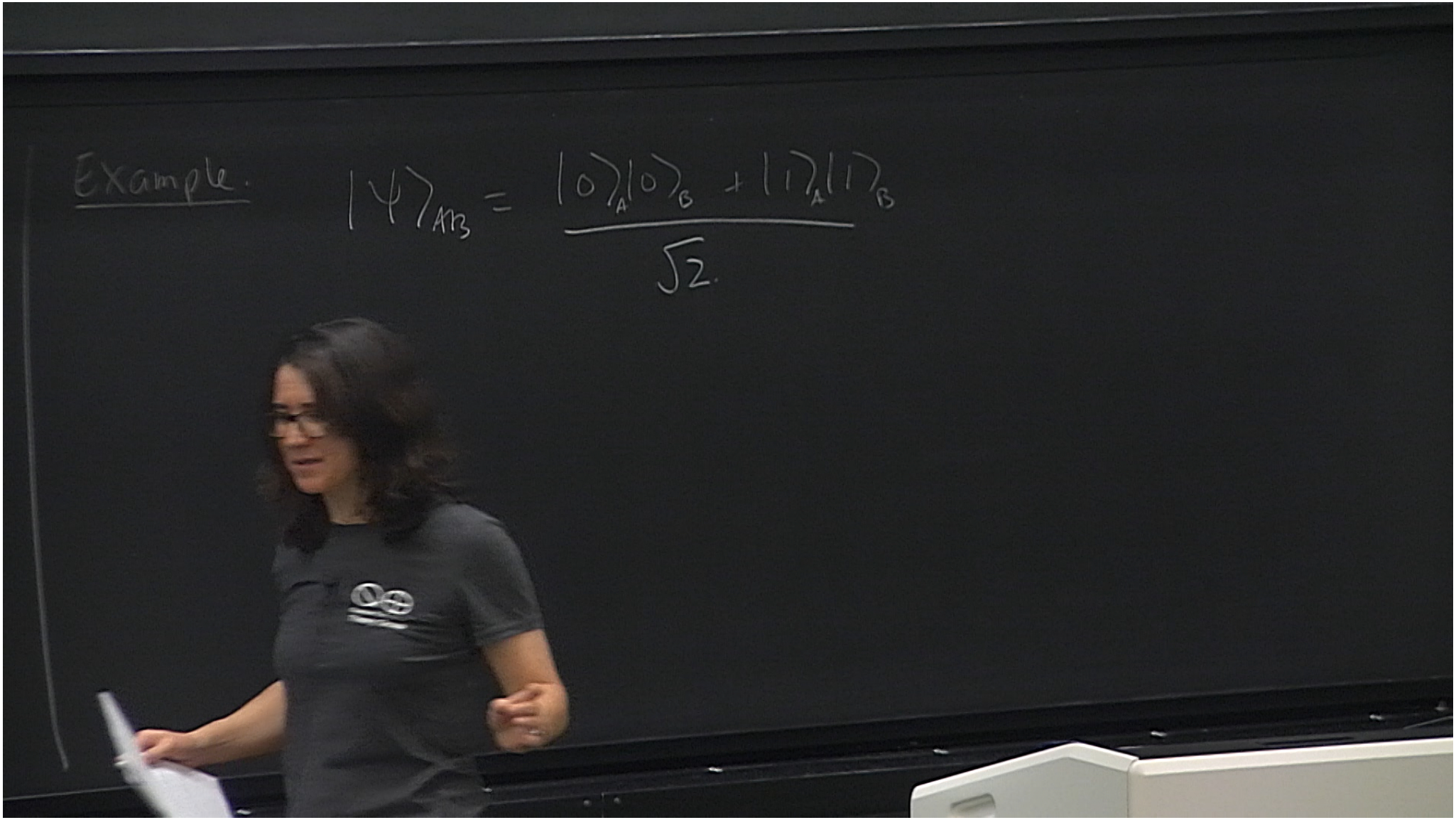

term in decomp.
Entanglement.
Pure states
A pure state $|\psi\rangle_{AB}$ of a composite system AB, that can be expressed as $|\psi\rangle_{AB} = |\phi\rangle_A |\varphi\rangle_B$ is called a "seperable" state; otherwise it is called "entangled".

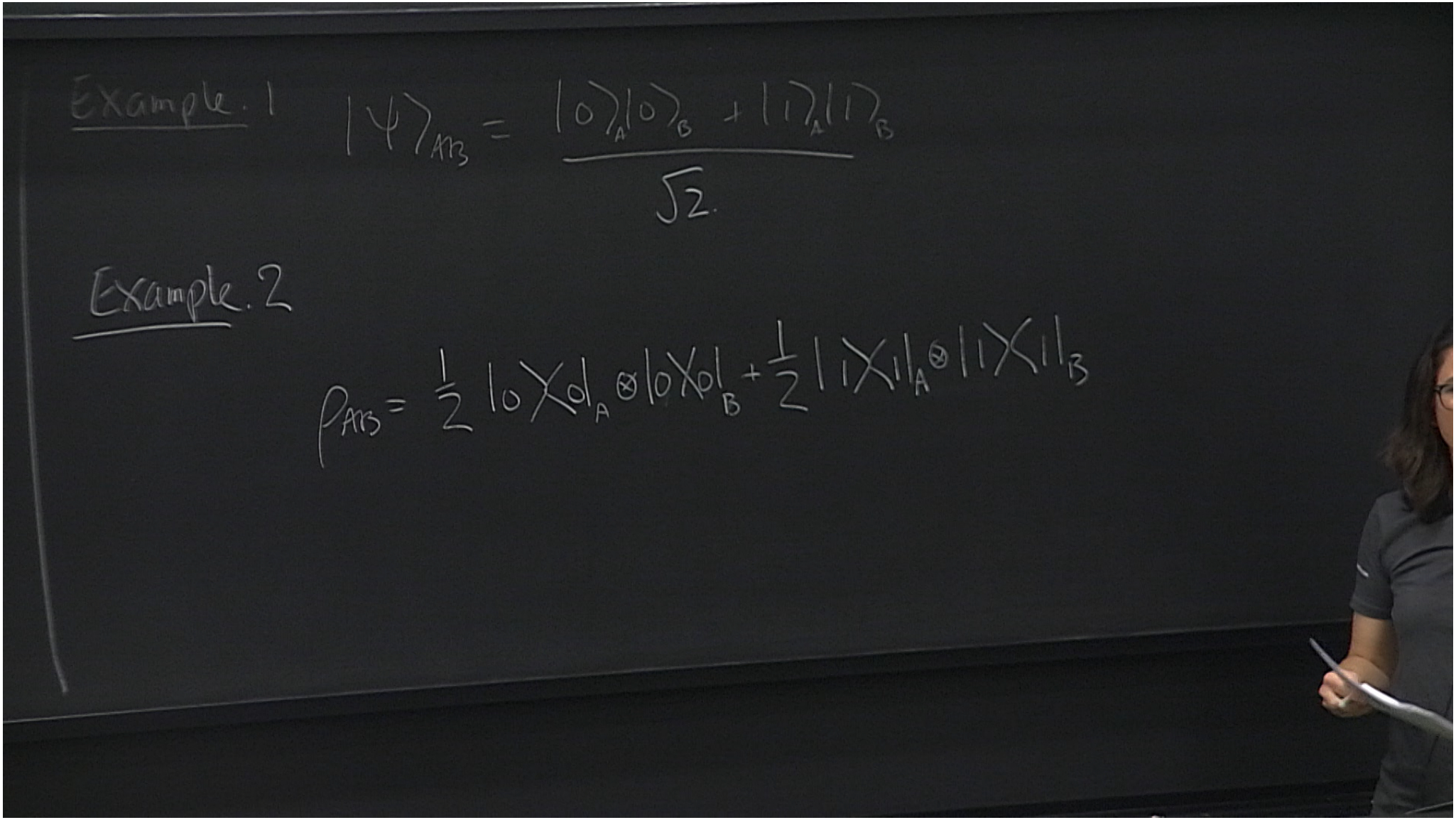
Entanglement: Mixed states.

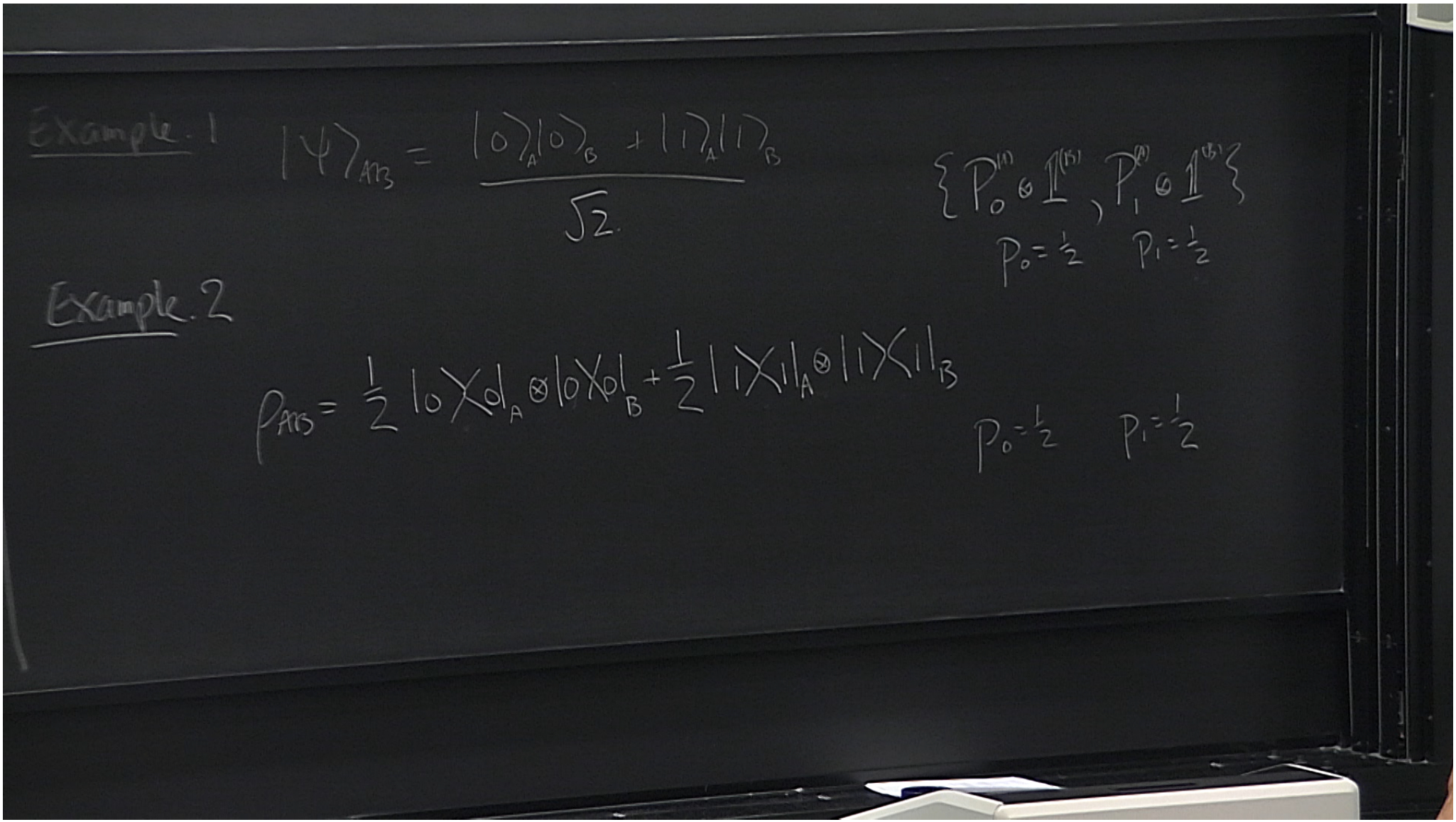
A mixed state ρ_{AB} of a composite system AB, that can be expressed as

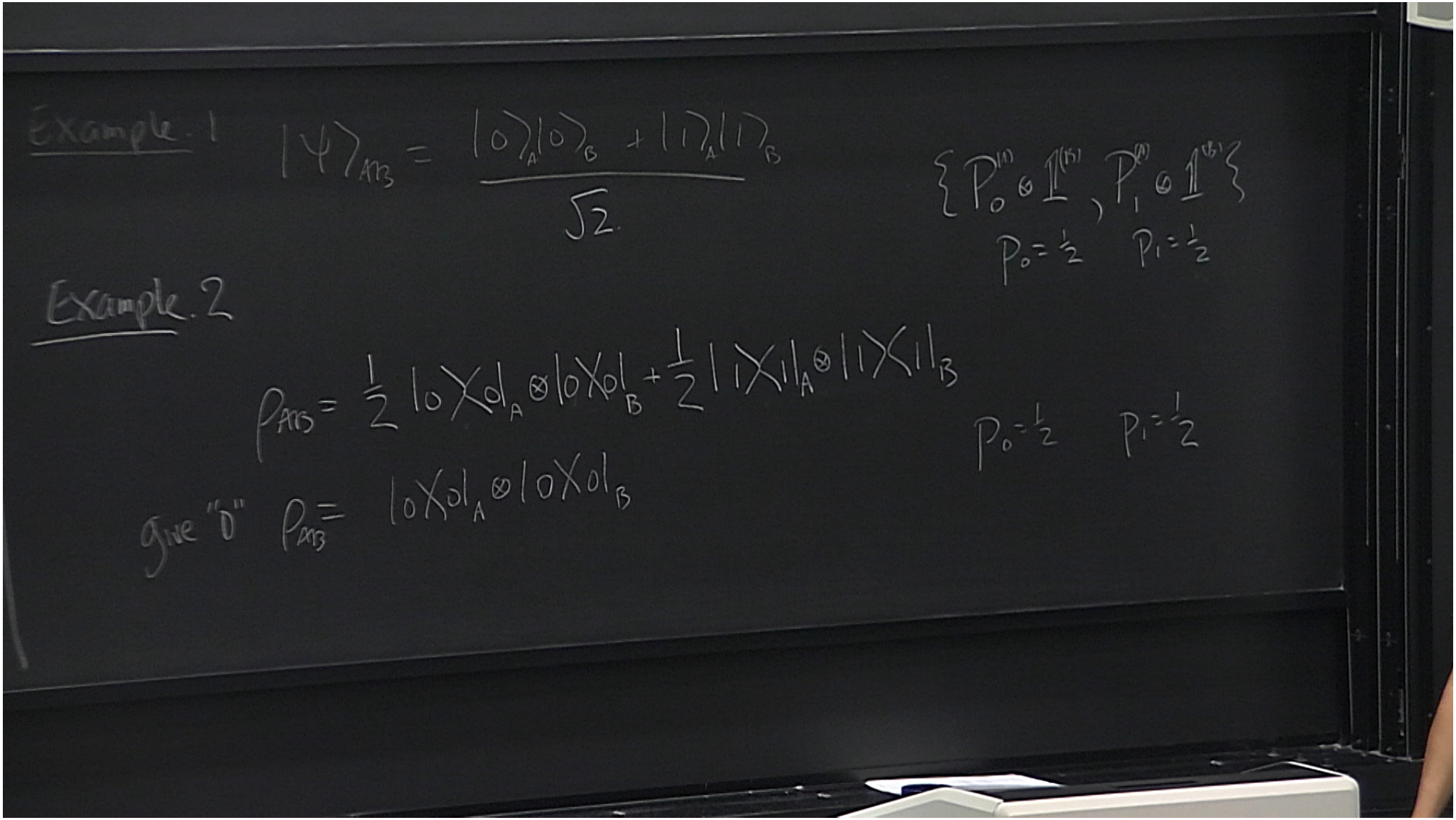
$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}$$

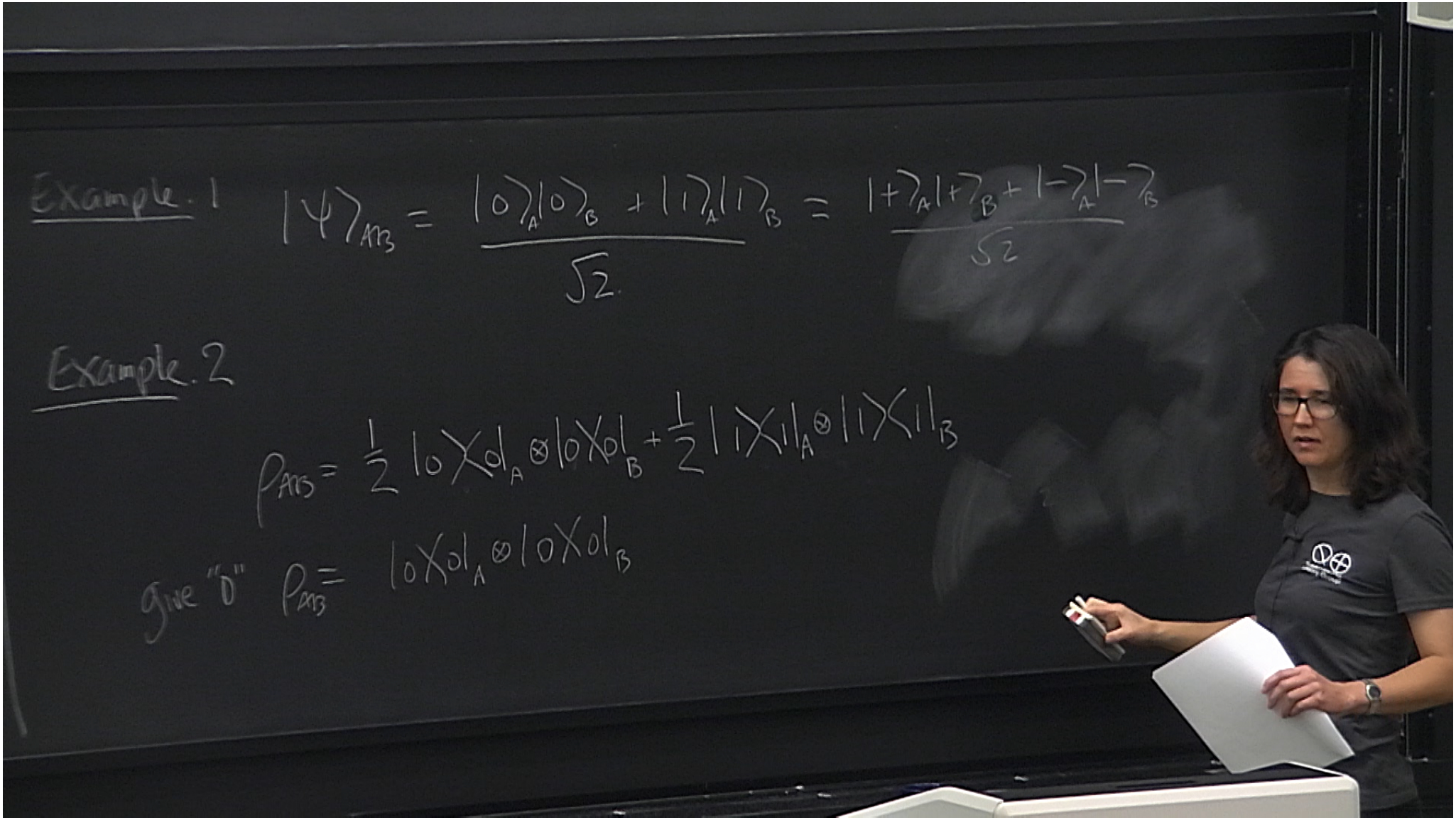
is called a "seperable" state; otherwise it is called "entangled".

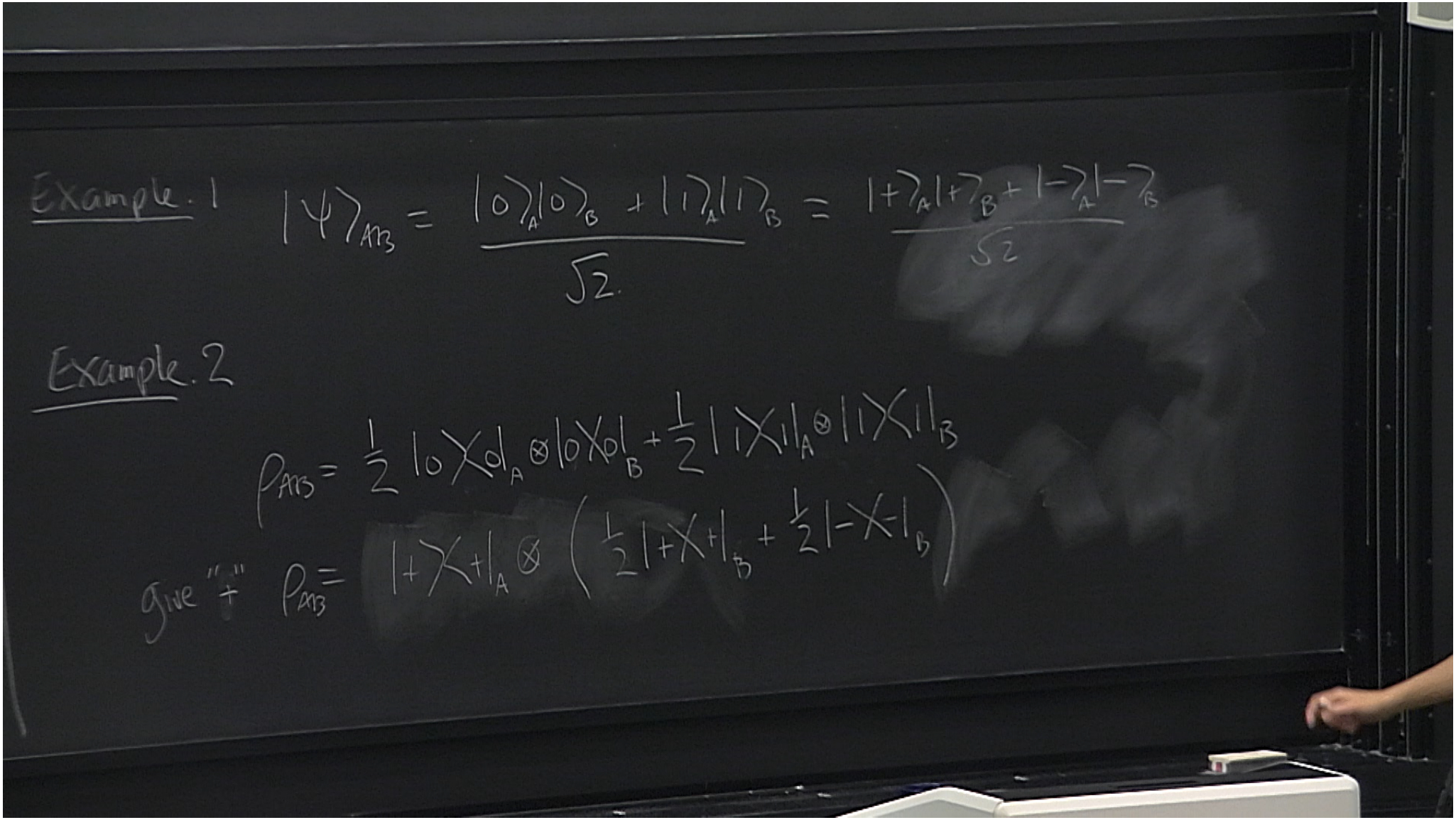

Example.
$|\psi\rangle_{AB} = \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}$

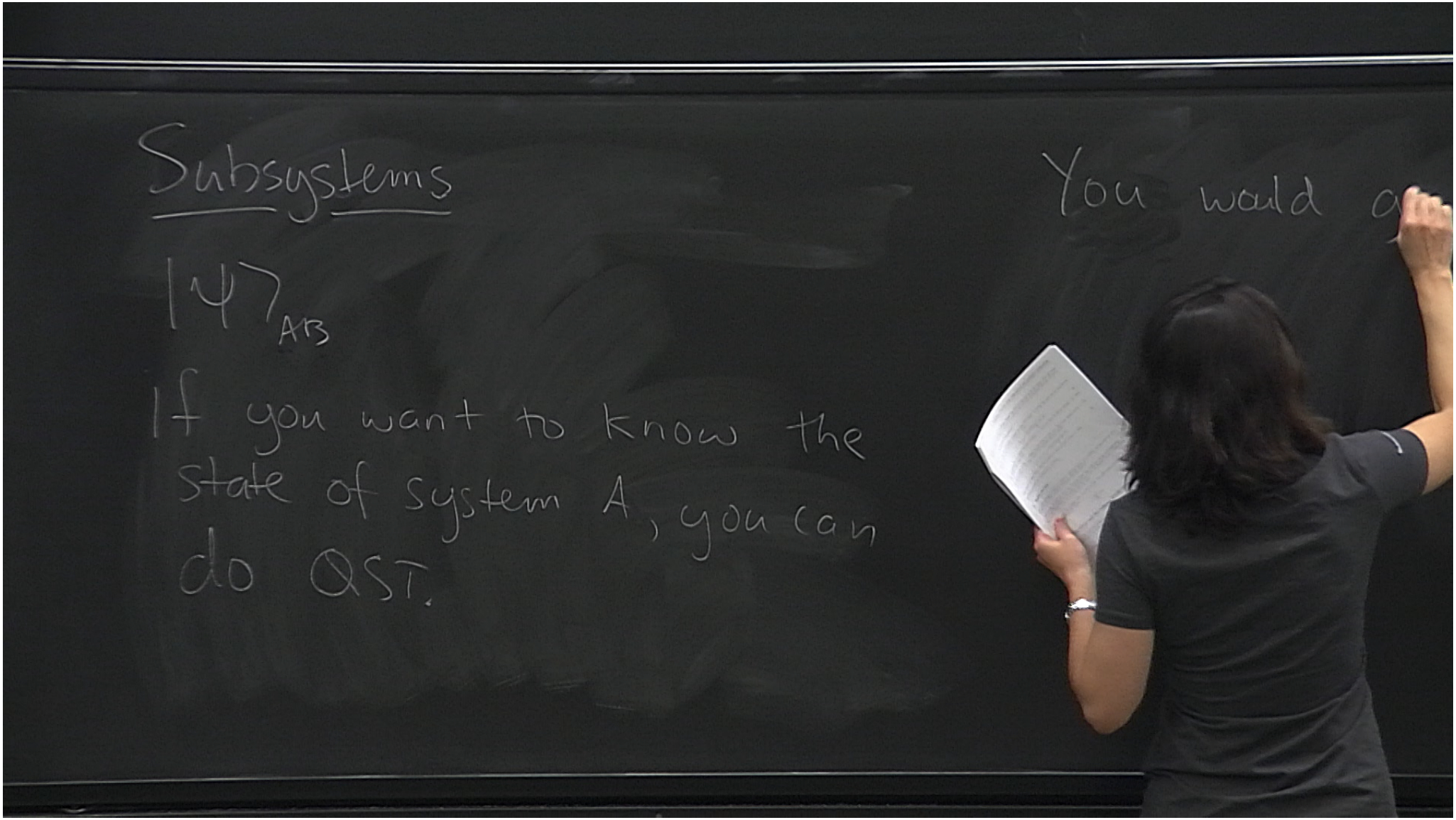

Example. 1
$|\psi\rangle_{AB} = \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}$
Example. 2
$\rho_{AB} = \frac{1}{2}|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + \frac{1}{2}|1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B$


Example. 1
$|\psi\rangle_{AB} = \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}$
$\{P_0^{(A)} \otimes \mathbb{1}^{(B)}, P_1^{(A)} \otimes \mathbb{1}^{(B)}\}$
$p_0 = \frac{1}{2}$ $p_1 = \frac{1}{2}$
Example. 2
$\rho_{AB} = \frac{1}{2}|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + \frac{1}{2}|1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B$
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Example. 1
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Example. 2
$\rho_{AB} = \frac{1}{2}|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + \frac{1}{2}|1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B$
$p_0 = \frac{1}{2}$ $p_1 = \frac{1}{2}$
give "0" $\rho_{AB} = |0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B$


Example 1
$|\psi\rangle_{AB} = \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}} = \frac{|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B}{\sqrt{2}}$
Example 2
$\rho_{AB} = \frac{1}{2}|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + \frac{1}{2}|1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B$
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Example. 1
$|\psi\rangle_{AB} = \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}} = \frac{|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B}{\sqrt{2}}$
Example. 2
$\rho_{AB} = \frac{1}{2}|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + \frac{1}{2}|1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B$
give "+" $\rho_{AB} = |+\rangle\langle +|_A \otimes \left(\frac{1}{2}|+\rangle\langle +|_B + \frac{1}{2}|-\rangle\langle -|_B\right)$


Subsystems
$|\psi\rangle_{AB}$
If you want to know the state of system A, you can do QST.
You would

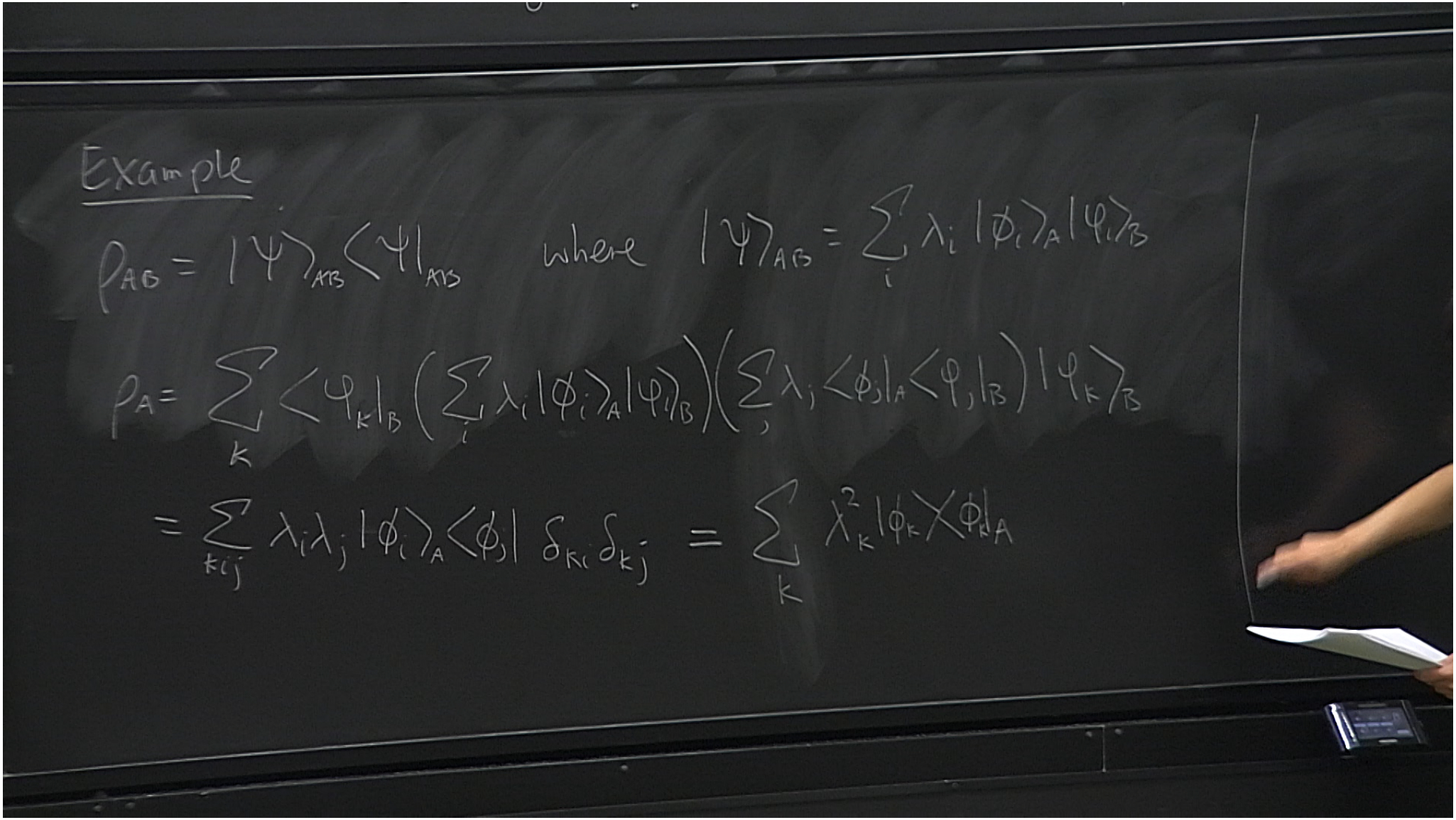
You would get ρ_A. How to get ρ_A from ρ_{AB}?
Do "partial trace".

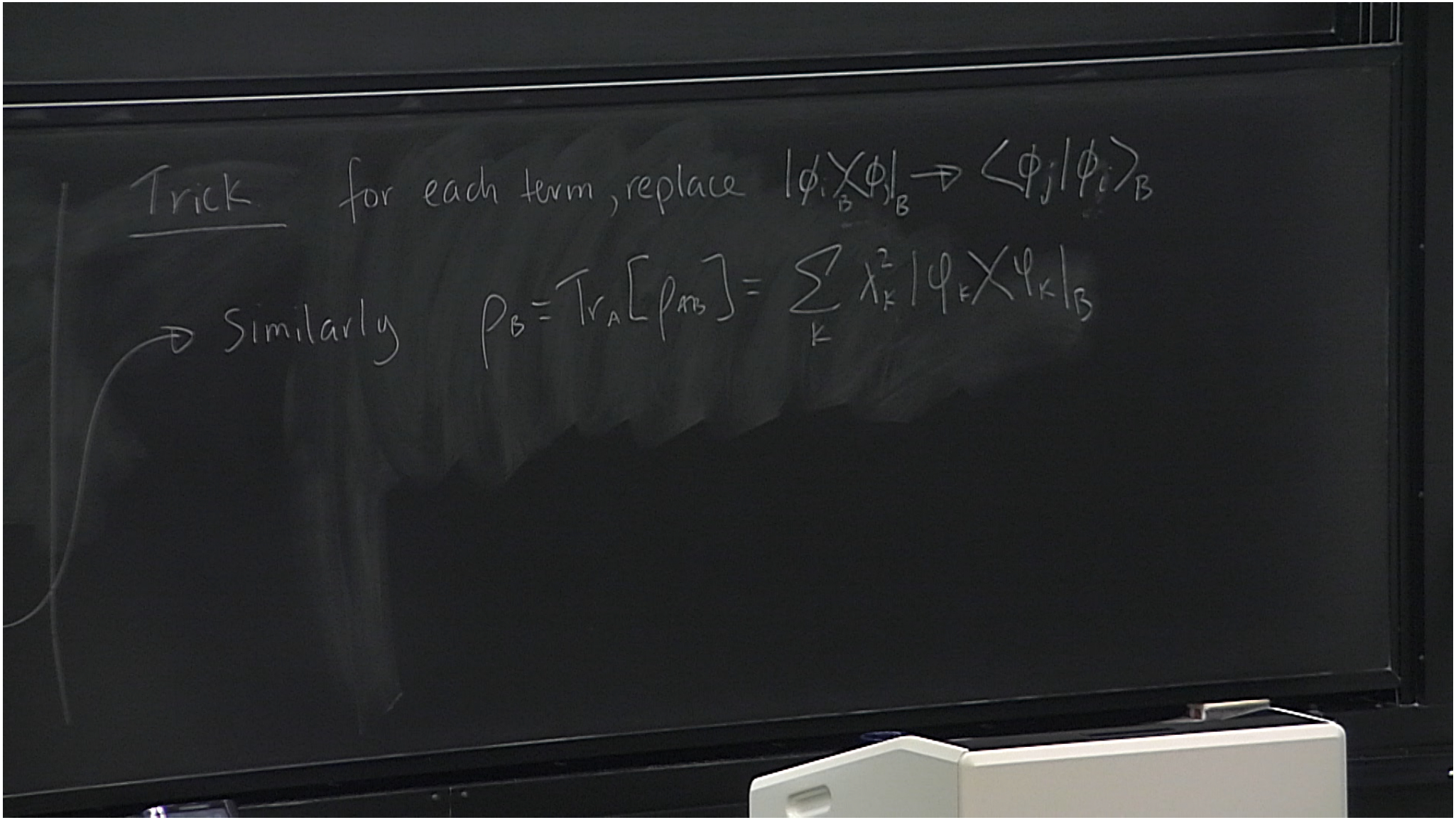
$$\rho_A = \mathrm{Tr}_B[\rho_{AB}] = \sum_i (\mathbb{1}_A \otimes \langle \phi_i |_B)\, \rho_{AB}\, (\mathbb{1}_A \otimes |\phi_i\rangle_B) = \sum_i \langle \phi_i |_B \rho_{AB} |\phi_i\rangle_B$$

Example

$$\rho_{AB} = |\psi\rangle_{AB}\langle\psi|_{AB} \quad \text{where} \quad |\psi\rangle_{AB} = \sum_i \lambda_i |\phi_i\rangle_A |\varphi_i\rangle_B$$

$$\rho_A = \sum_k \langle\varphi_k|_B \left(\sum_i \lambda_i |\phi_i\rangle_A |\varphi_i\rangle_B\right)\left(\sum_j \lambda_j \langle\phi_j|_A \langle\varphi_j|_B\right)|\varphi_k\rangle_B$$


Example
$\rho_{AB} = |\psi\rangle_{AB}\langle\psi|_{AB}$ where $|\psi\rangle_{AB} = \sum_i \lambda_i |\phi_i\rangle_A |\varphi_i\rangle_B$
$\rho_A = \sum_k \langle\varphi_k|_B \left(\sum_i \lambda_i |\phi_i\rangle_A |\varphi_i\rangle_B\right)\left(\sum_j \lambda_j \langle\phi_j|_A \langle\varphi_j|_B\right)|\varphi_k\rangle_B$
$= \sum_{kij} \lambda_i \lambda_j |\phi_i\rangle_A \langle\phi_j| \delta_{ki}\delta_{kj} = \sum_k \lambda_k^2 |\phi_k\rangle\langle\phi_k|_A$


Trick for each term, replace $|\phi_i\rangle_B\langle\phi_j|_B \Rightarrow \langle\phi_j|\phi_i\rangle_B$
Similarly $\rho_B = Tr_A[\rho_{AB}] = \sum_k \lambda_k^2 |\varphi_k\rangle_B\langle\varphi_k|_B$

Characterizing entanglement

The purity of the reduced state is related to the entropy of the system.

$$E_A = -\mathrm{Tr}[\rho_A \log(\rho_A)] = -\sum_k \lambda_k^2 \log \lambda_k^2 = -\mathrm{Tr}[\rho_B \log \rho_B] = E_B$$

If $E=0$ for either subsystem, the $|\psi\rangle_{AB}$ is seperable.

The higher E, the more entangle $|\psi\rangle_{AB}$ is.

BUT composite system is pure!

If $E=0$ for either subsystem, the $|\psi\rangle_{AB}$ is seperable.

The higher E, the more entangle $|\psi\rangle_{AB}$ is.

BUT Composite system is pure!

Definition: A maximally entangled state $|\psi\rangle_{AB}$ is one for which the reduced state is maximally mixed.

Characterizing entanglement

The purity of the reduced state is related to the entropy of the system.

$$E_A = -\mathrm{Tr}[\rho_A \log(\rho_A)] = -\sum_k \lambda_k^2 \log \lambda_k^2 = -\mathrm{Tr}[\rho_B \log \rho_B] = E_B$$

(λ_k^2: eigenvalues of ρ_A)

def of entropy

if $\rho_A = \mathrm{Tr}_B[|\psi\rangle_{AB}\langle\psi|_{AB}]$ λ_k are also Schmidt coeffs.

If

The

B

Def

It is called "entangled".

Charachterizing Mixed entangled states

Recall def. of completely positive map $\mathcal{E}_A(\rho_A)$

$\rho_{AB}(t) = \mathcal{E}_A \otimes \mathbb{1}_B(\rho_{AB}(0))$ is positive if $\rho_{AB}(0)$ is positive.

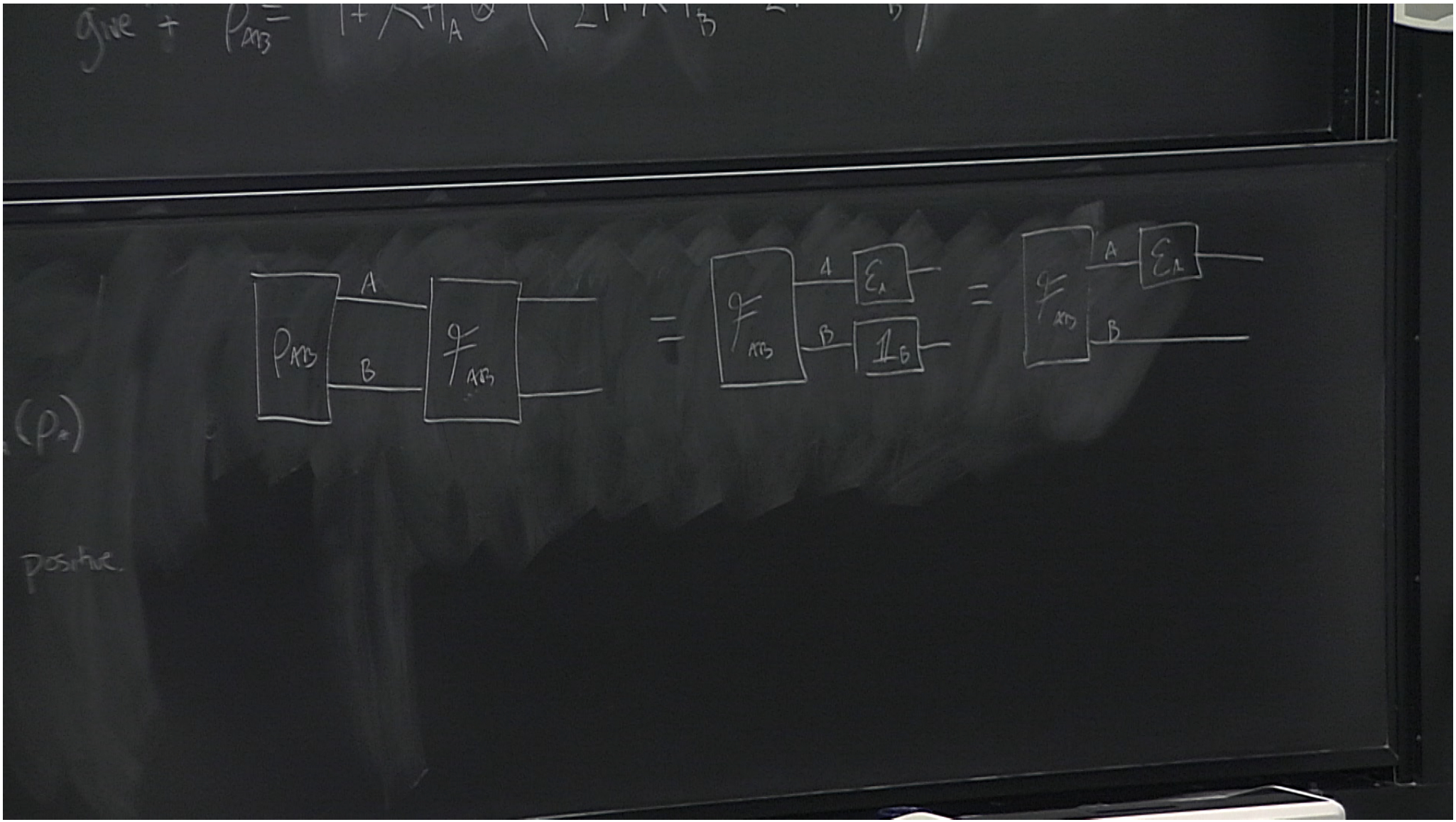
It is called "entangled".

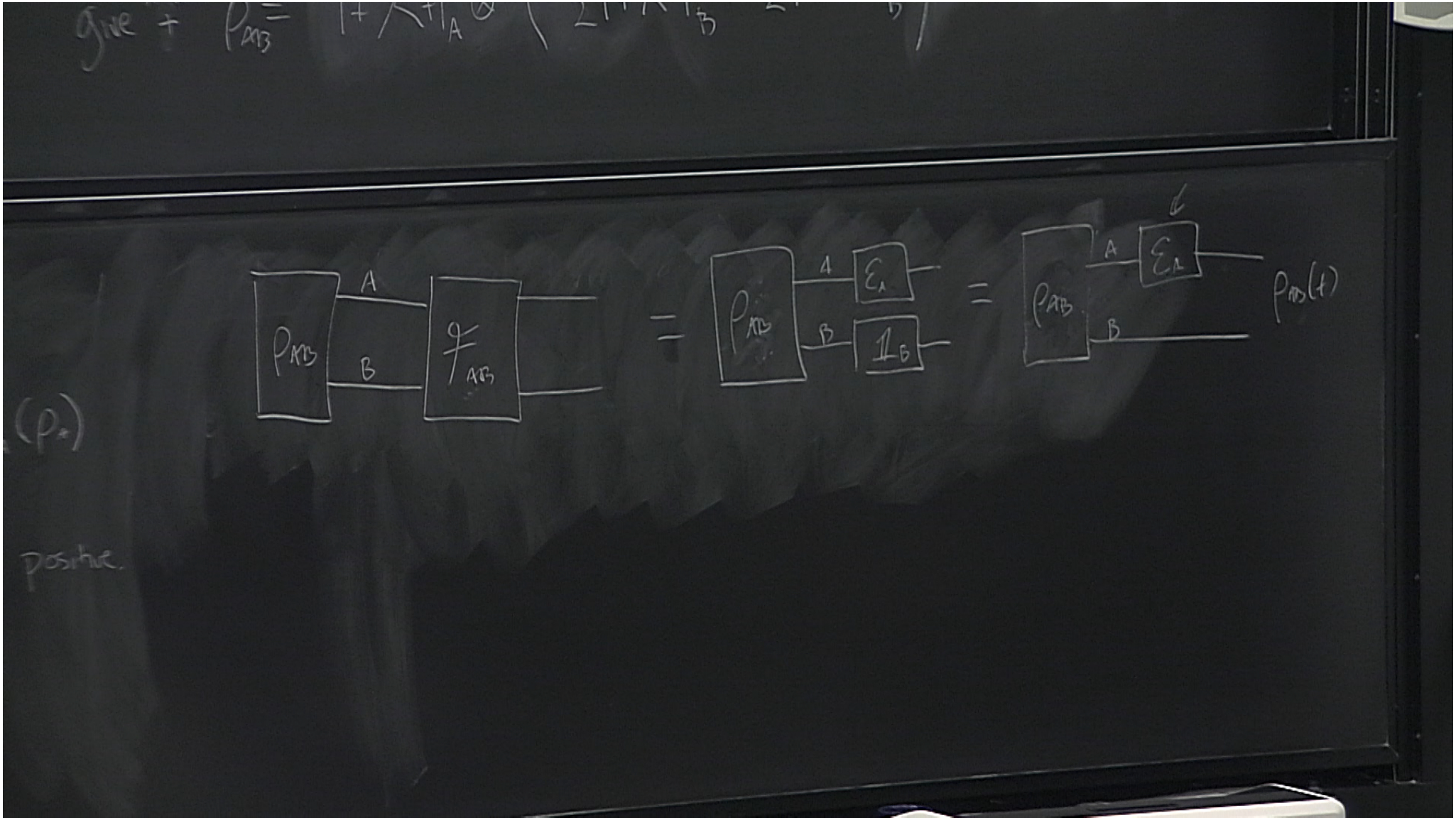
Charachterizing Mixed entangled states

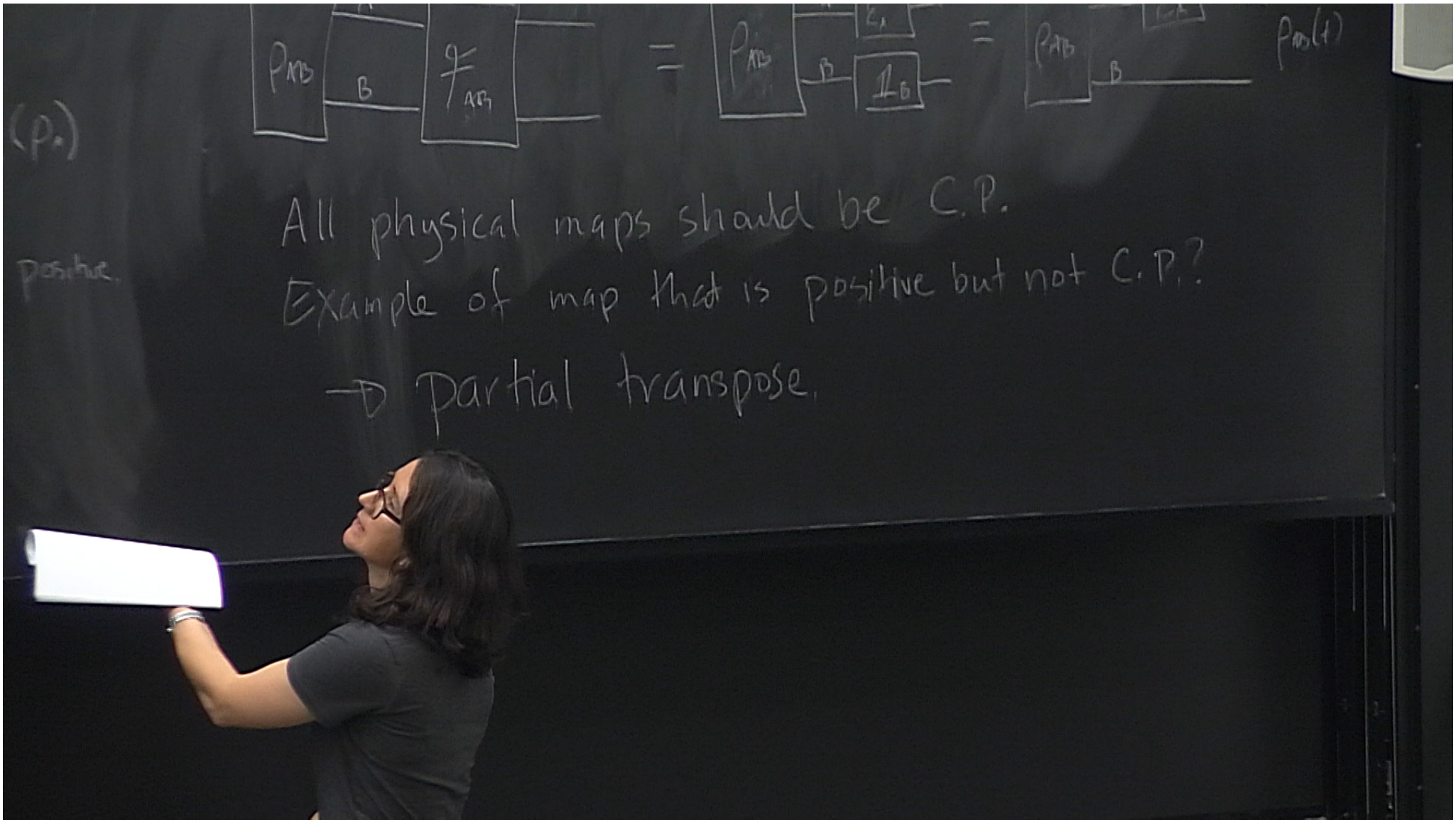
Recall def. of completely positive map $\mathcal{E}_A(\rho_A)$

$\rho_{AB}(t) = \underbrace{\mathcal{E}_A \otimes \mathbb{1}_B}(\rho_{AB}(0))$ is positive if $\rho_{AB}(0)$ is positive.

$= \mathcal{I}_{AB}(\rho_{AB}(0))$


A
B
A
B
A
B


A
B
ρ_{AB}
$\mathcal{E}_{AB}$
$\mathcal{E}_A$
$\mathbb{1}_B$
$\rho_{AB}(t)$

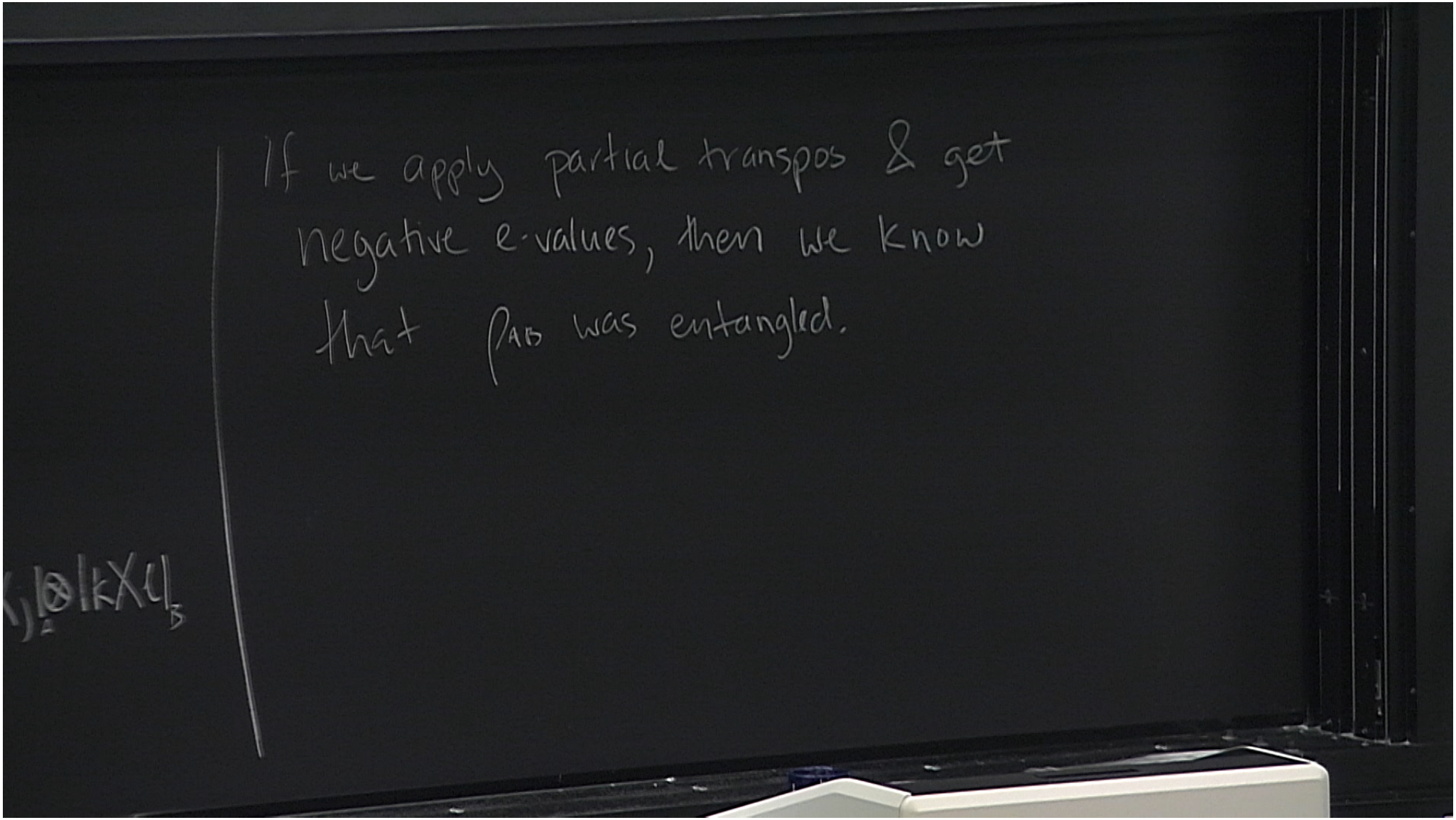

All physical maps should be C.P.
Example of map that is positive but not C.P.?
→ Partial transpose.

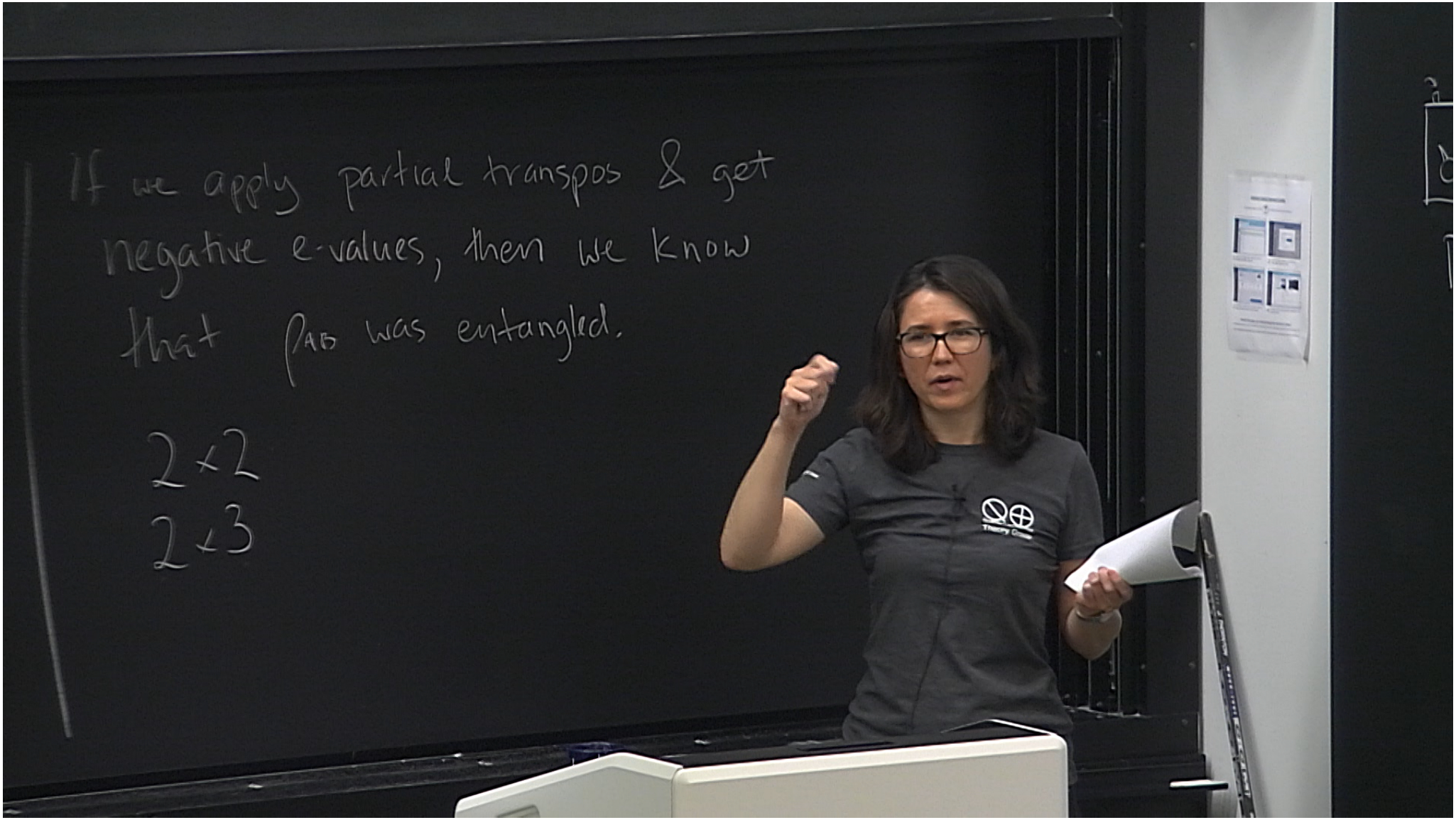
Can always write:

$$\rho_{AB} = \sum_{ijkl} p_{kl}^{ij} |i\rangle\langle j|_A \otimes |k\rangle\langle l|_B$$

Partial transpose wrt. B is.

$$\rho^{T_B} := \mathbb{1} \otimes T(\rho) = \sum_{ijkl} p_{kl}^{ij} |i\rangle\langle j|_A \otimes (|k\rangle\langle l|_B)^T = \sum_{ijkl} p_{kl}^{ij} |i\rangle\langle j|_A \otimes |l\rangle\langle k|_B = \sum_{ijkl} p_{il}^{jk} |i\rangle\langle j|_A \otimes |k\rangle\langle l|_B$$


If we apply partial transpos & get
negative e-values, then we know
that ρ_{AB} was entangled.


If we apply partial transpos & get
negative e-values, then we know
that ρ_{AB} was entangled.
2x2
2x3