Title: PSI 2019/2020 - Relativity (Kubiznak) - Lecture 13

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Collection: PSI 2019/2020 - Relativity (Kubiznak)

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URL: http://pirsa.org/19090040

b) SCHWARZSCHILD SOLUTION (1916)

SCHWARZSCHILD METRIC

$$\boxed{ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \qquad f = 1 - \frac{2M}{r}}$$

EXACT SOLUTION OF $R_{\mu\nu} = 0$

SPHERICALLY SYMMETRIC

- M ... INTEGRATION CONSTANT
- AS $r \to \infty$ WE RECOVER FLAT METRIC IN SPHERICAL COORDINATES.

MORE GENERALLY FOR $M \ll r$

$$ds^2 \simeq -\left(1-\frac{2M}{r}\right)dt^2+\left(1+\frac{2M}{r}\right)dr^2+r^2 d\Omega^2$$

$\Phi_N = -\frac{M}{r}$ .... GRAV. FIELD OF AN OBJECT WITH MASS $\underline{M}$.

BIRKHOFF'S THEOREM. SCHW. SOL. IS THE MOST GENERAL SPHERICALLY SYMMETRIC SOLUTION OF VACUUM E.E.

NOTE: WE HAVE EXTRA SYMMETRY. SOLUTION IS STATIC

$\boxed{\xi = \partial_t}$ .. KILLING VECTOR

PROOF: • LET'S START WITH THE MOST GENERAL SS METRIC

$$ds^2 = -e^{2\psi} f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2$$

g_{tt} $\quad$ g_{rr} $\quad$ ↖ AREA GAUGE

HAS 2 FUNCTIONS $\psi(r,t)$, $f(r,t)$

• LET $f = 1 - \frac{2m(r,t)}{r}$

• NOW IMPOSE $\boxed{R_{\mu\nu} = 0}$

MORE GENERALLY $\boxed{G^{\mu\nu} = 8\pi G T^{\mu\nu}}$

3 EQS ARE INDEP:

$$\frac{\partial m}{\partial r} = 4\pi r^2 \left(-T^t{}_t\right), \quad \frac{\partial m}{\partial t} = -4\pi r^2 \left(-T^t{}_r\right)$$

$$\frac{\partial \psi}{\partial r} = \frac{4\pi r}{f}\left(-T^t{}_t + T^r{}_r\right)$$

UNCTIONS

$f(r,t)$

g_{tt} g_{rr} AREA GAUGE

IN VACUUM $\underline{T^{\mu\nu} = 0}$

- $m = M =$ CONST
- $\psi = \psi(t)$

NOTE $e^{2\psi} dt^2 = d\tilde{t}^2$

$dt\, e^{\psi} = d\tilde{t}$

GAUGE

HAS 2 FUNCTIONS $\psi(r,t)$, $f(r,t)$

• UNEXPECTED THINGS

• $\boxed{r = r_+ = 2M}$ $f=0$ "SINGULARITY IN THE METRIC"

→ DOES NOT MATTER FOR PLANETS/STARS..

• $\boxed{r=0}$ ... "SINGULARITY IN METRIC"

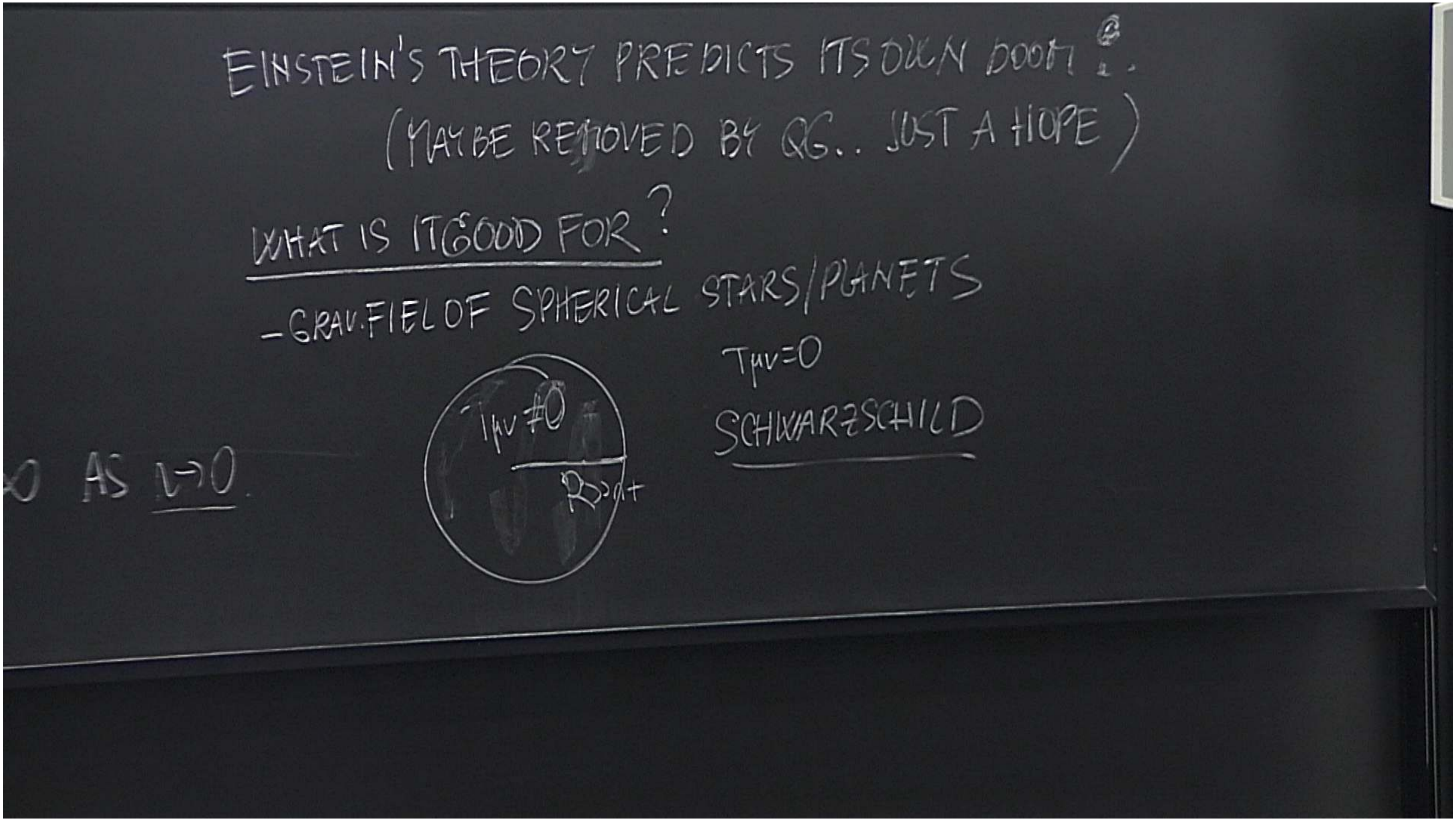
MAYBE EXOTIC OBJECT FOR WHICH THIS IS "PHYSICAL"

... BLACK HOLE

- $r = r_+ = 2M$ ... COORDINATE SINGULARITY
  (DISSAPEARS FOR NEW COORDINATES)
- $\boxed{r=0}$ ... TRUE SINGULARITY

E.G.: KRETSCHMANN SCALAR

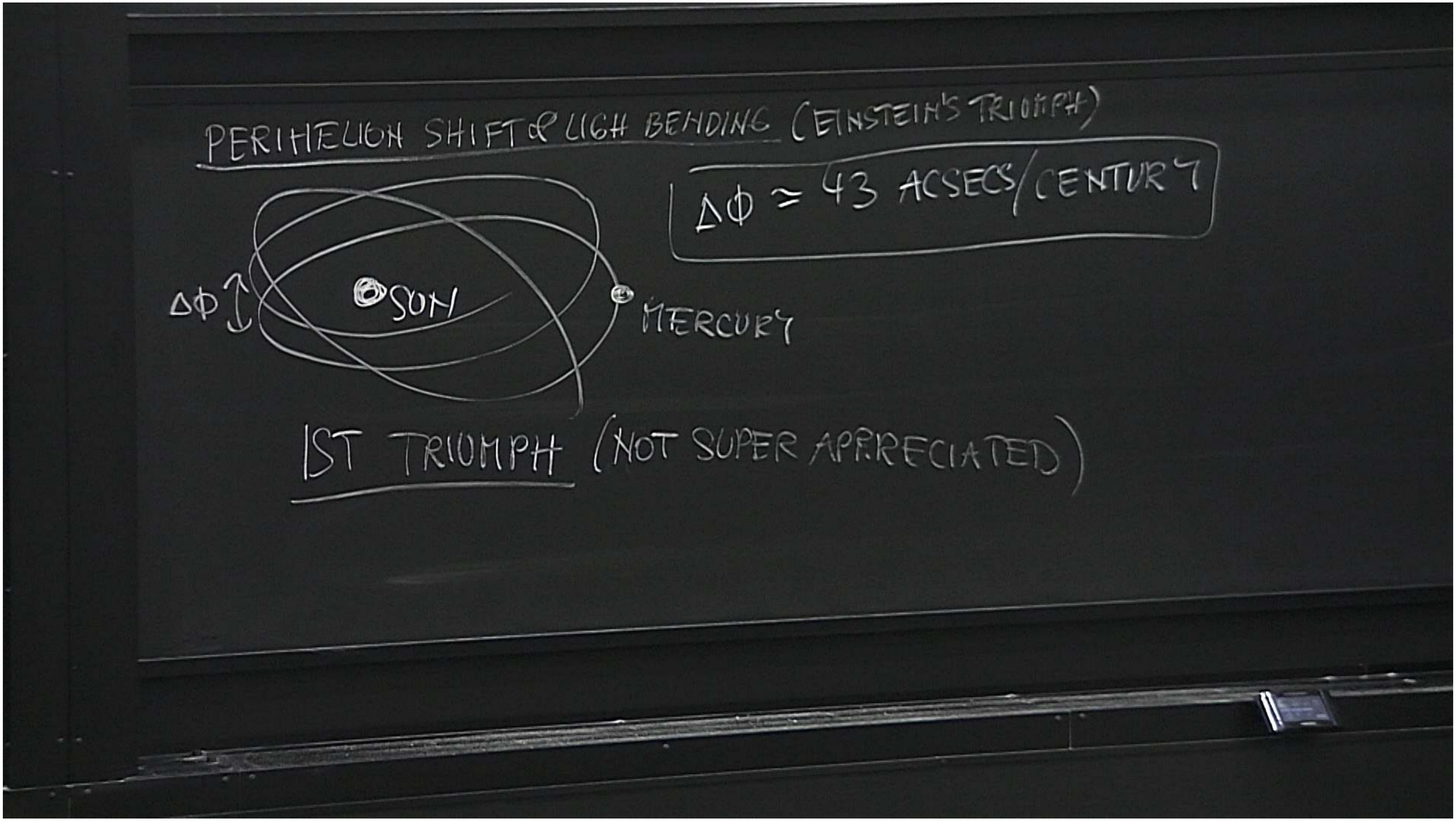
$$\boxed{R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6}} \nearrow \infty \text{ AS } r \to 0.$$


EINSTEIN'S THEORY PREDICTS ITS OWN DOOM !.
(MAYBE REMOVED BY QG.. JUST A HOPE)
WHAT IS IT GOOD FOR ?
- GRAV. FIEL OF SPHERICAL STARS/PLANETS
$T_{\mu\nu}=0$
SCHWARZSCHILD
$T_{\mu\nu}\neq 0$
$R>R_{st}$
AS $r\to 0$

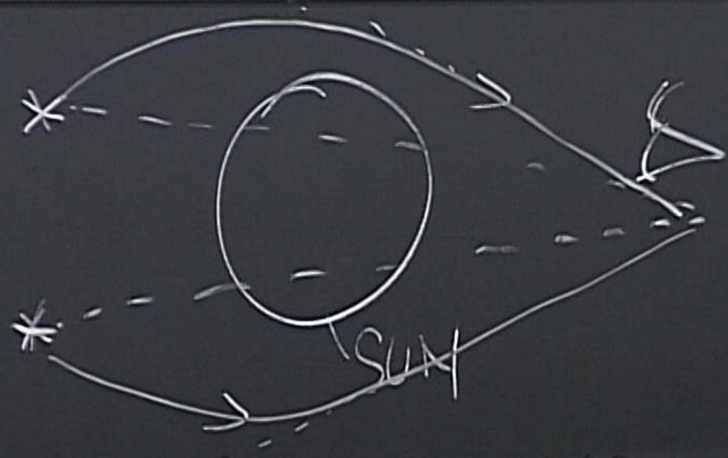
— BLACK HOLE (SOLUTION IS "VALID" ALL THE WAY TO $\Lambda = 0$)

r_+ ... RADIUS OF BLACK HOLE HORIZON

- PRIMORDIAL (TINY TINY) .. NOT OBSERVED SO FAR
- STELLAR MASS ($M \simeq M_\odot$) .. SEEN TO BE BIGGER THAN EXPECTED ?
  LESS SPINNING —"— ?
- SUPERMASSIVE ($M \simeq 10^8 M_\odot$) ... HOW ARE THEY FORMED ?


PERIHELION SHIFT & LIGH BENDING (EINSTEIN'S TRIUMPH)
ΔΦ ≃ 43 ACSECS/CENTURY
ΔΦ
SUN
MERCURY
1ST TRIUMPH (NOT SUPER APPRECIATED)

TURY



STARS GET APART
AS SUN CROSSES BY

EDDINGTON 1919. EINSTEIN BECAME CELEBRITY.

$\boxed{\xi = \partial_t}$ .. KILLING VECTOR

TO UNDERSTAND BOTH, WE STUDY GEODESICS IN SCHW. METRIC AROUND SUN.

- S.S. $\Rightarrow$ MOTION PLANAR. $\theta = \frac{\pi}{2}$
- 2 KILLING VECTORS

$$\underset{\text{STATIC}}{\xi = \partial_t}, \quad \underset{\text{AXISYMMETRIC}}{m = \partial_\phi}$$

E.E.

• LET $f = 1 - \frac{2m(r,t)}{\ldots}$

• $U^\mu = \frac{dx^\mu}{d\lambda} = \dot{x}^\mu = (\dot{t}, \dot{r}, 0, \dot{\varphi})$

• 2 INTEGRALS: $E = -U\cdot\xi = -U^\mu \xi_\mu = -U^\mu \xi^\nu g_{\mu\nu} =$

$= -U^\mu g_{t\mu} = f\dot{t}$

$L = U\cdot m = g_{\phi\phi}\dot{\phi} = r^2\dot{\phi}$

$U^2 = -\mathcal{H} = -f\dot{t}^2 + \frac{\dot{r}^2}{f} + r^2\dot{\varphi}^2$

$\mathcal{H} \begin{cases} 1 \ldots \text{MASSIVE (MERCURY)} \\ 0 \ldots \text{LIGHT} \end{cases}$

BLACK HOLE

• USING 1ST TWO EQS:

$$-\varkappa = -f\left(\frac{E}{f}\right)^2 + \frac{\dot{r}^2}{f} + r^2\left(\frac{L}{r^2}\right)^2$$

$$\boxed{\frac{1}{2}\dot{r}^2 + V = \frac{1}{2}E^2}$$

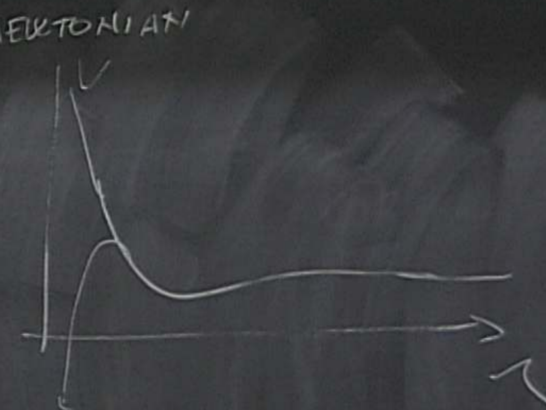
$$\boxed{V = \frac{\varkappa}{2} - \frac{\varkappa M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}}$$

EFFECTIVE POTENTIAL

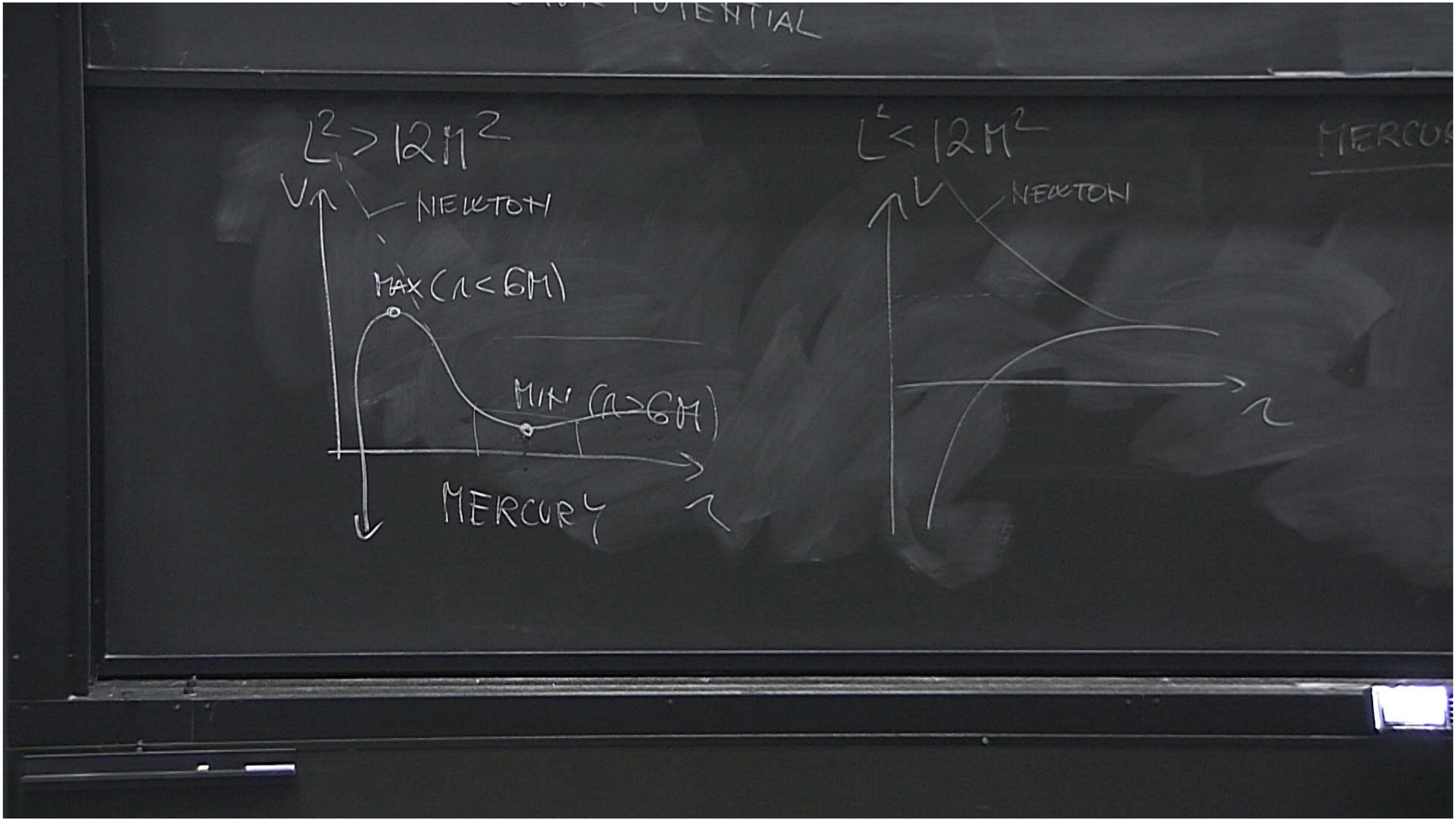
$\mathcal{H} < 0 \ldots$ LIGHT

• FOR MERCURY ($\mathcal{H}=1$)

$$V = \underbrace{\frac{1}{2} - \underbrace{\frac{M}{r}}_{\text{NEWTON}} + \underbrace{\frac{L^2}{2r^2}}_{\text{CENTRE}}}_{\text{NEWTONIAN}} - \underbrace{\frac{ML^2}{r^3}}_{\text{GR CORRECTION}}$$



INFLECTION POINT $\quad V' = 0 = V'' \iff L^2 = 12M^2,\ r = 6M$



POTENTIAL
L² > 12M²
V
NEWTON
MAX (r < 6M)
MIN (r > 6M)
MERCURY
r
L² < 12M²
V
NEWTON
r
MERCURY

MERCURY: TRAJECTORY ALMOST CIRCULAR $(r \gg M)$

- $V' = 0$ SOLVE FOR L^2

$$L^2 = \frac{M r_c^2}{r_c - 3M}$$

- $\omega_r^2 = V''\big|_{r=r_c}$ RADIAL OSCILLATIONS

$$= \frac{M(r_c - 6M)}{r_c^3 (r_c - 3M)}$$

$\rightarrow r$

- ANGULAR FREQ:

$$\omega_\phi^2 = \frac{L^2}{r_c^4} = \frac{M}{r_c^2(r_c-3M)}$$

- IF ω_ϕ MULTIPLE OF ω_r ... TRAJECTORY IS CLOSED.

PRECESSION FREQ.

$$\omega_p = \omega_\phi - \omega_r \simeq \frac{3M^{3/2}}{r_c^{5/2}}$$

KEPLER $T^2 = 4\pi r_c^3/M$

$$\Delta\phi = T\omega_p = \frac{6\pi M^2}{L^2}$$

ORBITAL PER.

EINSTE

WHAT

- GRA

← STELLAR MASS ($M \simeq M_\odot$) . SEEM TO BE BIGGER THAN

LESS SPINNING

SUPERMASSIVE ($M \sim 10^8 M$

HOW ARE THEY