Title: General Relativity for Cosmology - Lecture 7

Speakers: Achim Kempf

Collection: General Relativity for Cosmology (Kempf)

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GR for Cosmology, Achim Kempf — Lecture 7

Note Title

Recall: Physical motivation for the "Metric Tensor"

- In Minkowski space, in inertial and cartesian coordinates:

$$\left[\text{distance}\left(x,\hat{x}\right)\right]^2 = -\left(x^0-\hat{x}^0\right)^2+\left(x^1-\hat{x}^1\right)^2+\left(x^2-\hat{x}^2\right)^2+\left(x^3-\hat{x}^3\right)^2$$

(4-dim space-time distance!)

(indep. of choice of inertial cds.)

$$= \eta_{\mu\nu}\left(x^\mu-\hat{x}^\mu\right)\left(x^\nu-\hat{x}^\nu\right)$$

$$\text{with } \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- In Minkowski space, in an <u>arbitrary</u> coordinate system:

$$\left[\text{distance}\left(x,\hat{x}\right)\right]^2 = g_{\mu\nu}(x)\left(x^\mu-\hat{x}^\mu\right)\left(x^\nu-\hat{x}^\nu\right)+\mathcal{O}^3$$

(e.g. polar cds, or accelerated cds)

$$\text{with } g_{\mu\nu}(x)\neq\eta_{\mu\nu}$$

(complicated higher order terms)

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- In Minkowski space, in an arbitrary coordinate system:

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with $g_{\mu\nu}(x) \neq \eta_{\mu\nu}$

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(complicated higher order terms)

- Generalization to curved space-time, historically:

  Allow even such $g_{\mu\nu}(x)$ which in no coordinate system obey:

  $$g_{\mu\nu}(x) = \eta_{\mu\nu} \quad \text{for all } x \in M$$

  $\Rightarrow$ $g_{\mu\nu}(x)$ is not simply $\eta_{\mu\nu}$ in noninertial coordinates

  $\Rightarrow$ Such $g_{\mu\nu}(x)$ take us beyond special relativity!

- Enforce Einstein's equivalence principle:

  Require $g_{\mu\nu}$ to be such that for each $x \in M$ there exists a coordinate system so that at least at x:

  $$g_{\mu\nu}(x) = \eta_{\mu\nu}$$

  (i.e., locally, special relativity holds $\mathrm{dist}(x,\hat{x})^2 = \eta_{\mu\nu}(x-\hat{x})^\mu(x-\hat{x})^\nu + \mathcal{O}^3$ to lowest nontrivial order.)

(Recall equiv. principles (EP):

If freely falling small masses

fall equally $\Rightarrow$ "weak EP"

+ same internal non-grav. physics $\Rightarrow$ "Einst. EP"

+ same internal grav. physics $\Rightarrow$ "strong EP")

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⇝ Modern view of the Einsteinian equivalence principle:

Recall: We asked that for each point $p \in \mathcal{M}$ there is a coordinate system so that, when choosing the bases $\{dx^\mu\}, \{\frac{\partial}{\partial x^\mu}\}$ then $g(x) = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu$, $g_{\mu\nu}(x) = g(x)\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$ obeys: $g_{\mu\nu}(p) = \eta_{\mu\nu}$ (in general only at p)

Modern formulation of the Einsteinian equivalence principle:

Independently of any choice of coordinate system:

There are choices of dual bases $\{\theta^\mu(x)\}, \{e_\nu(x)\}$ of $T_x(\mathcal{M})$, $T_x(\mathcal{M})^*$,

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Independently of any choice of coordinate system:

There are choices of dual bases $\{\theta^\mu(x)\}, \{e_\nu(x)\}$ of $T_x(M), T_x(M)^*$, so that: $g_{\mu\nu}(x) = g\left(e_\mu(x), e_\nu(x)\right) = \eta_{\mu\nu}$ $\forall x \in M$

Now, knowing distances through $g_{\mu\nu}$, what else follows?

- Distances yield volumes, namely $g_{\mu\nu}(x)$ induces an $\Omega(x)$.
- g, g^{-1} yield duality of covariance and contravariance.
- g yields "Hodge star" $*: \Lambda_p \to \Lambda_{n-p}$ duality.
- $*$ yields $(\ ,\ )$ making the Λ_p Hilbert spaces. (for Riemannian manifolds)
- g yields co-derivative $\delta: \Lambda_p \to \Lambda_{p-1}$
- d, δ yield the Laplacian/d'Alembertian $\Delta: \Lambda_p \to \Lambda_p$

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Proposition:

Given a notion of distance, i.e., a metric, g, this also induces a volume form Ω. (i.e., a positive $\Omega \in \Lambda_n(M)$, i.e., that when integrated over any portion of M yields a positive number)

Namely:

- Assume, as always, that M is oriented.
- Consider a positive chart.

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Then:

$$\Omega := \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m$$

where $|g| := |\det(g_{ij}(x))|$

is a well-defined volume form.

Proof:

- Nonzero for all $p \in M$?
  Yes, because g is assumed non-degenerate.

- Well-defined, i.e., is definition chart-independent?
  Yes: To see this, change chart: $x \to \tilde{x}$

Then: $\tilde{g}_{i;j}(\tilde{x}(x)) = g_{rs}(x) \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j}$ because covariant

i.e., as matrices:

$$\tilde{g} = \left(\frac{\partial x}{\partial \tilde{x}}\right)^T g \left(\frac{\partial x}{\partial \tilde{x}}\right)$$ now take determinant:

$$\Rightarrow |\tilde{g}| = \left|\frac{\partial x}{\partial \tilde{x}}\right|^2 |g| \quad \text{i.e. } |\tilde{g}|^{1/2} = \left|\frac{\partial x}{\partial \tilde{x}}\right| |g|^{1/2}$$

Also: $d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^m = \det\left(\frac{\partial \tilde{x}}{\partial x}\right) dx^1 \wedge \ldots \wedge dx^m$

$$\Rightarrow |\tilde{g}|^{1/2} d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^m = \underbrace{\left|\frac{\partial \tilde{x}}{\partial x}\right| \left|\frac{\partial x}{\partial \tilde{x}}\right|}_{=1} |g|^{1/2} dx^1 \wedge \ldots \wedge dx^m \checkmark$$

Q: Other use of g ?

A: One needs g to formulate d'Alembertian Δ, or $\Box$, for wave equations.

Why? a) $\Box$ should be non-directional 2nd derivative, but $d^2 = 0$.

b) need e.g. $\Box : \Lambda^0 \to \Lambda^0$ for Klein Gordon, i.e., need degree of forms conserved by $\Box$.

Strategy:

A) Use g for a covariant $\leftrightarrow$ contravariant tensors relation

B) Define a map "Hodge" $* : \Lambda_r \to \Lambda_{n-r}$

C) Define the "Coderivative": $\delta : \Lambda_r \to \Lambda_{r-1}$

D) Define "Laplacian/d'Alembertian": $\Box := d\delta + \delta d$

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A) Covariant $\leftrightarrow$ contravariant tensors equivalence through g:

- $g(x)$ can be used as a map: by evaluation of one tensor factor:

$$g(x): T_x(M)' \to T_x(M),$$

$$g(x): \xi^i(x) e_i(x) \to g_{\mu\nu}(x)\,\theta^\mu(x) \otimes \theta^\nu(\cdot)(\xi^i(x) e_i(x))$$

($\theta^\nu(e_i) = \delta^\nu_i$)

$$= g_{\mu\nu}(x)\,\xi^\nu(x)\,\theta^\mu(x) \quad \in T_x(M),$$

($g_{\mu\nu}(x) \in \mathcal{F}_x(M)$, $\theta^\mu(x) \in T_x(M)_1$)

$\Rightarrow$ For the coefficient functions we have: $g: \xi^\mu(x) \to \omega_\nu(x) = g_{\nu\sigma}(x)\xi^\sigma(x)$ (relative to bases θ^i, e_j)

- Conversely, g^{-1} acts as:

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B) The Hodge $*$ map: $\Lambda_p \to \Lambda_{n-p}$

(Recall: $\dim(\Lambda_p) = \binom{n}{p} = \binom{n}{n-p} = \dim(\Lambda_{n-p})$)

Idea:

- each $\nu \in \Lambda_p$ is a covariant tensor
- through g it is equivalent to a contravariant tensor $\tilde{\nu}$
- can feed Ω with $\tilde{\nu}$ to obtain $*\nu \in \Lambda_{n-p}$.

Concretely:

- Consider any $\nu := \frac{1}{p!} \nu_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \Lambda_p$

  ($\frac{1}{p!}$: convenient because drops out here; $\nu_{i_1 \dots i_p}$: anything totally antisymmetric, coefficients as a covariant tensor; one could choose other bases.)

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- Use g^{-1} to define a contravariant image of ν:

$$\tilde{\nu} = \tilde{\nu}^{i_1 \dots i_n} \frac{\partial}{\partial x^{i_1}} \otimes \dots \frac{\partial}{\partial x^{i_p}}$$

where $\tilde{\nu}^{i_1, \dots, i_p} := g^{i_1 j_1} g^{i_2 j_2} \dots g^{i_p j_p} \nu_{j_1 \dots j_p}$

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$$\Omega(\tilde{v}) = \underbrace{\Omega_{i_1 \dots i_n} \tilde{v}^{i_1 \dots i_p}}_{!!} \overbrace{dx^{i_{p+1}} \otimes \dots \otimes dx^{i_n}}^{n-p \text{ factors}} \in \Lambda_{n-p}$$

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Use $*$ to turn $\Lambda(M)$ into an "Inner Product Space":

Definition: The Hodge $*$ provides a "scalar" (or also called "inner") product for $\Lambda(M)$:

$$(\alpha, \beta) := \int_M \alpha \wedge *\beta$$

(α: p-form, β: p-form, $*\beta$: (n-p) form, $\alpha \wedge *\beta$: n-form)

This definition is extended linearly to forms that are lin. comb. of forms of arb. degree, p.

Exercises:

1.) Write (α, β) in coordinates

2.) Show that $(,)$ is always positive definite on Λ_0, i.e., $(\alpha, \alpha) > 0 \ \forall \alpha \in \Lambda_0, \alpha \neq 0$.

Notes:

- If g is indefinite, then also $(,)$ is indefinite.
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c) $(\,,\,)$ yields an adjoint for d, the co-derivative δ:

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$$D_{A^+} := \{ v \in \mathcal{H} \mid \exists w \in \mathcal{H} \ \ \forall z \in D_A : \ \langle v, Az \rangle = \langle w, z \rangle \}$$

and this action: $A^+ v := w$. We then have:

$$\langle A^+ v, z \rangle = \langle v, Az \rangle \quad \forall z \in D_A, v \in D_{A^+}$$

Definition:

The co-derivative, δ, is the (anti-) adjoint of d with

c) The Codifferential δ explicitely

Clearly: $\delta : \Lambda^p(M) \to \Lambda^{p-1}(M)$

Proposition: $\delta : \nu \to \left((-1)^{np+n+s} * d *\right)\nu$ (Some authors define δ as the negative of this)

Properties:

- $\delta^2 = 0$
- In coordinates:
$$(\delta\omega)^{i_1 \dots i_{p-1}} = \frac{1}{\sqrt{|g|}}\left(\sqrt{g}\,\omega^{k\, i_1 \dots i_{p-1}}\right)_{,k}$$
- If M is contractible (and in every contractible part):
$$\delta\nu = 0 \Rightarrow \exists \omega : \nu = \delta\omega$$

Exercises:

- Show the above.
- Determine whether or not δ is a derivation.

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Exercises:

- Show the above.
- Determine whether or not δ is a derivation.

Use d and δ to obtain the Maxwell equations on M

Define:

"Field strength": $F_{\mu\nu}(x) := \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$, $F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

(E_i: electric field; B_i: magn. field; F: "Field strength" 2-form)

"Current" 3-form $\longrightarrow$ $j(x) := \frac{1}{3!}\varepsilon_{\mu\nu\lambda\sigma}\, j^\tau\, dx^\nu \wedge dx^\lambda \wedge dx^\sigma$

(j^τ: "current 4-vector")

Then: The Maxwell Eqns read:

"Homogeneous Maxwell equations" (indep. of metric)

$$dF = 0, \quad \delta F = *j$$

"Inhomogeneous Maxwell equations" (dependent on the metric)

(j: current 1-form, i.e., cotangent vector field)

- General relativity also possesses structure equations.

Remark:

The gauge principle of electrodynamics is the observation that, for any $\omega \in \Lambda_0$:

$$A \text{ and } \tilde{A} := A + d\omega$$

describe the same physics.

The Aharonov-Bohm effect and topological phases in general, can make A directly visible when Poincaré Lemma doesn't apply.

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D The Laplacian/d'Alembertian, Δ, $\Box$:

- Definition of the Laplacian:

$$\Delta := \delta d + d\delta$$

Some authors define Δ as the negative of this and let δ be the adjoint of d.

- Clear: $\Delta : \Lambda^p(M) \to \Lambda^p(M)$
- If signature $s=1$: Then also called d'Alembertian and denoted $\Box := d\delta + \delta d$.
- Action on, e.g., $f \in \Lambda_0(M)$ in a chart:

$$\Box f = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, g^{\mu\nu} f_{,\nu} \right)_{,\mu}$$

Exercise: verify

$$\left(= \left(-\frac{\partial^2}{\partial x^{0\,2}} + \frac{\partial^2}{\partial x^{1\,2}} + \frac{\partial^2}{\partial x^{2\,2}} + \frac{\partial^2}{\partial x^{3\,2}} \right) f \quad \text{if } g = \eta \right)$$

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Properties of the d'Alembertian, $\Box$ in the Hilbert space $\Lambda(M)$:

(if Λ is a Hilbert space)

* **Defined:**

$$\Box : \Lambda_r(M) \to \Lambda_r(M)$$

$$\Box : \nu \to (\delta d + d\delta)\nu$$

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* **Defined:** $\Box : \Lambda_r(\mathcal{M}) \to \Lambda_r(\mathcal{M})$

  $\Box : \omega \to (\delta d + d\delta)\omega$

* In the Hilbert space $\Lambda(\mathcal{M})$:

  $\Box = \delta d + d\delta$ obeys $(\alpha, \Box\beta) = (\Box\alpha, \beta)$

* $\Box$ is self-adjoint, $\Box = \Box^\dagger$, for suitable boundary conditions, or if $\partial\mathcal{M} = \phi$, and assuming $(,)$ is positive definite.

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Consequences of the self-adjointness of $\Box$: – if Λ is a Hilbert space

A) The operators Δ and $\Box$ can be diagonalized, with real spectrum.

B) For Riemannian manifolds, $\mathrm{spec}(\Delta) \subset [0, \infty)$.

C) For compact Riem. manifolds (of finite volume): $\mathrm{spec}(\Delta)$ is discrete.

(Still the finite volume Riemannian case.)

D) Then, $\mathrm{spec}(\Delta)$ is carrying a lot of information about (M, g)!

Remark: There exists a related mathematical discipline, called "Spectral Geometry", combining differential geometry (of Riemannian manifolds) with functional analysis, i.e., the languages of general relativity and quantum theory!

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Application: Klein-Gordon "action":

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of Riemannian manifolds →

Application: Klein-Gordon "action":

$$g^{\mu\delta}\, e_\mu \otimes e_\delta \;(\phi_{,\alpha}\,\theta^\alpha \otimes \phi_{,\beta}\,\theta^\beta)$$

($g^{\mu\delta} \in T^2$, $\phi_{,\alpha} \in T_1$, $\phi_{,\beta} \in T_1$)

$$S[\phi] := \frac{1}{2}\int_M g^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu}\,\Omega$$

($g^{\mu\nu} = \Lambda_0$, $\phi_{,\mu} = \Lambda_0$, $\phi_{,\nu} = \Lambda_0$, $\Omega = \Lambda_n$)

($S[\phi]$: Klein Gordon field $\phi \in \mathcal{F}(M)$)

(Recall special relativity: $S'[\phi] = \int_{\mathbb{R}^4} \eta^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu}\,d^4x$)

$$= \frac{1}{2}\int g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi\right)\left(\frac{\partial}{\partial x^\nu}\phi\right)\underbrace{\sqrt{|g(x)|}\,d^n x}_{=\Omega}$$

Application: Klein-Gordon "action":

$$\overbrace{g^{\gamma\delta}\, e_\gamma \otimes e_\delta}^{\in T^2} \; (\overbrace{\phi_{,\alpha}\,\theta^\alpha}^{\in T_1} \otimes \overbrace{\phi_{,\beta}\,\theta^\beta}^{\in T_1})$$

$$S[\phi] := \frac{1}{2}\int_M \overbrace{\underset{\Lambda_0}{g^{\mu\nu}}\;\underset{\Lambda_0}{\phi_{,\mu}}\;\underset{\Lambda_0}{\phi_{,\nu}}}^{=}\;\underset{\Lambda_n}{\Omega}$$

Klein Gordon field $\phi \in \mathcal{F}(M)$

(Recall special relativity: $S[\phi] = \int_{\mathbb{R}^4} \eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\, d^4x$)

$$= \frac{1}{2}\int_M g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi\right)\left(\frac{\partial}{\partial x^\nu}\phi\right)\overbrace{\sqrt{|g(x)|}\, d^n x}^{=\Omega}$$

next: integrate by parts!

$$= \frac{1}{2}\int_M -\phi\, \underbrace{\frac{\partial}{\partial x^\nu}\left(\sqrt{|g|}\, g^{\mu\nu}\frac{\partial}{\partial x^\mu}\phi\right)\frac{1}{\sqrt{|g|}}}_{=\Box\phi}\; \underbrace{\sqrt{|g|}\, d^n x}_{=\Omega}$$

$$= -\frac{1}{2}\int \phi(\Box\phi)\,\Omega$$