Title: PSI 2019/2020 - Quantum Theory (Branczyk) - Lecture 4

Speakers: Agata Branczyk

Collection: PSI 2019/2020 - Quantum Theory (Branczyk/Dupuis)

Date: September 06, 2019 - 9:00 AM

URL: http://pirsa.org/19090046

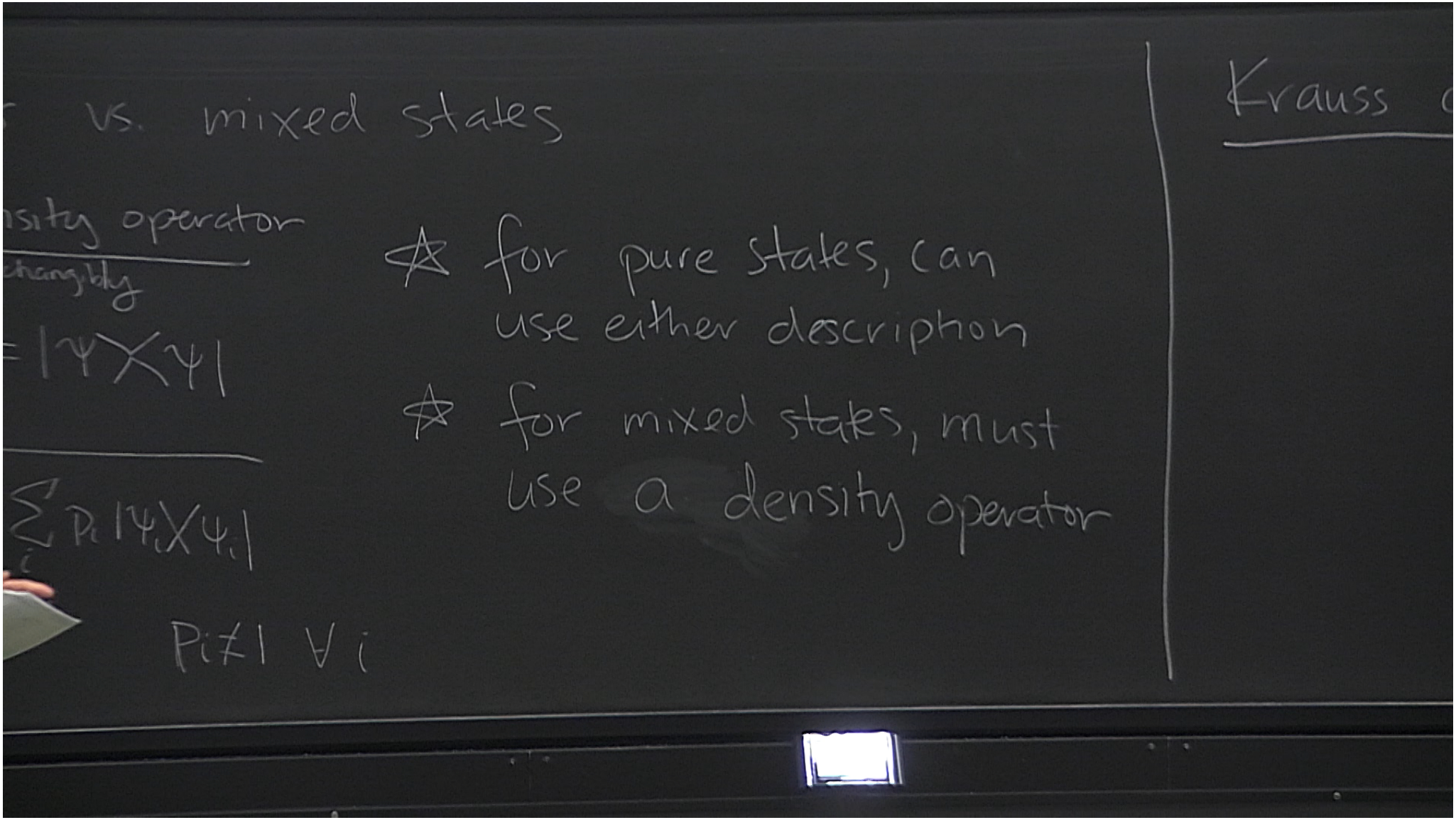
Review: Pure states vs. mixed states

	state vector	density operator
PURE	$\|\psi\rangle$	$\hat{\rho} = \|\psi\rangle\langle\psi\|$
MIXED (NOT PURE)	no possible description	$\hat{\rho} = \sum_i p_i \|\psi_i\rangle\langle\psi_i\|$ $\sum_i p_i = 1$ $p_i \neq 1 \ \forall i$

use those interchangibly

★ for pu
use eit

★ for mi
use a


vs. mixed states
nsity operator
echangibly
$= |\psi\rangle\langle\psi|$
$\sum_i p_i |\psi_i\rangle\langle\psi_i|$
$p_i \neq 1 \ \forall i$
★ for pure states, can use either description
★ for mixed states, must use a density operator
Krauss

Kraus decomposition

$\hat{\rho}$, apply $\hat{U}_j$ with prob p_j

$$\sum_j p_j \hat{U}_j \rho \hat{U}_j^\dagger = \mathcal{E}(\rho)$$

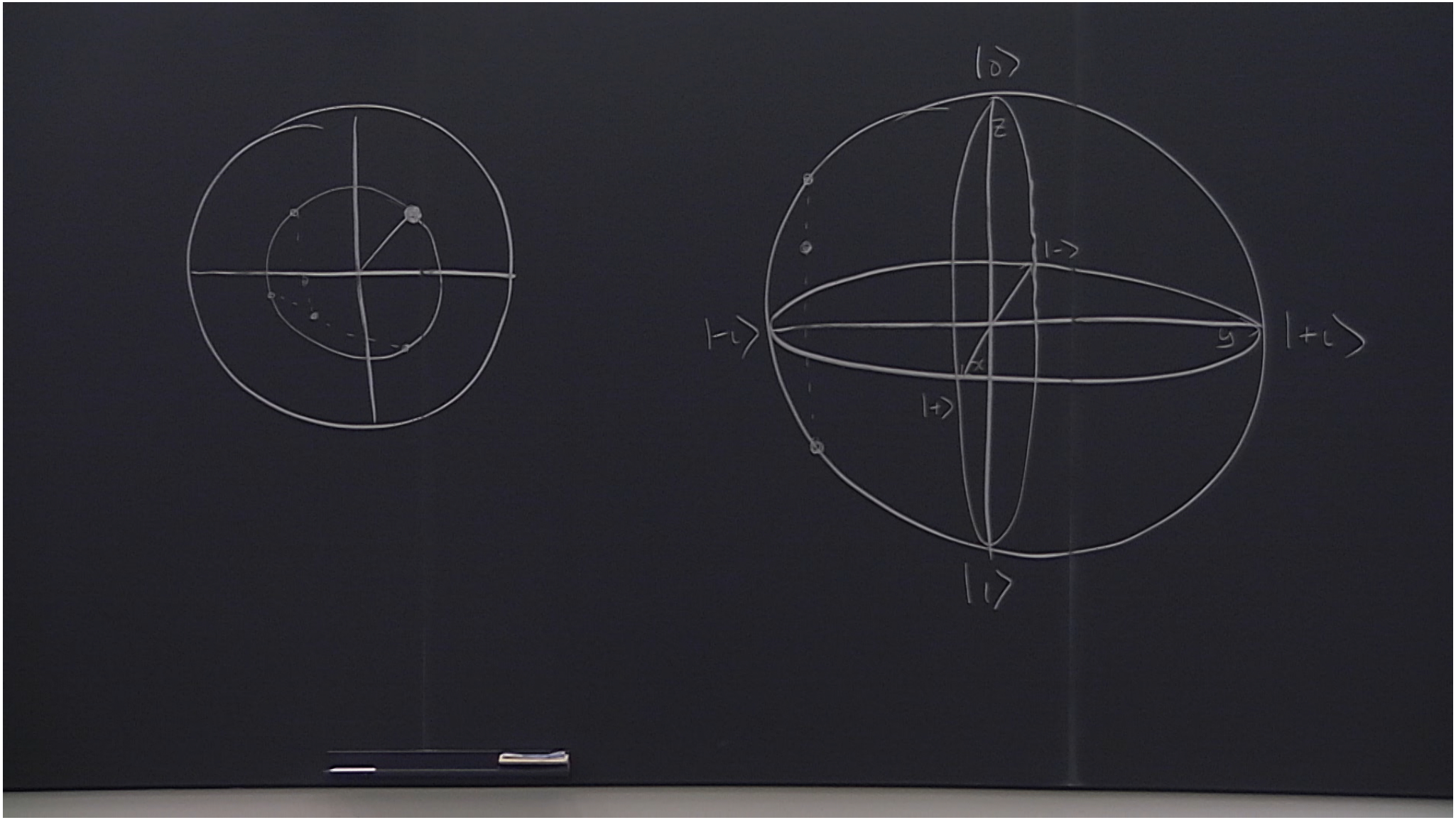
Kraus operators

$$= \sum_j \hat{A}_j \rho \hat{A}_j^\dagger \quad \text{where} \quad \hat{A}_j = \sqrt{p_j}\,\hat{U}_j$$

Map non-unitary $\rightarrow$ does not preserve purity.

But preserves normalization of $\hat{\rho}$

$\sum_j A_j^\dagger A_j = \mathbb{1}$ ensures $Tr[\rho(t)] = Tr[\rho(0)]$


|0>
z
|->
|-i>
|+i>
y
x
|+>
|1>

Map non-unitary → does not preserve purity.

But preserves normalization of $\hat{\rho}$

$\sum_j A_j^\dagger A_j = \mathbb{1}$ ensures $\mathrm{Tr}[\rho(t)] = \mathrm{Tr}[\rho(0)]$

↖ all physical maps satisfy this

Unital maps

Do not increase purity $\sum_j A_j A_j^\dagger = \mathbb{1}$

↙ unital maps satisfy this

Kraus decomposition

$\hat{\rho}$, apply $\hat{U}_j$ with prob p_j

$$\sum_j p_j \hat{U}_j \rho \hat{U}_j^\dagger = \mathcal{E}(\rho)$$

Kraus operators

$$= \sum_j \hat{A}_j \rho \hat{A}_j^\dagger \quad \text{where} \quad \hat{A}_j = \sqrt{p_j}\,\hat{U}_j$$

all maps have this decomp.

only for unital maps

Examples: quantum channels

1. Bit flip channel — flips bit with prob p, otherwise does nothing.

$$A_0 = \sqrt{1-p}\,\sigma_0 \quad ; \quad A_1 = \sqrt{p}\,\sigma_x$$

2. Phase-flip

$$A_0 = \sqrt{1-p}\,\sigma_0 \quad ; \quad A_1 = \sqrt{p}\,\sigma_z$$

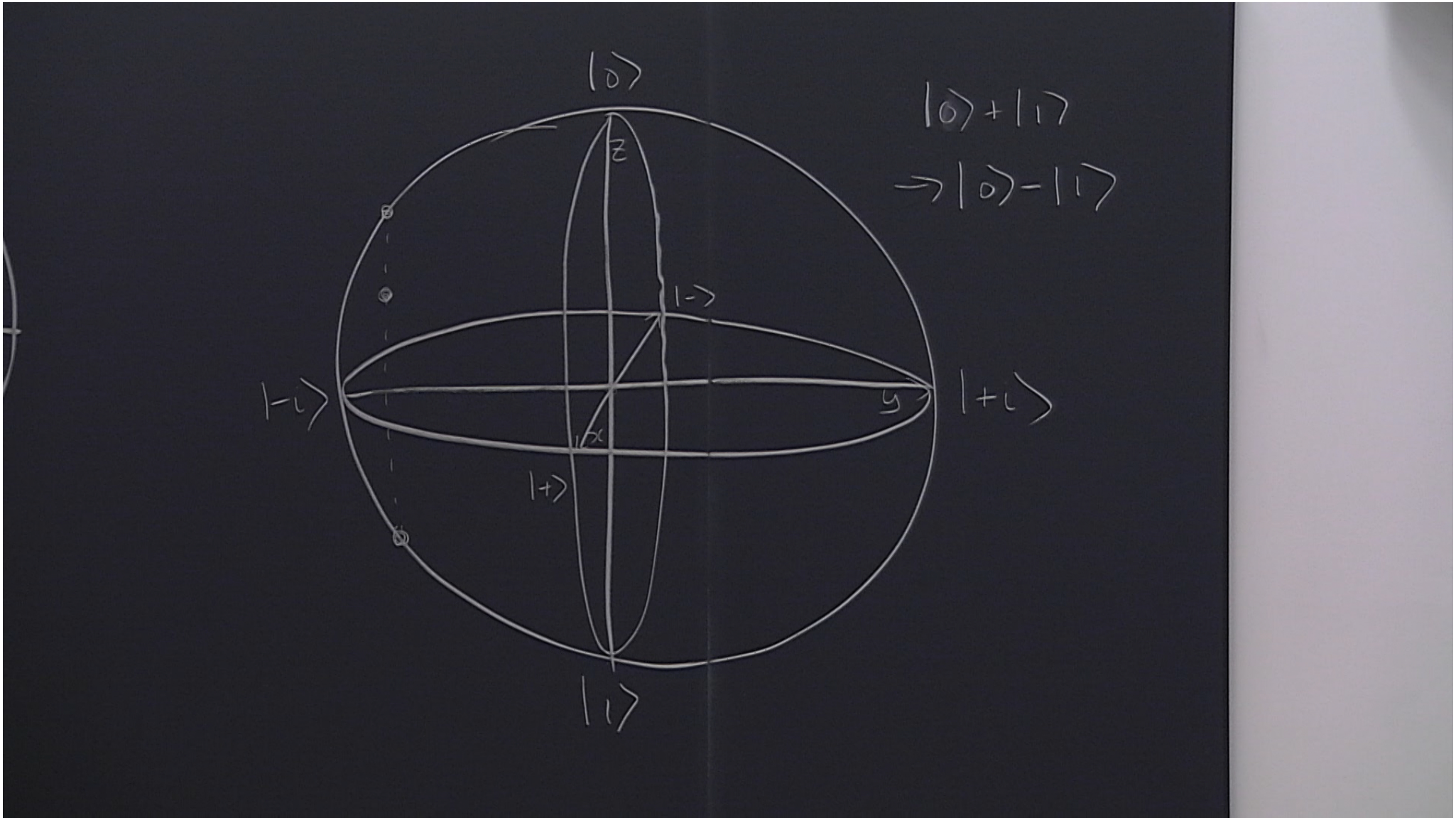
Examples: quantum channels

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2. Phase-flip

$$A_0 = \sqrt{1-p}\,\sigma_0 \quad ; \quad A_1 = \sqrt{p}\,\sigma_z$$


|0⟩
|0⟩+|1⟩
→|0⟩-|1⟩
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|->
|-i⟩
y
|+i⟩
x
|+⟩
|1⟩

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Out[12]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\,(1+\text{Cos}[\theta]-2\,p\,\text{Cos}[\theta]) & \frac{1}{2}\,e^{-i\,\phi}\,\left(1+\left(-1+e^{2\,i\,\phi}\right)\,p\right)\,\text{Sin}[\theta] \\ \frac{1}{2}\,e^{-i\,\phi}\,\left(-e^{2\,i\,\phi}\,(-1+p)+p\right)\,\text{Sin}[\theta] & \frac{1}{2}+\left(-\frac{1}{2}+p\right)\,\text{Cos}[\theta] \end{pmatrix}$$

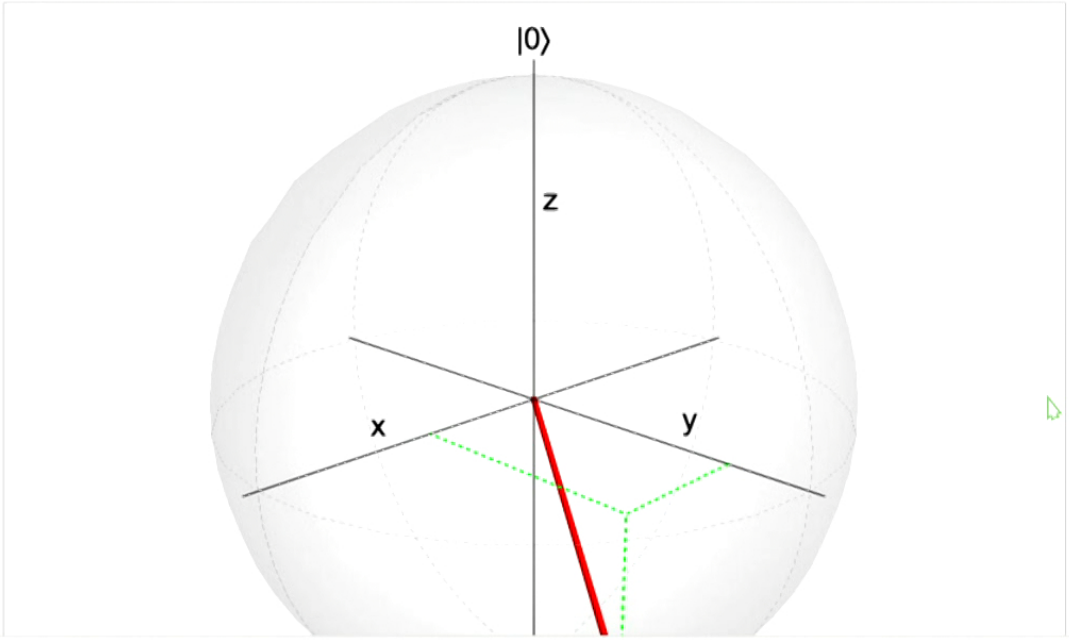
Plot the density operator on the Bloch Sphere

In[13]:=
```
Manipulate[Show[BlochSphere, PlotDM[ρol[θ, ϕ, p]], ImageSize → 800], {{θ, 4π/3}, 0, 2π}, {{ϕ, 4π/3}, 0, 2π}, {{p, 0}, 0, 1/2}]
```

θ

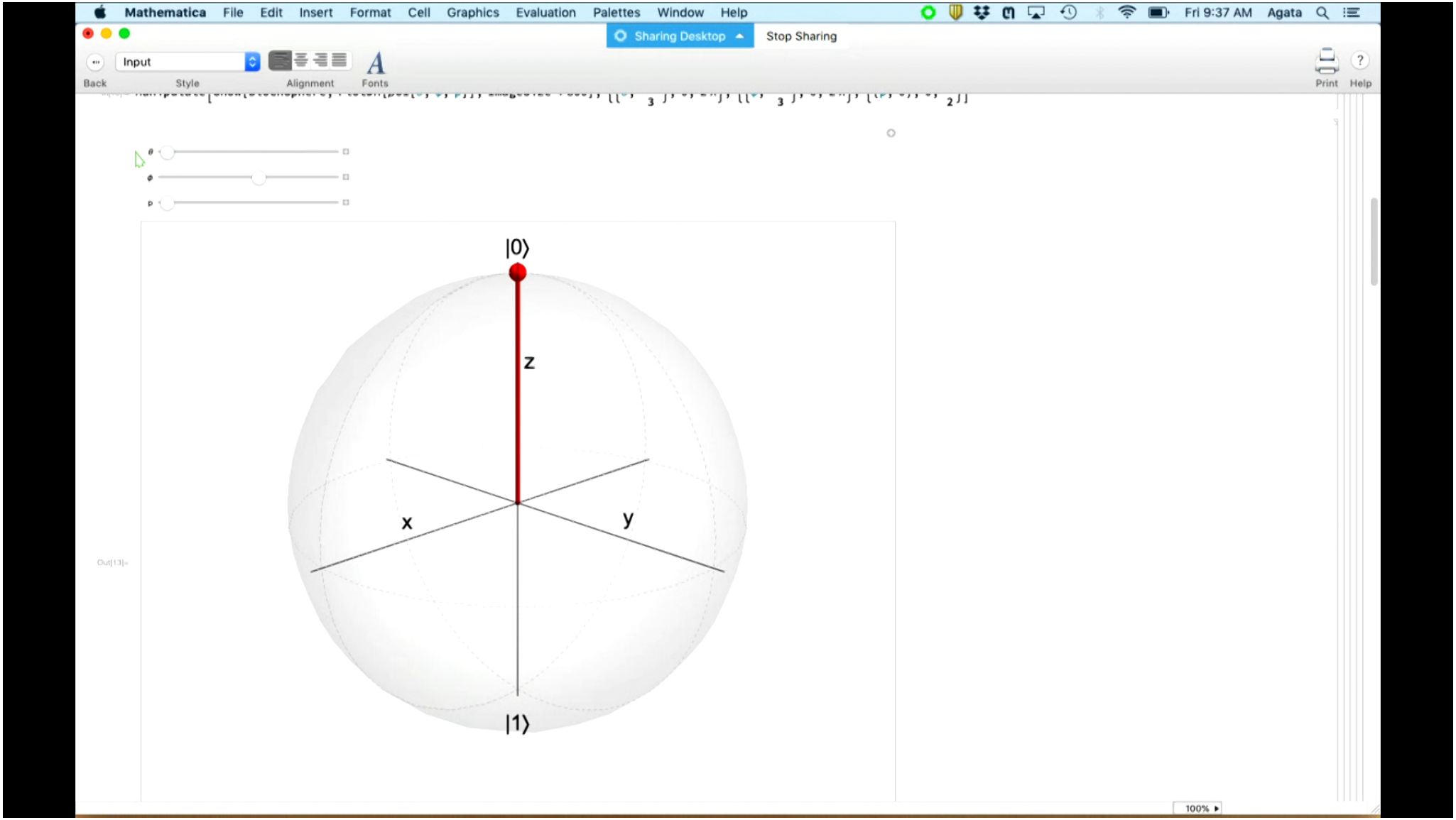
ϕ

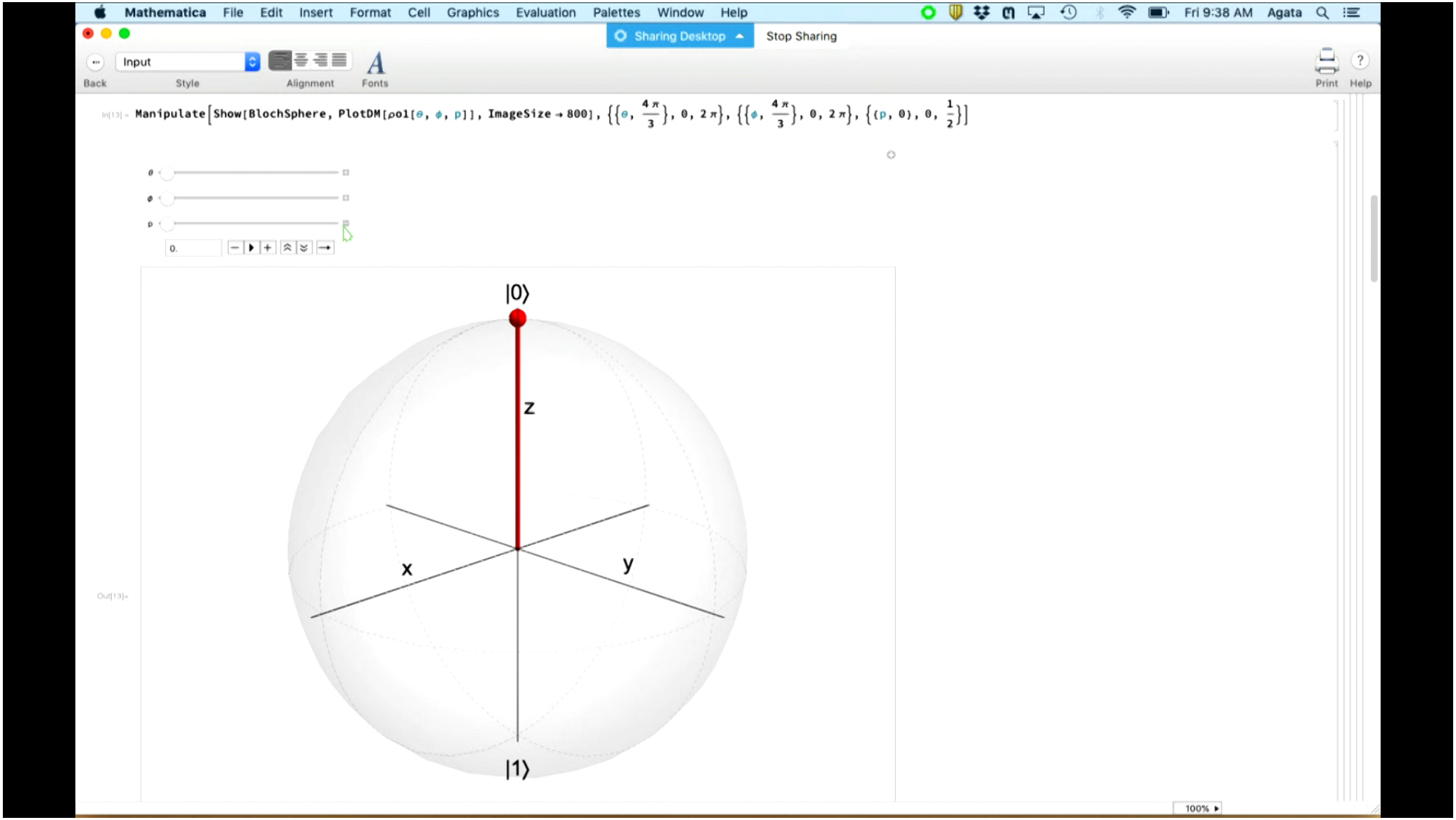
p

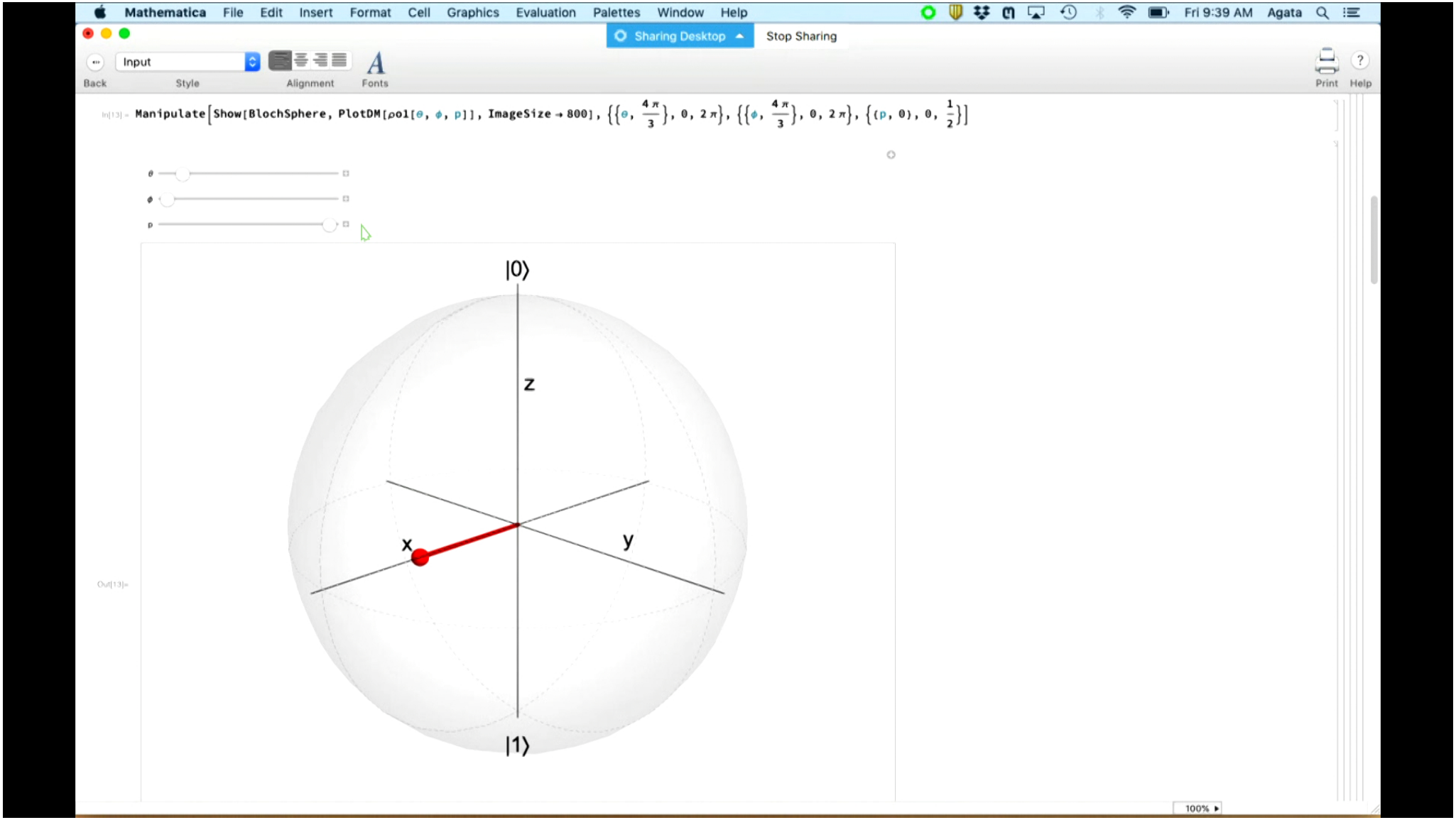


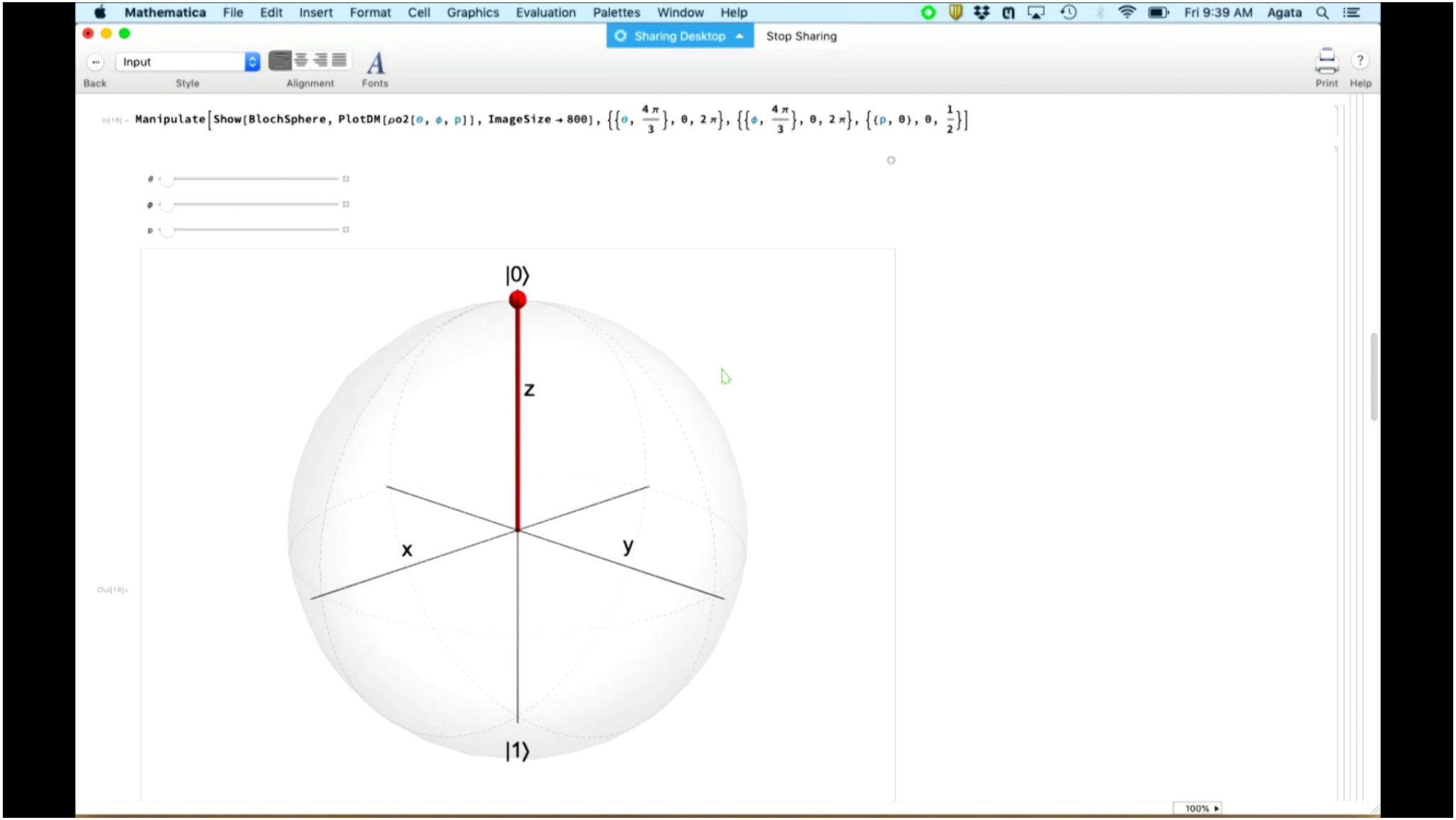
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In[13]:= Manipulate[Show[BlochSphere, PlotDM[ρ0[θ, ϕ, p]], ImageSize → 800], {{θ, 4π/3}, 0, 2π}, {{ϕ, 4π/3}, 0, 2π}, {{p, 0}, 0, 1/2}]
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ϕ
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Out[13]=
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In[13]:= Manipulate[Show[BlochSphere, PlotDM[ρ0[θ, ϕ, p]], ImageSize → 800], {{θ, 4π/3}, 0, 2π}, {{ϕ, 4π/3}, 0, 2π}, {{p, 0}, 0, 1/2}]
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In[18]:= Manipulate[Show[BlochSphere, PlotDM[ρo2[θ, ϕ, p]], ImageSize → 800], {{θ, 4π/3}, 0, 2π}, {{ϕ, 4π/3}, 0, 2π}, {{p, 0}, 0, 1/2}]
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Phase-flip channel

Define the Kraus operators

```
A2₀[p_] = √(1 - p) PauliMatrix[0];
A2₁[p_] = √p PauliMatrix[3];
```

Calculate the output state

```
ρo2[θ_, ϕ_, p_] = Simplify[Sum[A2ₖ[p].ρi[θ, ϕ].ConjugateTranspose[A2ₖ[p]], {k, 0, 1}], (p) ∈ Reals && 1 > p > 0];
% // MatrixForm
```

Out[17]//MatrixForm=

$$\begin{pmatrix} \cos\left[\frac{\theta}{2}\right]^2 & \frac{1}{2} e^{-i\phi} (1-2p) \sin[\theta] \\ \frac{1}{2} e^{i\phi} (1-2p) \sin[\theta] & \sin\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

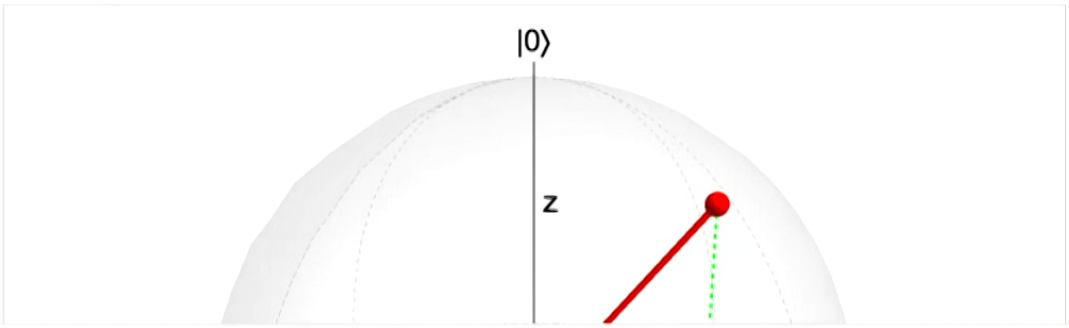
Plot the density operator on the Bloch Sphere

```
Manipulate[Show[BlochSphere, PlotDM[ρo2[θ, ϕ, p]], ImageSize → 800], {{θ, 4π/3}, 0, 2π}, {{ϕ, 4π/3}, 0, 2π}, {{p, 0}, 0, 1/2}]
```

θ

ϕ

p



100%

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Phase-flip channel

Define the Kraus operators

```
A2₀[p_] = √(1 - p) PauliMatrix[0];
A2₁[p_] = √p PauliMatrix[3];
```

Calculate the output state

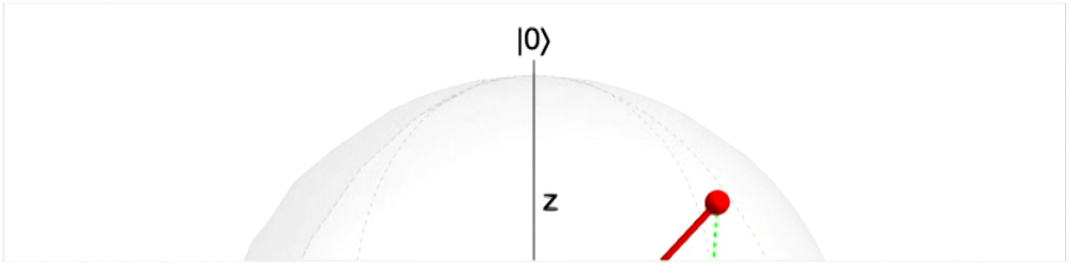
```
ρo2[θ_, ϕ_, p_] = Simplify[Sum[A2ₖ[p].ρi[θ, ϕ].ConjugateTranspose[A2ₖ[p]], {k, 0, 1}], (p) ∈ Reals && 1 > p > 0];
% // MatrixForm
```

$$\begin{pmatrix} \cos\left[\frac{\theta}{2}\right]^2 & \frac{1}{2} e^{-i\phi} (1-2p) \sin[\theta] \\ \frac{1}{2} e^{i\phi} (1-2p) \sin[\theta] & \sin\left[\frac{\theta}{2}\right]^2 \end{pmatrix}$$

Plot the density operator on the Bloch Sphere

```
Manipulate[Show[BlochSphere, PlotDM[ρo2[θ, ϕ, p]], ImageSize → 800], {{θ, 4π/3}, 0, 2π}, {{ϕ, 4π/3}, 0, 2π}, {{p, 0}, 0, 1/2}]
```

θ
ϕ
p



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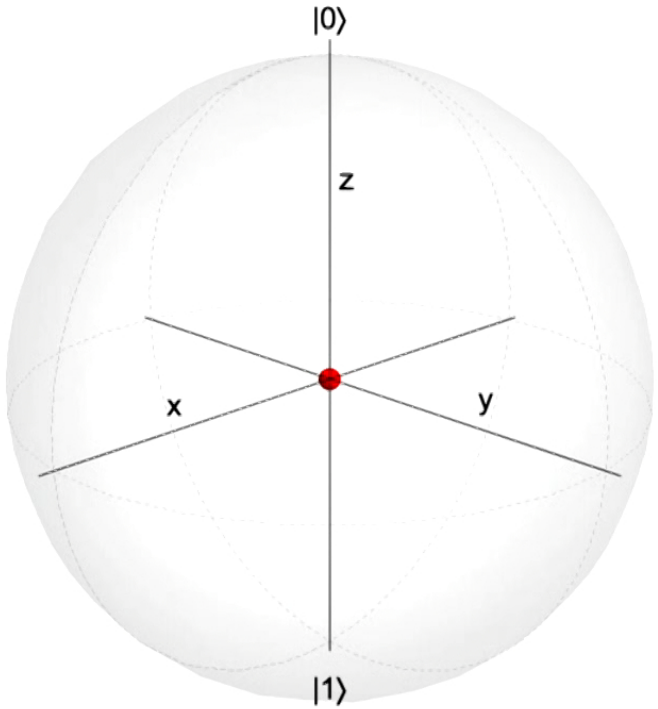
Plot the density operator on the Bloch Sphere

```
In[25]:= Manipulate[Show[BlochSphere, PlotDM[ρo3[θ, ϕ, p]], ImageSize → 800], {{θ, 4π/3}, 0, 2π}, {{ϕ, 4π/3}, 0, 2π}, {{p, 0}, 0, 1/3}]
```

θ

ϕ

p



Out[25]=

100%

nothing.

Measurements

Postulate 3: A realistic measurement procedure is represented by a POVM $\{E_m\}$. The index "m" refers to the measurement outcomes that may occur in the experiment. The prob of finding "m", given $\hat{\rho}$, is $Pr(m) = Tr[E_m \hat{\rho}]$.

A discrete POVM (positive operator valued measure) is any set of operators $\{E_m\}$ that satisfy:

$$\sum_m E_m = \mathbb{1} \qquad E_m \geq 0 \;\; \forall\, m$$

a.k.a. generalized measurements

Example. With prob $\frac{1}{4}$ meas in σ_z basis
With prob $\frac{3}{4}$ meas in σ_x basis.

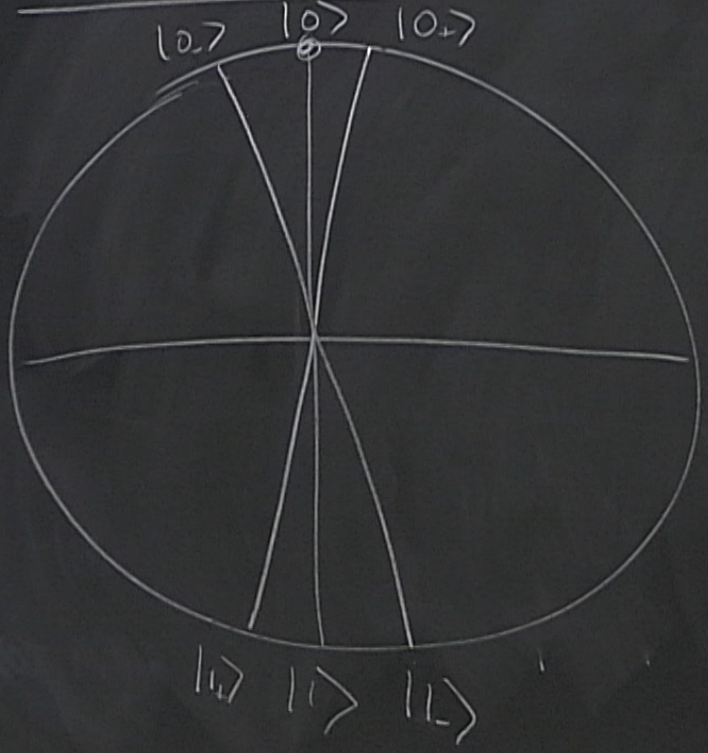
$E_0 = \frac{1}{4}|0\rangle\langle 0|$
$E_1 = \frac{1}{4}|1\rangle\langle 1|$
$E_+ = \frac{3}{4}|+\rangle\langle +|$
$E_- = \frac{3}{4}|-\rangle\langle -|$

Given $\hat{\rho} = |0\rangle\langle 0|$, what is $Pr(+)$

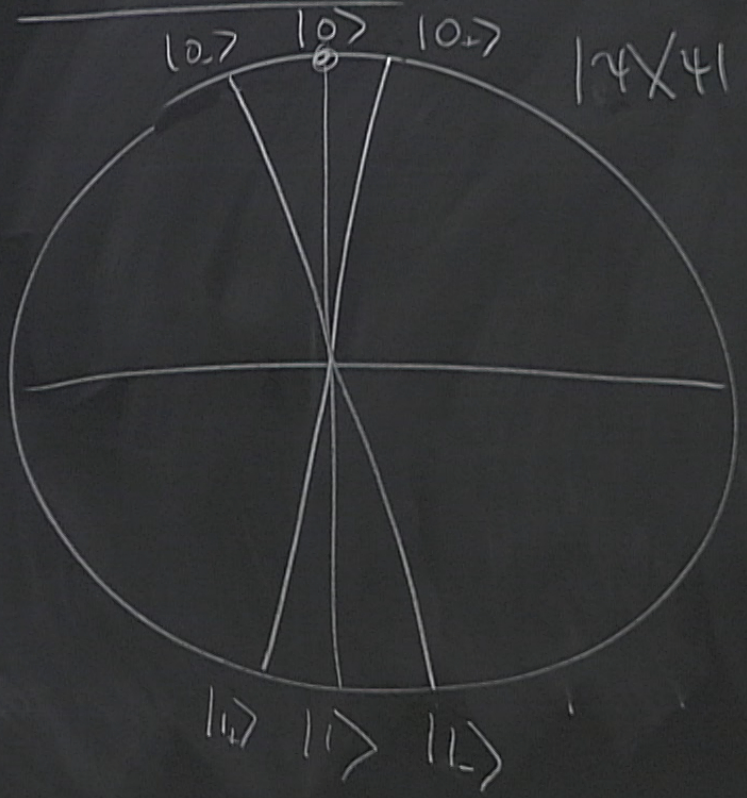
$$Pr(+) = Tr\left[\left(\tfrac{3}{4}|+\rangle\langle +|\right)\left(|0\rangle\langle 0|\right)\right]$$

$$= \frac{3}{4}\langle 0|+\rangle\langle +|0\rangle = \frac{3}{4}\times\frac{1}{2} = \frac{3}{8}$$

Example 2.



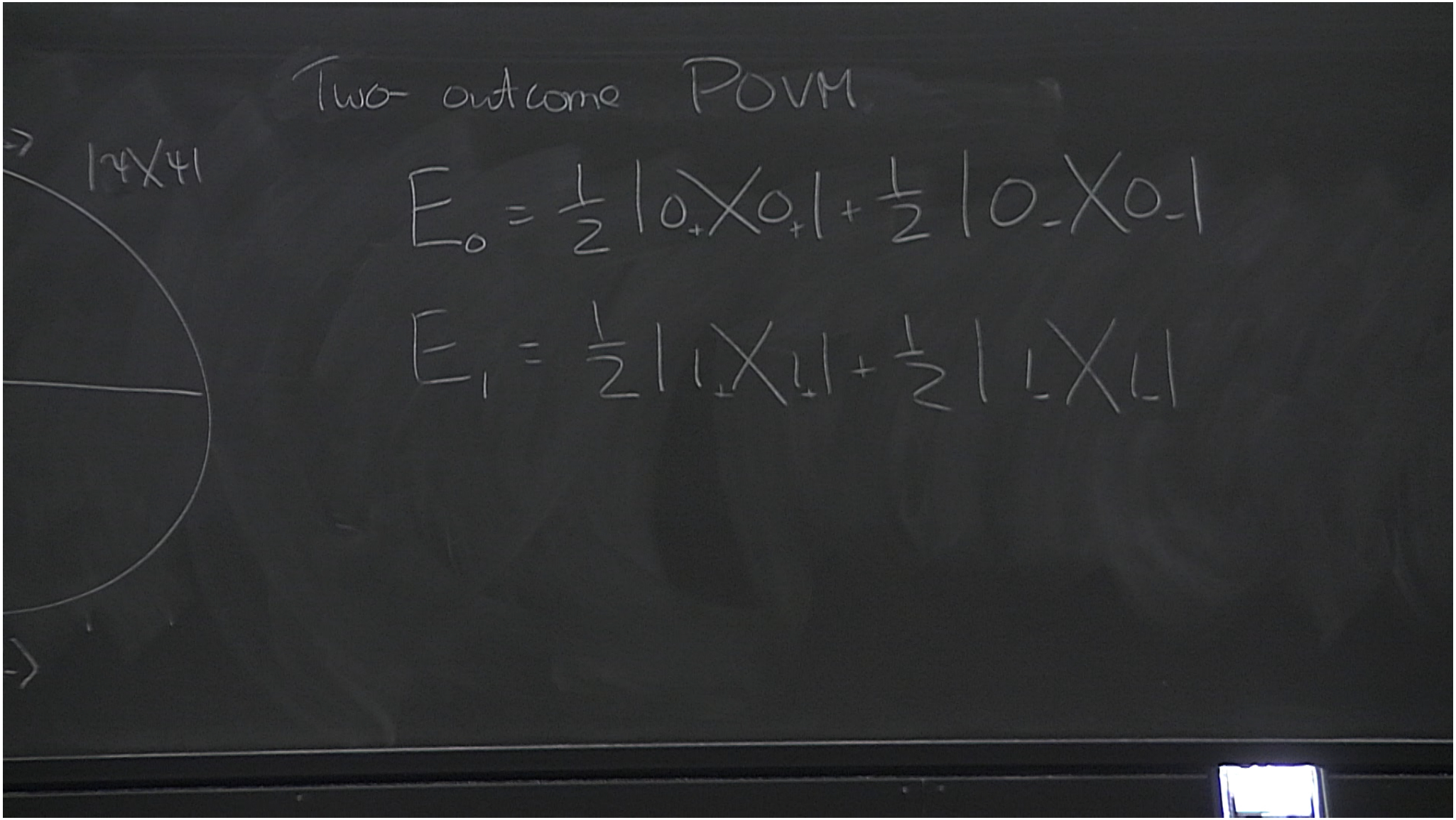
Example 2.



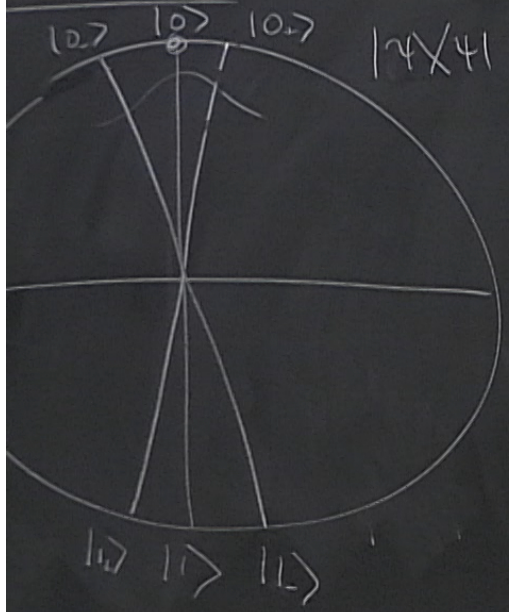
Two-outcome POVM

$$E_0 = \frac{1}{2}|0_+\rangle\langle 0_+| + \frac{1}{2}$$

$$E_1 = \frac{1}{2}|1_-\rangle\langle 1_-| + \frac{1}{2}$$


Two- outcome POVM
$E_0 = \frac{1}{2}|0_+\rangle\langle 0_+| + \frac{1}{2}|0_-\rangle\langle 0_-|$
$E_1 = \frac{1}{2}|1_+\rangle\langle 1_+| + \frac{1}{2}|1_-\rangle\langle 1_-|$

Example 2.



Two outcome POVM

$$E_0 = \frac{1}{4}|0_+\rangle\langle 0_+| + \frac{1}{4}|0_-\rangle\langle 0_-| + \frac{1}{2}|0\rangle\langle 0|$$

$$E_1 = \frac{1}{2}|1_+\rangle\langle 1_+| + \frac{1}{2}|1_-\rangle\langle 1_-|$$

Two outcome POVM

$$E_0 = \frac{1}{2}|0_+\rangle\langle 0_+| + \frac{1}{2}|0_-\rangle\langle 0_-|$$

$$E_1 = \frac{1}{2}|1_+\rangle\langle 1_+| + \frac{1}{2}|1_-\rangle\langle 1_-|$$

Given prep $\rho = |\psi\rangle\langle\psi|$

$$P_r(0) = \mathrm{Tr}\left[\left(\tfrac{1}{2}|0_+\rangle\langle 0_+| + \tfrac{1}{2}|0_-\rangle\langle 0_-|\right)|\psi\rangle\langle\psi|\right]$$

$$= \tfrac{1}{2}\left(|\langle 0_+|\psi\rangle|^2 + |\langle 0_-|\psi\rangle|^2\right)$$

$$|0_+\rangle = p|0\rangle + \sqrt{1-p}\,|1\rangle$$

$\rho = |1\rangle\langle 1|$

$\left[\frac{1}{2}|0_+\rangle\langle 0_+| + \frac{1}{2}|0_-\rangle\langle 0_-|\right]|1\rangle\langle 1|$

$\left(|\langle 0_\perp|1\rangle|^2 + |\langle 0_-|1\rangle|^2\right) \neq 0 \quad (\text{for } p<1)$

$+\sqrt{1-p}\,|1\rangle$

State update rule.

POVM does not contain enough info for this.
Need to know how POVM was implemented.
This is given by "measurement operators" $M_{m,i}$

$$E_m = \sum_i M_{m,i}^\dagger M_{m,i}$$

$$\rho_m = \sum_i \frac{M_{m,i}\,\rho\,M_{m,i}^\dagger}{Pr(m)}$$

$M_{m,i}$ are <u>not</u> in general projectors.

If $M_{m,i} = M_{m,i}^\dagger = P_m$ (and the sum over i has one term)

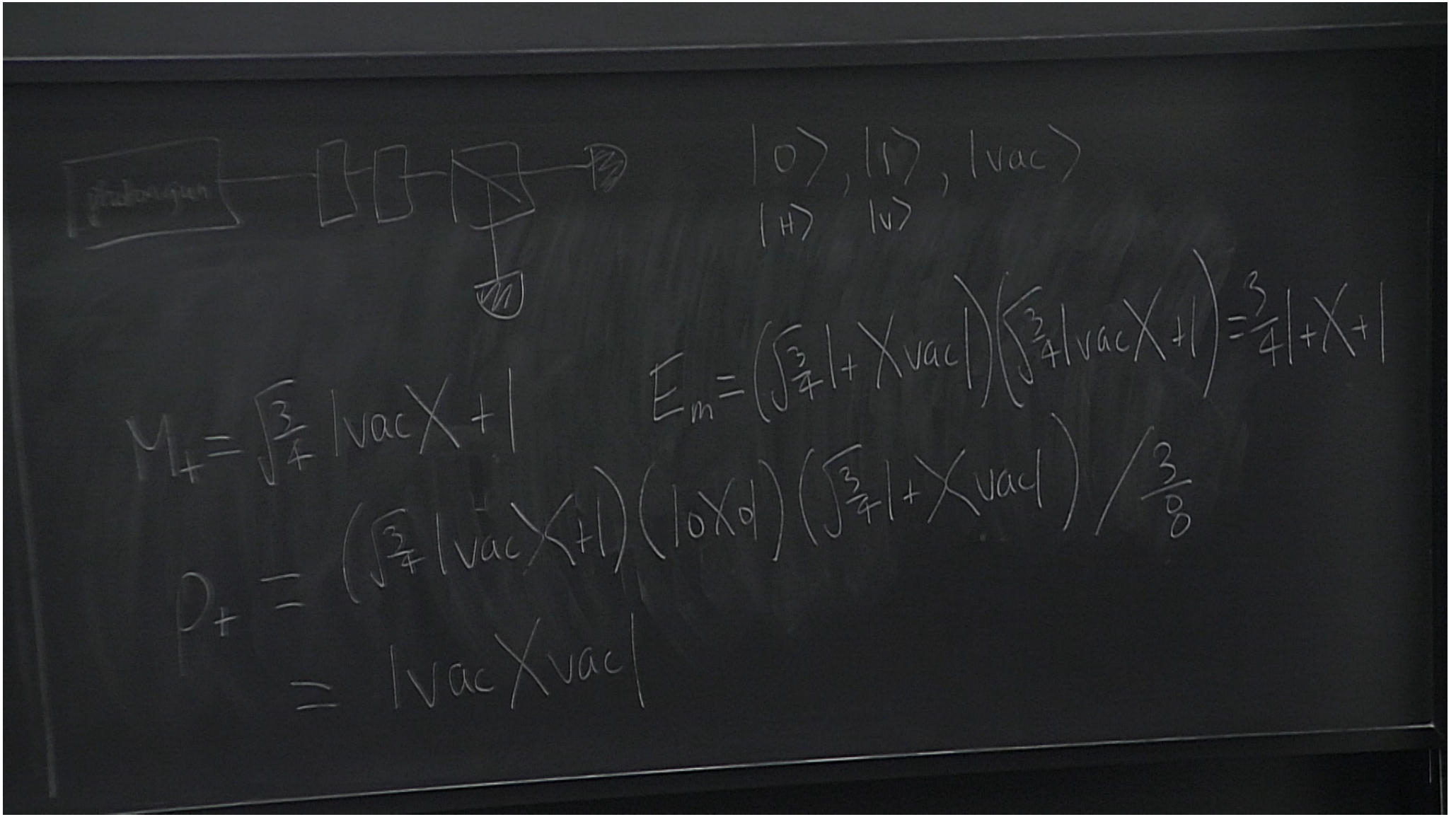
$\Rightarrow$ ideal meas postulate!

Example 1 cont.

Given outcome "+", what is ρ?

$M_+ = \sqrt{\frac{3}{4}}\,|+\rangle\langle X+|$ Check $E_m = M_m^\dagger M = \frac{3}{4}|+\rangle\langle X+|$

$$\rho_+ = \frac{\left(\sqrt{\frac{3}{4}}\,|+\rangle\langle X+|\right)|0\rangle\langle X0|\left(\sqrt{\frac{3}{4}}\,|+\rangle\langle X+|\right)}{\frac{3}{8}} = |+\rangle\langle X+|$$


photon gun
D
|0⟩, |1⟩, |vac⟩
|H⟩ |V⟩
$M_+ = \sqrt{\frac{3}{4}}|vac\rangle X + 1|$
$E_m = \left(\sqrt{\frac{3}{4}}|+\rangle X|vac|\right)\left(\sqrt{\frac{3}{4}}|vac\rangle X + 1|\right) = \frac{3}{4}|+\rangle X + 1|$
$\rho_+ = \left(\sqrt{\frac{3}{4}}|vac\rangle X + 1|\right)(|0\rangle X\langle 0|)\left(\sqrt{\frac{3}{4}}|+\rangle X|vac|\right) / \frac{3}{0}$
$= |vac\rangle X\langle vac|$

Postulate 1:
A realistic preparation procedure is described by a density operator, which is a positive semi-definite operator ρ with unit trace, acting on Hilbert space $\mathcal{H}$.

Postulate 2:
Realistic dynamical transformations are prescribed by completely positive trace preserving maps $\mathcal{E}(\rho)$.

