

Title: PSI 2019/2020 - Relativity (Kubiznak) - Lecture 9

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Collection: PSI 2019/2020 - Relativity (Kubiznak)

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YESTERDAY MATTER IN CURVED SPACE

$$S_m = \int d^4x \sqrt{-g} \mathcal{L}_m(\phi, \nabla\phi, g)$$

$$\delta_\phi S_m = \int \frac{\delta S_m}{\delta \phi} \delta \phi = \int d^4x \sqrt{-g} \left(\frac{\partial \mathcal{L}_m}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}_m}{\partial (\nabla_\mu \phi)} \right) \right) \delta \phi + \text{BDR}$$

(E-L) FIELD EQUATIONS

$$\frac{\partial \mathcal{L}_m}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}_m}{\partial (\nabla_\mu \phi)} \right) = 0$$

$$\delta_g S_m = -\frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu}$$

DEF OF (ROSENFELD'S) ENERGY-MOMENTUM TENSOR

- SYMMETRIC

- IF FIELD EQUATIONS ARE SATISFIED (& S_m IS DIFF. INVARIANT)

THEN

$$\nabla_\mu T^{\mu\nu} = 0$$

"CONSERVATION OF $T^{\mu\nu}$ "

$\delta\phi$ + BDRY

EQUATIONS

PROOF: Show INVARIANT UNDER COORDINATE
TRANSF (DIFFEOMORPHISMS)

$$X^M \rightarrow X^M - \xi^M$$

INDUCES: $\delta\phi, \delta g^{\mu\nu} = \xi^\mu g^{\mu\nu} = 2 \nabla^{(\mu} \xi^{\nu)}$

$$X^M \rightarrow X^M - \xi^M$$

INDUCES: $\delta\phi, \delta g^{\mu\nu} = \xi^\mu g^{\mu\nu} = 2 \nabla^{(\mu} \xi^{\nu)}$

$$0 = \delta S_m = \int \left(\frac{\delta S_m}{\delta \phi} \delta\phi + \frac{\delta S_m}{\delta g^{\mu\nu}} \delta g^{\mu\nu} \right)$$

0 . FIELD EQS SATISFIED

$$= \int \frac{\delta S_m}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = -\frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu}$$

$$= - \int d^4x \sqrt{-g} T_{\mu\nu} \nabla^\mu \xi^\nu \stackrel{\text{BY PARTS}}{=} \int d^4x \sqrt{-g} \underbrace{\nabla_\mu T^{\mu\nu}}_{=0} \xi_\nu$$

$$\text{IF } \boxed{\nabla_\mu T^{\mu\nu} = 0} \stackrel{(\text{?})}{\Leftrightarrow} \boxed{\frac{\delta S_m}{\delta \phi} = 0}$$

$\updownarrow$ (?)
 FIELD EQUATIONS ???

EX1. SCALAR FIELD

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi)$$

$$(E-L) \quad \nabla^\mu \nabla_\mu \phi - \frac{dV}{d\phi} = 0$$

$$V = m^2 \phi^2$$

$$(\nabla^2 - m^2) \phi = 0$$

$$S_M = \int d^4x \sqrt{-g} \frac{c}{2}$$

WE NEED:

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$$

$$\delta g_{\mu\nu} = -g_{\mu\sigma} g_{\nu\sigma} \delta g^{\sigma\sigma}$$

$$S_m = \int d^4x \sqrt{-g} \mathcal{L}$$

WE NEED:

$$\begin{aligned} \delta \sqrt{-g} &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \\ \delta g_{\mu\nu} &= -g_{\mu\sigma} g_{\nu\sigma} \delta g^{\sigma\sigma} \end{aligned}$$

$$\delta S_m = \int d^4x \left[\underbrace{(\delta g \mathcal{L})}_{-\frac{1}{2} D_\mu \phi D_\nu \phi \delta g^{\mu\nu}} \sqrt{-g} + \mathcal{L} (\delta \sqrt{-g}) \right]$$

SATISFIED

$$S_m = \int d^4x \sqrt{-g} \mathcal{L}$$

WE NEED:

$$\begin{aligned}\delta \sqrt{-g} &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \\ \delta g_{\mu\nu} &= -g_{\mu\sigma} g_{\nu\sigma} \delta g^{\sigma\sigma}\end{aligned}$$

$$\delta S_m = \int d^4x \left[\underbrace{(\delta g^{\mu\nu}) \sqrt{-g}}_{-\frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g^{\mu\nu}} + \sqrt{-g} \delta \mathcal{L} \right]$$

$$= -\frac{1}{2} \int d^4x \sqrt{-g} \left(\nabla_\mu \phi \nabla_\nu \phi + \mathcal{L} g_{\mu\nu} \right) \delta g^{\mu\nu}$$

$$\int d^4x \sqrt{g} \mathcal{L}$$

WE NEED:

$$\begin{aligned} \delta \sqrt{-g} &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \\ \delta g_{\mu\nu} &= -g_{\mu\sigma} g_{\nu\sigma} \delta g^{\sigma\sigma} \end{aligned}$$

$$\begin{aligned} & \int d^4x \left[\underbrace{(\delta g^{\mu\nu}) \sqrt{g}}_{\equiv D_\mu \phi D_\nu \phi \delta g^{\mu\nu}} + \mathcal{L} (\delta \sqrt{-g}) \right] \\ & \equiv D_\mu \phi D_\nu \phi \delta g^{\mu\nu} \end{aligned}$$

$$\int d^4x \sqrt{g} \underbrace{(D_\mu \phi D_\nu \phi + \mathcal{L} g_{\mu\nu})}_{T_{\mu\nu}} \delta g^{\mu\nu}$$

$$\text{EX: } D_\mu T^{\mu\nu} = 0 \Leftrightarrow \square \phi = m$$

$$\delta g_{\mu\nu} = g_{\mu\sigma} \delta g^{\sigma}_{\nu} + g^{\sigma}_{\nu} \delta g_{\mu\sigma}$$

$$\delta \left(\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} g_{\mu\nu} \right) \delta g^{\mu\nu}$$

⑦

EX: $\boxed{\nabla_{\mu} T^{\mu\nu} = 0 \Leftrightarrow (\square + m^2) \phi = 0}$

EX: ELECTRIC FIELD

$\mathcal{L}_{EM}(A, \nabla A, \nabla^2 A) + \text{GAUGE INVARIANCE},$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

EX2: ELIAC FIELD

$\mathcal{L}_{EM}(A, \nabla A, \nabla^2 A) + \text{GAUGE INVARIANCE,}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

~~$F_{\mu\nu} g^{\mu\nu}$~~

$$F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (E^2 - B^2)$$

$$G = \frac{1}{4} F_{\mu\nu} (*F)^{\mu\nu} \quad (E \cdot B)$$

$\mathcal{L}_{EM}(F, G^2)$.. GENERAL ELMAG THEORY,

MAXWELL $\mathcal{L}_{EM} = F$

OTHER CHOICES YIELD NON-LINEAR ELECTROMAGNETISM,

EX. BORN-INFELD (EVERY 20 YEARS)

$$\mathcal{L}_{BI} = b^2 \left(\sqrt{1 + \frac{2F}{b^2} - \frac{G^2}{b^4}} - 1 \right)$$

$$\mathcal{L}_{EM} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\mu\sigma} g^{\nu\sigma} - J^\nu A_\nu$$

δA_μ : $\nabla_\mu F^{\mu\nu} = -4\pi J^\nu$

$$\frac{1}{\sqrt{-g}} (\sqrt{-g} F^{\mu\nu})_{;\mu} = -4\pi J^\nu$$

NOTE: $\partial_\mu \partial^\mu A^\nu = -4\pi J^\nu$
 $\partial_\mu A^\mu = 0$

$\rightarrow \nabla_\mu \nabla^\mu A^\nu = -4\pi J^\nu$
 NOT TRUE!

$\nabla_\mu \nabla^\mu A^\nu \pm R^\nu{}_\alpha A^\alpha = -4\pi J^\nu$

NOTE: $\partial_\mu \partial^\mu A^\nu = -4\pi J^\nu$ $\rightarrow \nabla_\mu \nabla^\mu A^\nu = -4\pi J^\nu$
 $\partial_\mu A^\mu = 0$ NOT TRUE $\circ$

$$\nabla_\mu \nabla^\mu A^\nu \pm R^\nu_\alpha A^\alpha =$$

EX: 5g $T_{\mu\nu} = \frac{1}{4\pi} \left(F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right)$

$$-4\pi \left[\mathcal{L}_{EM} = \frac{1}{4} F_{\mu\nu} F_{\sigma\delta} g^{\mu\sigma} g^{\nu\delta} \right] - J^\nu A_\nu$$

δA_μ : $\nabla_\mu F^{\mu\nu} = -4\pi J^\nu$

$$\frac{1}{\sqrt{-g}} (\sqrt{-g} F^{\mu\nu})_{,\mu} = -4\pi J^\nu$$

NOTE: $\partial_\mu \partial^\mu A^\nu = -4\pi J^\nu$ $\rightarrow \nabla_\mu \nabla^\mu A^\nu = -4\pi J^\nu$
 $\partial_\mu A^\mu = 0$ NOT TRUE!

$$\nabla_\mu \nabla^\mu A^\nu \pm R^\nu{}_\alpha A^\alpha = -4\pi J^\nu$$

EX: Eq

$$T_{\mu\nu} = \frac{1}{4\pi} \left(F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right)$$

~~$$F_{\mu\nu} g^{\mu\nu}$$~~

$$F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (E^2 - B^2)$$

EX. BOE

EX3. PERFECT FLUID

$$T^{\mu\nu} = (\rho + P) U^{\mu} U^{\nu} + P g^{\mu\nu}$$

U^{μ} ... 4-VELOCITY OF FLUID

ρ ... ENERGY DENSITY

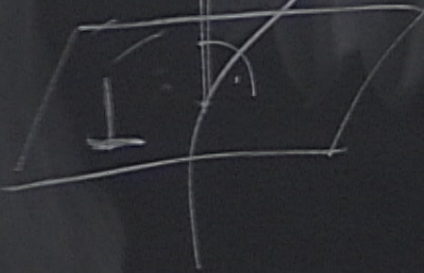
P ... PRESSURE

} OBSERVER MOVING
WITH THE FLUID.

HAT DO WE GET?

$$0 = \nabla_\mu T^{\mu\nu} = (\rho + p)_{;\mu} U^\mu U^\nu + (\rho + p) (\nabla_\mu U^\mu) U^\nu + (\rho + p) U^\mu \nabla_\mu U^\nu + P_{;\mu} g^{\mu\nu} + 0$$

U^μ



$$\delta^\alpha_\beta = \underbrace{h^\alpha_\beta}_I + \underbrace{(-U^\alpha U_\beta)}_{II}$$

EX2. ELECTRIC FIELD

EX3. PERFECT FLUID

WHAT DO

$$T^{\mu\nu} = (\rho + P) U^\mu U^\nu + P g^{\mu\nu}$$

U^μ ... 4-VELOCITY OF FLUID ($U^2 = -1$)

ρ ... ENERGY DENSITY

P ... PRESSURE

} OBSERVER MOVING
WITH THE FLUID.

$$= (\rho + p)_{, \mu} U^\mu U^\nu + (\rho + p) (\nabla_\mu U^\mu) U^\nu + (\rho + p) U^\mu \nabla_\mu U^\nu + p_{, \mu} g^{\mu \nu} + 0$$

$$\delta^\alpha_\beta = \underbrace{h^\alpha_\beta}_{\perp} + \underbrace{(-U^\alpha U_\beta)}_{\parallel}$$

CAN CHECK.

$$h^\alpha_\beta h^\beta_\gamma = h^\alpha_\gamma$$

$$h^\alpha_\beta U^\beta = (\delta^\alpha_\beta + U^\alpha U_\beta) U^\beta = U^\alpha - U^\alpha = \underline{\underline{0}}$$

$$\nabla_\mu T^{\mu\nu} = 0 + 0 + (g + p) U^\mu \nabla_\mu U^\nu \underbrace{h^\alpha_\nu}_{\delta^\alpha_\nu + U^\alpha U_\nu} + h^\alpha_\nu P_{;\mu} g^{\mu\nu}$$

$$U_\nu \nabla_\mu U^\nu = \frac{1}{2} \nabla_\mu (U^2)$$

$$\underline{1} \quad \nabla_\mu T^{\mu\nu} = 0: \quad 0 = h^\alpha_\nu \nabla_\mu T^{\mu\nu} = 0 + 0 + (s+p) U^\mu$$

$$0 = (s+p) U^\mu \nabla_\mu U^\alpha + h^\alpha_\nu P_{\mu\nu} g^{\mu\nu} U_\nu \nabla_\mu U^\nu = \frac{1}{\Sigma}$$

$$(s+p) \frac{DU^\alpha}{d\tau}$$

$$\underline{\perp \nabla_\mu T^{\mu\nu} = 0.} \quad 0 = h^\alpha{}_\nu \nabla_\mu T^{\mu\nu} = 0 + 0 +$$

$$0 = (S + P) U^\mu \nabla_\mu U^\alpha + h^\alpha{}_\nu P_{\mu}{}^{\nu} g^{\mu\lambda} U_\lambda \nabla_\mu$$

$$\boxed{\left(S + \frac{P}{c^2}\right) \frac{DU^\alpha}{d\tau} = - \left(\nabla^\alpha P \right)_\perp}$$

RELATIVISTIC
NAVIER-

RELATIVISTIC STARS HAVE HARD LIFE

$$T^{\mu\nu} = 0 + 0 + (\rho + p) U^\mu \underbrace{D_\mu U^\nu}_{\delta^\mu_\nu + U^\mu U_\nu} + h^\mu_\nu P_{; \mu} g^{\mu\nu}$$

$$U_\nu D_\mu U^\nu = \frac{1}{2} D_\mu (U^2) = 0$$

RELATIVISTIC
NAVIER-STOKES

$$\rho \frac{D\pi}{Dt} = -Dp + \dots$$

PRESSURE
SELF-GRAVITATES

THE HARD LIFE (P/c^2 TERM..)

II PART. $0 = U_\nu \nabla_\mu T^{\mu\nu} = -(\rho + P)_{,\mu} U^\mu - (\rho + P) \nabla_\mu U^\mu + 0 +$

$$g_{\mu\mu} U^\mu + (\rho + P) \nabla_\mu U^\mu = 0$$

$$\boxed{\frac{d\rho}{dr} + (\rho + P) \nabla_\mu U^\mu = 0}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\boxed{\nabla_\mu (U^\mu (\rho + P)) = \frac{dP}{dr}}$$

CONTINUITY EQ.

$$U_\nu \nabla_\mu T^{\mu\nu} = -(\rho + P)_{, \mu} U^\mu - (\rho + P) \nabla_\mu U^\mu + 0 + \cancel{P_{, \mu} U^\mu}$$

$$+ (\rho + P) \nabla_\mu U^\mu = 0$$

$$+ (\rho + P) \nabla_\mu U^\mu = 0$$

$$U^\mu (\rho + P) = \frac{dP}{d\tau}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

CONTINUITY EQ.

$$\boxed{\nabla_\mu T^{\mu\nu} = 0}$$

↓
FIELD EQ.

d) EINSTEIN - HILBERT ACTION

$$\mathcal{L} = \mathcal{L}(g, \partial g)$$

THERE IS NO SUCH
SCALAR? (EQUIVALENCE
PRINCIPLE)

LET'S GO TO $\partial^3 g$

... SIMPLE CHOICE R . RICCI
SCALAR

$$S_{EH}[g] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R(g, \partial g, \partial^3 g)$$

EINSTEIN-HILBERT ACTION

δg :

$\mathcal{L} = \mathcal{L}(g, \partial g) \dots$ THERE IS NO SUCH
SCALAR? (EQUIVALENCE
PRINCIPLE)
LET'S GO TO $\partial^2 g \dots$ SIMPLE CHOICE R .

RICCI
SCALAR

$$S_{EH}[g] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R(g, \partial g, \partial^2 g)$$

TRICK:

$$R = R_{\mu\nu} g^{\mu\nu}$$

$$16\pi G \delta S_{EH} = \int d^4x \left[\underbrace{\delta(\sqrt{-g}) R}_{-\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \right]$$

$$= \int d^4x \sqrt{-g} \left(\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{G_{\mu\nu}} \right) \delta g^{\mu\nu} + \int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}$$

$$\boxed{G_{\mu\nu} = 0}$$

$$V_\mu = \nabla^\beta \delta g_{\mu\beta} - g^{\alpha\beta} \nabla_\mu (\delta g_{\alpha\beta})$$

$\nwarrow \nabla_\mu V^\mu$
 - BOUNDARY TERM

ILL POSED. ON BOUNDARY CANNOT IMPOSE THAT
BOTH $\delta g| = 0$, $\delta Vg| = 0$

-CAN UPGRADE ACTION, BY ADDING BOUNDARY
TERM WHICH CANCELS $\delta Vg|$

GIBBONS-HAWKING TERM

2. (FR) SELF-GRAVITATES

IN VACUUM: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$

By $g^{\mu\nu}$

$$R - \frac{1}{2} R d = \left(1 - \frac{d}{2}\right) R = 0$$

$d \neq 2$ $R = 0 \Rightarrow \boxed{R_{\mu\nu} = 0} \quad \text{EE IN VACUUM}$

$\boxed{G_{\mu\nu} = 0}$