

Title: PSI 2019/2020 - Relativity (Kubiznak) - Lecture 3

Speakers: David Kubiznak

Collection: PSI 2019/2020 - Relativity (Kubiznak)

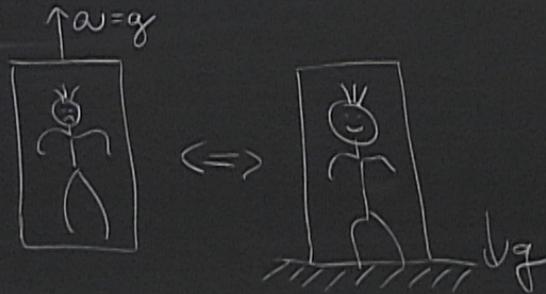
Date: September 05, 2019 - 10:45 AM

URL: <http://pirsa.org/19090031>

YESTERDAY: EQUIVALENCE PRINCIPLE

$$m\mathbf{I} = mg \Rightarrow \boxed{a = g}$$

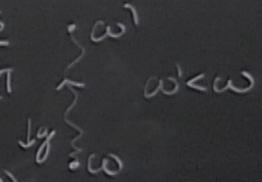
• EINSTEIN'S ELEVATOR



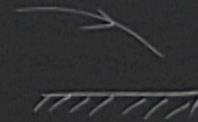
• IMMEDIATE CONSEQUENCES

• GRAVITATIONAL REDSHIFT

POUND-REBKA (1960)



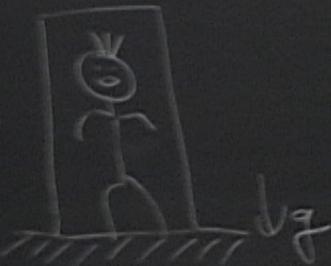
• BENDING OF LIGHT



EDDINGTON (1913)

WAVE PRINCIPLE

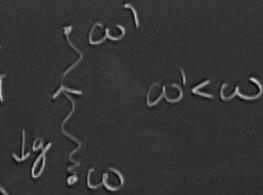
$$a = g$$



IMMEDIATE CONSEQUENCES

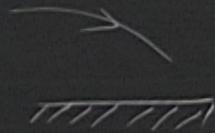
GRAVITATIONAL REDSHIFT

POUND-REBKA (1960)

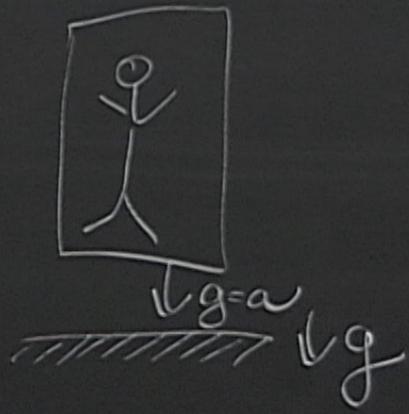
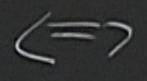


BENDING OF LIGHT

EDDINGTON (1913)



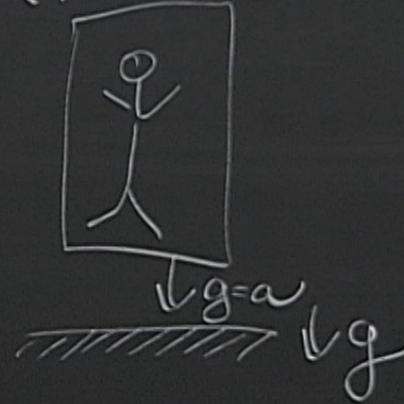
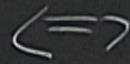
• LOCAL INERTIAL FRAME



ω'
 $\omega' < \omega$
 g
 ω

IGTDH (1913)

• LOCAL INERTIAL FRAME (FREELY FALLING FRAME)



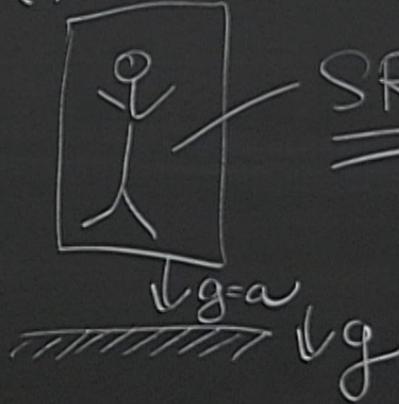
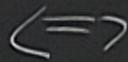
ω'

$\omega' < \omega$

ω

(1913)

• LOCAL INERTIAL FRAME (FREELY FALLING FRAME)



SR IS VALID

ω'

$\omega' < \omega$

ω

(1913)

LOCAL INERTIAL FRAME (FREELY FALLING FRAME)



SR IS VALID

CAN SOLVE PHYSICS IN FREELY FALLING FRAME

→ MAKE A COORDINATE TRANSFORM TO ANY OTHER FRAME (LAB ON EARTH)

ω
 $\omega' < \omega$

ω

(1913)

CONSEQUENCE 3, GEODESIC EQUATION

- LET ξ^μ BE COORDS IN LOCAL INERTIAL FRAME

$$\frac{d^2 \xi^\mu}{d\tau^2} = 0 \quad (\text{SR GEOD EQ.})$$

CONSEQUENCE 3. GEODESIC EQUATION

- LET ξ^M BE COORDS IN LOCAL INERTIAL FRAME

$$\frac{d^2 \xi^M}{d\tau^2} = 0 \quad (\text{SR GEOD EQ.})$$

- IN SOME OTHER FRAME

$$0 = \frac{d}{d\tau} \left(\frac{\partial \xi^M}{\partial x^\alpha} \frac{dx^\alpha}{d\tau} \right) = \frac{\partial^2 \xi^M}{\partial x^\beta \partial x^\alpha} \frac{dx^\beta}{d\tau} \frac{dx^\alpha}{d\tau} + \frac{\partial \xi^M}{\partial x^\alpha} \frac{d^2 x^\alpha}{d\tau^2}$$

CONSEQUENCE 3. GEODESIC EQUATION

- LET ξ^M BE COORDS IN LOCAL INERTIAL FRAME

$$\frac{d^2 \xi^M}{d\tau^2} = 0 \quad (\text{SR GEOD EQ.})$$

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$$0 = \frac{d}{d\tau} \left(\frac{\partial \xi^M}{\partial x^\alpha} \frac{dx^\alpha}{d\tau} \right) = \frac{\partial^2 \xi^M}{\partial x^\beta \partial x^\alpha} \frac{dx^\beta}{d\tau} \frac{dx^\alpha}{d\tau} + \frac{\partial \xi^M}{\partial x^\alpha} \frac{d^2 x^\alpha}{d\tau^2}$$

USE: $\frac{\partial x^\nu}{\partial x^\alpha} = \delta^\nu_\alpha = \frac{\partial x^\nu}{\partial \xi^M} \frac{\partial \xi^M}{\partial x^\alpha}$

• MULTIPLY BY $\frac{\partial X^V}{\partial \xi^M}$

$$0 = \frac{d^2 X^V}{d\tau^2} + \underbrace{\frac{\partial X^V}{\partial \xi^M} \frac{\partial^2 \xi^M}{\partial X^\alpha \partial X^\beta}}_{\nabla_{\alpha\beta}^V} \frac{dX^\alpha}{d\tau} \frac{dX^\beta}{d\tau}$$

$$\frac{dX^\alpha}{d\tau} = u^\alpha$$

$$\boxed{\frac{du^V}{d\tau} + \nabla_{\alpha\beta}^V u^\alpha u^\beta = 0}$$



• MULTIPLY BY $\frac{\partial x^\nu}{\partial \xi^\mu}$

$$0 = \frac{d^2 x^\nu}{d\tau^2} + \underbrace{\frac{\partial x^\nu}{\partial \xi^\mu} \frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\beta}}_{\Gamma^\nu_{\alpha\beta}} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$

$$\frac{dx^\alpha}{d\tau} = u^\alpha$$

$$\boxed{\frac{du^\nu}{d\tau} + \Gamma^\nu_{\alpha\beta} u^\alpha u^\beta = 0}$$

GEODESIC EQUATION

CHRISTOFFEL SYMBOL

• MULTIPLY BY $\frac{\partial x^\nu}{\partial \xi^\mu}$

$$0 = \frac{d^2 x^\nu}{d\tau^2} + \underbrace{\frac{\partial x^\nu}{\partial \xi^\mu} \frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\beta}}_{\nabla_{\alpha\beta}^\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \quad \frac{dx^\alpha}{d\tau} = u^\alpha$$

LT. $\xi = \Lambda \cdot x$

$$\frac{du^\nu}{d\tau} + \nabla_{\alpha\beta}^\nu u^\alpha u^\beta = 0$$

GEODESIC EQUATION

CHRISTOFFEL SYMBOL

• CURVED METRIC • LOCAL INERTIAL FRAME

$$ds^2 = -d\tau^2 \stackrel{\checkmark}{=} \eta_{\mu\nu} d\xi^\mu d\xi^\nu = \underbrace{\eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}}_{g_{\alpha\beta} \dots \text{"CURVED METRIC"}} dx^\alpha dx^\beta$$

$$\boxed{ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta}$$

• CURVED METRIC · LOCAL INERTIAL FRAME

$$ds^2 = -d\tau^2 \stackrel{\vee}{=} \eta_{\mu\nu} d\xi^\mu d\xi^\nu \stackrel{\vee}{=} \underbrace{\eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}}_{g_{\alpha\beta}} dx^\alpha dx^\beta$$

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

$$g_{\alpha\beta} = g(\alpha\beta) = g_{\beta\alpha}$$

$g_{\alpha\beta}$... "CURVED METRIC"

• CURVED METRIC . LOCAL INERTIAL FRAME

$$ds^2 = -d\tau^2 \stackrel{\vee}{=} \eta_{\mu\nu} d\xi^\mu d\xi^\nu \stackrel{\vee}{=} \underbrace{\eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}}_{g_{\alpha\beta}} dx^\alpha dx^\beta$$

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

$g_{\alpha\beta}$... "CURVED METRIC"

$g_{\alpha\beta} = g(\alpha\beta) = g_{\beta\alpha}$... IN GENERAL ... 10 COMPONENTS
(DEPEND ON COORDINATES)

• CURVED METRIC . LOCAL INERTIAL FRAME

$$ds^2 = -d\tau^2 \stackrel{\checkmark}{=} \eta_{\mu\nu} d\xi^\mu d\xi^\nu \stackrel{\checkmark}{=} \underbrace{\eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}}_{g_{\alpha\beta}} dx^\alpha dx^\beta$$

$$\boxed{ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -d\tau^2}$$

$g_{\alpha\beta}$... "CURVED METRIC"

$g_{\alpha\beta} = g(\alpha\beta) = g_{\beta\alpha}$... IN GENERAL... 10 COMPONENTS
(DEPEND ON COORDINATES)

$$\bullet \boxed{u^2 = u^\alpha u^\beta g_{\alpha\beta} = -1}$$

• INVERSE METRIC $g^{\alpha\beta}$: $g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$

$\nabla \sim \partial g$

STATEMENT:

$$\nabla^{\nu} \alpha_{\beta} = \frac{1}{2} g^{\nu\mu} \left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu} \right)$$

2ND KIND

$$\nabla^{\nu} \alpha_{\beta} = g^{\nu\mu} \nabla_{\mu} \alpha_{\beta}$$

↑ 1ST KIND



• INVERSE METRIC $g^{\alpha\beta}$: $g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$

$$\nabla \sim \partial g$$

STATEMENT:

$$\nabla^{\nu} \alpha_{\beta} = \frac{1}{2} g^{\nu\mu} \left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu} \right)$$

2ND KIND

$$\nabla^{\nu} \alpha_{\beta} = g^{\nu\mu} \alpha_{\mu\beta}$$

1ST KIND

$$\boxed{u = u^\alpha u^\beta g_{\alpha\beta} = -1}$$

(DEPEND ON COORDINATES)

• PROOF: PAIN?

$$g_{\mu\alpha, \beta} = \frac{\partial}{\partial x^\beta} \left(m_{pe\delta} \frac{\partial z^\alpha}{\partial x^\mu} \frac{\partial z^\delta}{\partial x^\alpha} \right) = \dots = g_{\alpha\alpha} \Gamma_{\beta\mu}^\alpha + g_{\mu\alpha\alpha} \Gamma_{\alpha\beta}^\alpha$$

+ 2 OTHER TERMS



c) NEWTONIAN LIMIT OF GEODESIC EQ.

• SLOW MOTION

$$x^M = (t, x^i)$$

$$\left| \frac{dx^i}{d\tau} \right| \ll \frac{dt}{d\tau} \quad (v \ll c)$$

$$\frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{\mu\sigma} \frac{dx^\mu}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

c) NEWTONIAN LIMIT OF GEODESIC EQ.

• SLOW MOTION

$$x^M = (t, x^i)$$

$$\left| \frac{dx^i}{d\tau} \right| \ll \frac{dt}{d\tau} \quad (v \ll c)$$

$$\frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

0th ORDER IN \vec{v}

• STATIC FIELD $g_{\mu\nu,t} = 0$

$$\nabla^{\nu}_{tt} = \frac{1}{2} g^{\nu\mu} \left(\cancel{g_{\mu,t}} + \cancel{g_{t,\mu}} - g_{tt,\mu} \right)$$

• STATIC FIELD

$$g_{\mu\nu,t} = 0$$

$$\nabla^{\nu}_{tt} = \frac{1}{2} g^{\nu\mu} \left(\cancel{g_{\mu t,t}} + \cancel{g_{t\mu,t}} - g_{tt,\mu} \right) = -\frac{1}{2} g^{\nu\mu} g_{tt,\mu}$$

• WEAK FIELD

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2)$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

$$(\eta + h)(\eta - h) = 1 \quad \checkmark$$

$$\nabla_{tt}^V = -\frac{1}{2} \eta^{\nu\mu} h_{tt,\mu} + O(h^2)$$

$$\boxed{u^{\alpha} = u^{\alpha} u^{\beta} g_{\alpha\beta} = -1}$$

(DEPEND ON COORDINATES)

$$\nabla_{tt}^V = -\frac{1}{2} \eta^{\nu\mu} h_{tt,\mu} + O(h^2)$$

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2}$$

$$\boxed{\frac{d^2 x^{\nu}}{d\tau^2} - \frac{1}{2} \eta^{\nu\mu} h_{tt,\mu} \frac{dt}{d\tau} \frac{dt}{d\tau} = 0}$$

$$\underline{V=t} \quad \frac{d^2 t}{d\tau^2} + 0 = 0 \quad \Rightarrow \quad \frac{dt}{d\tau} = C = \text{CONST}$$

$$\underline{V=1} \quad \frac{d^2 x^i}{dt^2} C^2 - \frac{1}{2} h_{tt,ij} \eta^{ij} C^2 = 0$$

COORDINATES)

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} h_{tt, i}^{\cdot}$$

IST

IST

COORDINATES)

IST

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} h_{tt, i}^{\cdot}$$

? COMPARE TO NEWTON

$$\frac{d^2 x^i}{dt^2} = -\nabla^i \Phi_H$$

IST

$$\delta B = \frac{1}{2} g \quad (g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu})$$

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} h_{tt,i}^{\quad i}$$

$$= \frac{1}{2} \nabla^i h_{tt}$$

$$\phi_N = -\frac{1}{2} h_{tt}$$

$$g_{tt} = - \left(1 + \frac{2\phi_N}{c^2} \right)$$

? COMPARE TO NEWTON

$$\frac{d^2 x^i}{dt^2} = -\nabla^i \phi_N$$

RIEDED METRIC

STATEMENT:

$$\chi_B = \frac{1}{2} g^{\nu\mu} (g_{\mu\alpha}\beta + g_{\mu\beta\alpha} - g_{\alpha\beta\mu})$$

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} h_{tt,i}^i$$
$$= \frac{1}{2} \nabla^i h_{tt}$$

? COMPARE TO NEWTON

$$\frac{d^2 x^i}{dt^2} = -\nabla^i \phi_N$$

$$\phi_N = -\frac{1}{2} h_{tt}$$

$$g_{tt} = -\left(1 + \frac{2\phi_N}{c^2}\right)$$

$$ds^2 = -\left(1 + \frac{2\phi_N}{c^2}\right) dt^2 + \left(1 - \frac{2\phi_N}{c^2}\right) \delta_{ij} dx^i dx^j$$



$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} h_{tt,i}^{\cdot}$$

$$= \frac{1}{2} \nabla^i h_{tt}$$

$$\phi_N = -\frac{1}{2} h_{tt}$$

$$g_{tt} = -\left(1 + \frac{2\phi_N}{c^2}\right)$$

? COMPARE TO NEWTON

$$\frac{d^2 x^i}{dt^2} = -\nabla^i \phi_N$$

$$ds^2 = -\left(1 + \frac{2\phi_N}{c^2}\right) dt^2 + \left(1 - \frac{2\phi_N}{c^2}\right) \delta_{ij} dx^i dx^j$$

GOOD EQ.

TRUE BUT I DID NOT JUSTIFY YET

• INVERSE METRIC $g^{\alpha\beta}$: $g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$

$$\underline{V=1}: \frac{d^2 x^i}{dt^2} c^2 - \frac{1}{2} h_{tt,ij} \eta^{ij} c^2 = 0$$

$$g_{tt} = - (1 + \dots) c^2$$

CAN USE THIS (WEAK FIELD) METRIC

$$\boxed{\frac{\Phi_N}{c^2} \ll 1}$$



$$\Phi_N \sim \frac{GM}{R}$$

$$\boxed{10^{-9} \approx \frac{\Phi_N}{c^2}}$$

$$V^{-1} = \frac{dx^i}{dt^2} c^2 - \frac{1}{2} h_{tt,ij} \dot{\gamma}^{ij} c^2 = 0$$

CAN USE THIS (WEAK FIELD) METRIC

$$\frac{\Phi_H}{c^2} \ll 1$$



$$\Phi_H \sim \frac{GM}{R}$$

$$10^{-9} \approx \frac{\Phi_H}{c^2}$$

• SUN 10^{-6}

• NEUTRON STARS

NOTE HOWEVER GPS

$$V^{-1} = \frac{\partial x^\mu}{\partial t^2} c^2 - \frac{1}{2} h_{tt,ij} \gamma^{ij} c^2 = 0$$

CAN USE THIS (WEAK FIELD) METRIC

$$\frac{\Phi_H}{c^2} \ll 1$$



$$\Phi_H \sim \frac{GM}{R}$$

$$10^{-9} \approx \frac{\Phi_H}{c^2}$$

NOTE HOWEVER GPS

• SUN 10^{-6}

• NEUTRON STARS $10^{-2} - 10^{-1}$

• BHs $10^{-1} - 1$

• BACK TO GRAVITATIONAL REDSHIFT

STARS $10^{-2} - 10^{-1}$
 $(10^{-1} - 1)$

$$O_1 \\ X + d\tau$$

$$O_2 \\ + \\ X' d\tau'$$

$$\frac{d\tau}{d\tau'}$$

USE THIS (WEAK FIELD) METRIC

$$\frac{\Phi_H}{c^2} \ll 1$$



$$\Phi_H \sim \frac{GM}{R}$$

$$10^{-9} \approx \frac{\Phi_H}{c^2}$$

NOTE HOWEVER GPS

$$ds^2 = -d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$$

SUN: 10^{-6}

NEUTRON STARS $10^{-2} - 10^{-1}$

BHS $10^{-1} - 1$

BACK TO

$$x + d\tau$$

$$\frac{d\tau}{dt} = 1$$

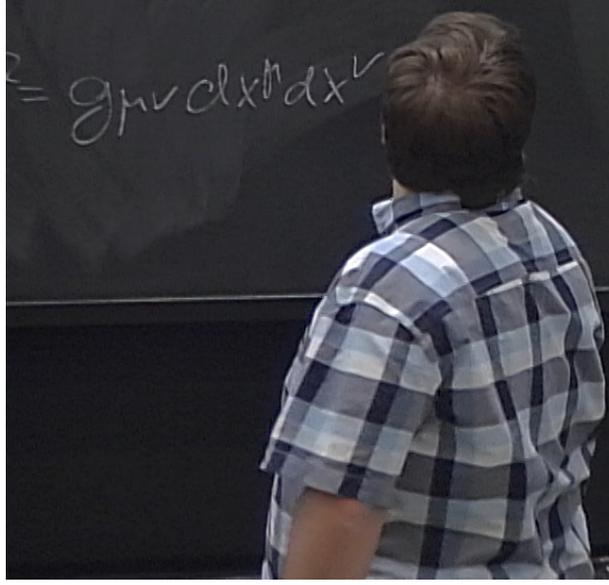
10^{-6}
 NEUTRON STARS $10^{-2} - 10^{-1}$
 BHS $10^{-1} - 1$

• BACK TO GRAVITATIONAL REDSHIFT

O_1 $x^{\mu} dx^{\nu}$ O_2 $x^{\mu} dx^{\nu}$
 $+ dt$ $+ dt'$

$$\frac{dt}{dt'} = \frac{\sqrt{-g_{\mu\nu} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt}} dt / x}{\sqrt{-g_{\mu\nu} \frac{dx^{\mu}}{dt'} \frac{dx^{\nu}}{dt'}} dt' / x'}$$

STATIC OBSERVERS
 $\frac{dx^{\mu}}{dt} = (1, 0, 0, 0)$



$$ds^2 = -d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$\frac{d\tau}{dx^1} = \frac{\sqrt{-g_{tt}(x)}}{\sqrt{-g_{tt}(x')}} = \frac{\omega'}{\omega}$$

GRAVITATIONAL
REDSHIFT



NOTE HOWEVER GPS

$$ds^2 = -c^2 dt^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$\sqrt{-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}}$$

$$\frac{dt}{dt'} = \frac{\sqrt{-g_{tt}(x)}}{\sqrt{-g_{tt}(x')}} = \frac{\omega'}{\omega}$$

GRAVITATIONAL
REDSHIFT
FOR STATIC OBSERVERS

$$z = \frac{\omega' - \omega}{\omega} = \frac{\phi_N(x) - \phi_N(x')}{c^2}$$

WEAK FIELD

$$\frac{\omega'}{\omega} = \frac{\sqrt{1 + \frac{2\phi_N(x)}{c^2}}}{\sqrt{1 + \frac{2\phi_N(x')}{c^2}}} \approx \left(1 + \frac{\phi_N(x)}{c^2}\right) \left(1 - \frac{\phi_N(x')}{c^2}\right) \approx 1 + \frac{\phi_N(x) - \phi_N(x')}{c^2}$$

$$z = \frac{\omega' - \omega}{\omega} = \frac{\phi_H(x) - \phi_H(x')}{c^2}$$

• SPEC HOMOGENEOUS FIELD

$$z = -\frac{g \Delta h}{c^2}$$

$$\approx \left| + \frac{\phi_H(x) - \phi_H(x')}{c^2} + \dots \right.$$