Title: General Relativity for Cosmology - Lecture 6

Speakers: Achim Kempf

Collection: General Relativity for Cosmology (Kempf)

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GR for Cosmology, Achim Kempf — Lecture 6

Note Title

Integration

Q: What is special about totally antisymmetric covariant tensors, i.e., about differential forms?

A: Antisymmetry ⇒ special transformation property under chart changes:
~ det(Jacobian)

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Q: What is special about totally antisymmetric covariant tensors, i.e., about differential forms?

A: Antisymmetry ⇒ special transformation property under chart changes:

~ det (Jacobian)

⇒ suitable for integration:

s-forms have natural integrals in s-dimensional manifolds

Except: Depending on charts, sign of Jacobian may be wrong!

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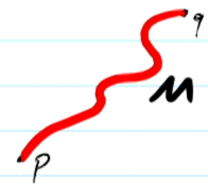
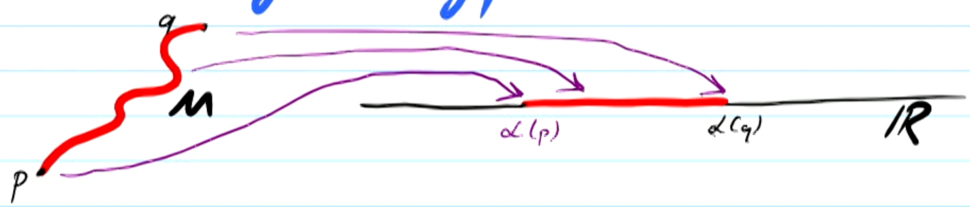
Thus: Before defining integration on mflds, must study notion of "Orientation" of the mfld.

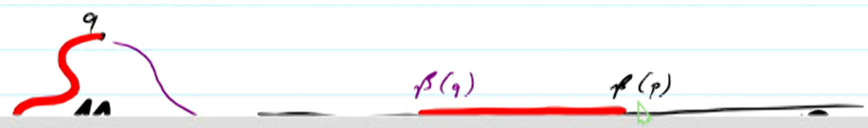
Namely: Consider e.g. 1-dim mfld:



- Could have charts of the type



- or charts of the type



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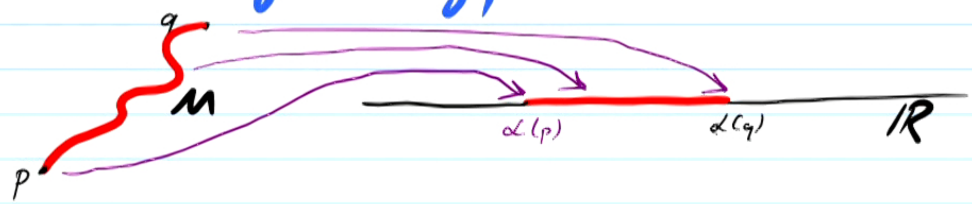
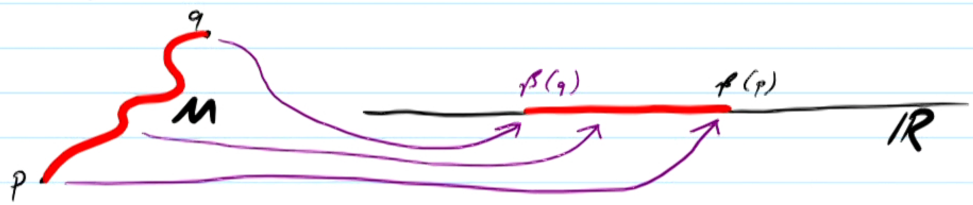


- or charts of the type



- But, since $\int_a^b f(t)\,dt = -\int_b^a f(t')\,dt'$ one needs to decide!

because $\frac{dt}{dt'} = -1$ (which is det[Jacobian])

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For n-dim mflds, may need several charts.

Definitions:

- A complete collection of charts, i.e., an Atlas, A, is called oriented if among all overlapping charts with coordinates say $x, \tilde{x}$ the Jacobi determinants are positive:

$$\det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) > 0$$

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charts with coordinates say $x, \tilde{x}$ the Jacobi determinants are positive:

$$\det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) > 0$$

☐ A mfld M is called orientable if it possesses an oriented atlas.

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are not orientable.

- A mfld, M, together with a choice of oriented atlas, A, is called an oriented manifold.

- Then, an <u>arbitrary</u> chart is called positive (or negative) if its Jacobian determinant with charts of the atlas A is positive (or negative).

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Definition:

An n-form $\Omega \in \Lambda_n(M)$ is called a volume form if it nowhere vanishes. We will later find a preferred volume form for space-time (using the metric).

Proposition:

M possesses a volume form

$\Uparrow$

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Proposition:

M possesses a volume form

$\Updownarrow$

M is orientable

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Integration:

- Recall change of cds in integration in $\mathbb{R}^n$:

For $(x^1, \dots, x^n) \to (\tilde{x}^1, \dots, \tilde{x}^n)$:

$g: \mathbb{R}^n \to \mathbb{R}$

Riemann or Lebesgue integrals →

$$\int_{\mathbb{R}^n} g(x^1, \dots, x^n)\, dx^1 \dots dx^n \overset{(*)}{=} \int_{\mathbb{R}^n} g(x(\tilde{x}))\, \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \dots d\tilde{x}^n$$

↑ Jacobian determinant is negative if coordinate systems change handedness.

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Riemann or Lebesgue integrals →

$g: \mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} g(x^1, \ldots, x^n)\, dx^1 \ldots dx^n = \int_{\mathbb{R}^n} g(x(\tilde{x}))\, \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \ldots d\tilde{x}^n \quad (*)$$

Jacobian determinant is negative if coordinate systems change handedness.

Now for a general n-dimensional diff'able mfld M, consider an n-form ω in a chart:

$$\omega = f(x)\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

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Then what is ω in an overlapping, second chart?

$$\omega = f(x(\tilde{x}))\frac{\partial x^1}{\partial \tilde{x}^{i_1}}\frac{\partial x^2}{\partial \tilde{x}^{i_2}}\cdots\frac{\partial x^n}{\partial \tilde{x}^{i_n}}\underbrace{d\tilde{x}^{i_1}\wedge d\tilde{x}^{i_2}\wedge\ldots\wedge d\tilde{x}^{i_n}}_{\text{totally antisymmetric!}}$$

- terms are nonzero only if contain each number $1,\ldots,n$ exactly once, e.g. $d\tilde{x}^1\wedge d\tilde{x}^3\wedge d\tilde{x}^2\wedge d\tilde{x}^4\wedge d\tilde{x}^5\wedge\ldots\wedge dx^n$.
- Reorder those terms - they are all

$$d\tilde{x}^1\wedge d\tilde{x}^2\wedge\ldots\wedge d\tilde{x}^n$$

up to a possible factor -1 because $dx^i\wedge dx^j = -dx^j\wedge dx^i$

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up to a possible factor -1 because $dx^i \wedge dx^j = -dx^j \wedge dx^i$

$$\Rightarrow \quad \omega = f(x(\tilde{x}))\, \det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \dots \wedge d\tilde{x}^n$$

Compare with equation (*) above $\Rightarrow$

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The following definition of the integral of n-forms in an n-dim. diffable mfld is chart-independent, i.e., is well-defined:

Definition:

Assume M is an oriented n-dim mfld and $\omega \in \Lambda_n(M)$ reads in a chart α: $\omega = f(x)\, dx^1 \wedge \dots \wedge dx^n$.

Then, if one chart suffices:

$$\int \omega := \underbrace{\int f(x)\, dx^1 dx^2 \dots dx^n}_{\text{usual Riemann or Lebesgue integral}}$$

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Definition:

Assume M is an oriented n-dim mfld and $\omega \in \Lambda_n(M)$ reads in a chart d: $\omega = f(x)\, dx^1 \wedge \dots \wedge dx^n$.

Then, if one chart suffices:

$$\int_M \omega := \underbrace{\int_{d(M)} f(x)\, dx^1 dx^2 \dots dx^n}_{\text{usual Riemann or Lebesgue integral}}$$

$\nwarrow$ image of M in $\mathbb{R}^n$

Else: Piece right hand side together from several charts

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and $\omega \in \Lambda_n(M)$ reads in a chart α: $\omega = f(x)\, dx^1 \wedge \dots \wedge dx^n$.

Then, if one chart suffices:

$$\int_M \omega := \underbrace{\int_{\alpha(M)} f(x)\, dx^1 dx^2 \dots dx^n}_{\text{usual Riemann or Lebesgue integral}}$$

$\alpha(M)$ ↖ image of M in $\mathbb{R}^n$

Else: Piece right hand side together from several charts

Note: how to piece together does not matter as long as charts are from the atlas that M is equiped with. That's why orientation is important.

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Definition: The boundary operator, ∂

- Assume $G \subset M$ is a region (i.e. an n-dim., open and connected subset) of the n-dim mfld M.

We denote the $(n-1)$ dim. boundary manifold of G by ∂G:

the boundary operator

$$\partial G := \text{boundary}(G)$$

- We say that ∂G is smooth if locally

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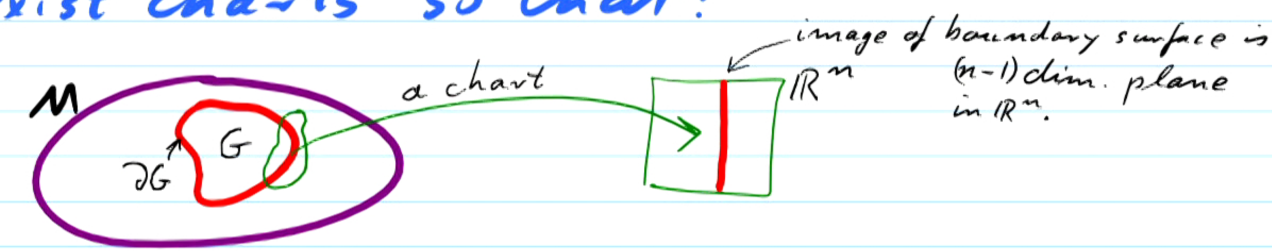
We denote the $(n-1)$ dim. boundary manifold of G by ∂G:

(the boundary operator)

$$\partial G := \text{boundary}(G)$$

We say that ∂G is smooth if locally there exist charts so that:



Proposition: If M is orientable, then so is G. Also, the orientation of G induces an orientation of ∂G.

We finally have all ingredients for one of Math's most important theorems:

Stokes' theorem: If closure $\bar{G}$ of G is a compact n-dim region, then:

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We finally have all ingredients for one of Math's most important theorems:

Stokes' theorem: If closure $\bar{G}$ of G is a compact n-dim region, then:

$$\int_G d\omega = \int_{\partial G} \omega \quad \text{for all } \omega \in \Lambda_{n-1}(M)$$

Definition: d is also called "co-boundary operator".

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Remark:

- Let us try iterating Stokes!
- Assume $G = \partial H$.
- Then, by Stokes we obtain $0 = 0$:

$$\int_H d d\omega \overset{\text{Stokes}}{=} \int_{\partial H} d\omega \overset{\text{Stokes}}{=} \int_{\partial\partial H} \omega$$

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- Let us try iterating Stokes!
- Assume $G = \partial H$.
- Then, by Stokes we obtain $0 = 0$:

$$\int_H \underbrace{dd}\omega \overset{\text{Stokes}}{=} \int_{\partial H} d\omega \overset{\text{Stokes}}{=} \int_{\underbrace{\partial\partial H}} \omega$$

(dd) $= 0$ always for algebraic reasons.

($\partial\partial H$) $= 0$ for geometric reasons because, indeed, boundaries don't possess boundaries:

E.g.

G = half sphere
∂G = equator
$\partial\partial G = \emptyset$

i.e.: Stokes implies $d^2 = 0 \iff \partial^2 = 0$

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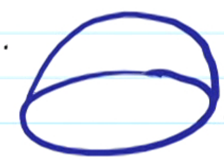

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$\partial\partial G = \emptyset$

i.e.: Stokes implies $d^2 = 0 \iff \partial^2 = 0$

- Stokes links homology (geometric) to cohomology (algebraic).

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Special case I:

Assume: $M = \mathbb{R}$, $G = (a, b)$

Therefore: $\partial G = \{a, b\}$

Then, Stokes' theorem is $\int_G df = \int_{\partial G} f$, namely:

$$\int_a^b df = f\Big|_a^b$$

(fund. thm of calculus)

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Then, Stokes' theorem is $\int_G df = \int_{\partial G} f$, namely:

$$\int_a^b df = f\Big|_a^b \qquad \text{(fund. thm of calculus)}$$

$$df = \frac{df}{dx}\, dx$$

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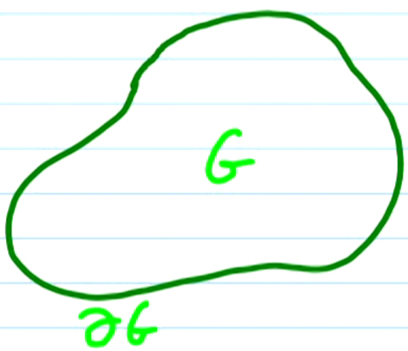
Special case II: "Green's theorem".

- $M = \mathbb{R}^2$, $G \subset \mathbb{R}^2$ a region with (closed) boundary curve ∂G.

  ↑ recall: this is automatic because $\partial \circ \partial = 0$



- Consider an arbitrary 1-form $\omega \in \Lambda_1(M)$:

$$\omega = \omega_1(x)\, dx^1 + \omega_2(x)\, dx^2$$

Then: $d\omega = d\omega_1(x) \wedge dx^1 + d\omega_2(x) \wedge dx^2$

$$= \left(\frac{\partial \omega_1}{\partial x^1} dx^1 + \frac{\partial \omega_1}{\partial x^2} dx^2\right) \wedge dx^1$$

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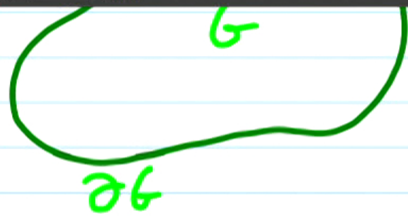

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Then: $d\omega = d\omega_1(x) \wedge dx^1 + d\omega_2(x) \wedge dx^2$

$$= \left(\frac{\partial \omega_1}{\partial x^1} dx^1 + \frac{\partial \omega_1}{\partial x^2} dx^2 \right) \wedge dx^1$$

$$+ \left(\frac{\partial \omega_2}{\partial x^1} dx^1 + \frac{\partial \omega_2}{\partial x^2} dx^2 \right) \wedge dx^2$$

$$= \frac{\partial \omega_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial \omega_1}{\partial x^1} dx^1 \wedge dx^2$$

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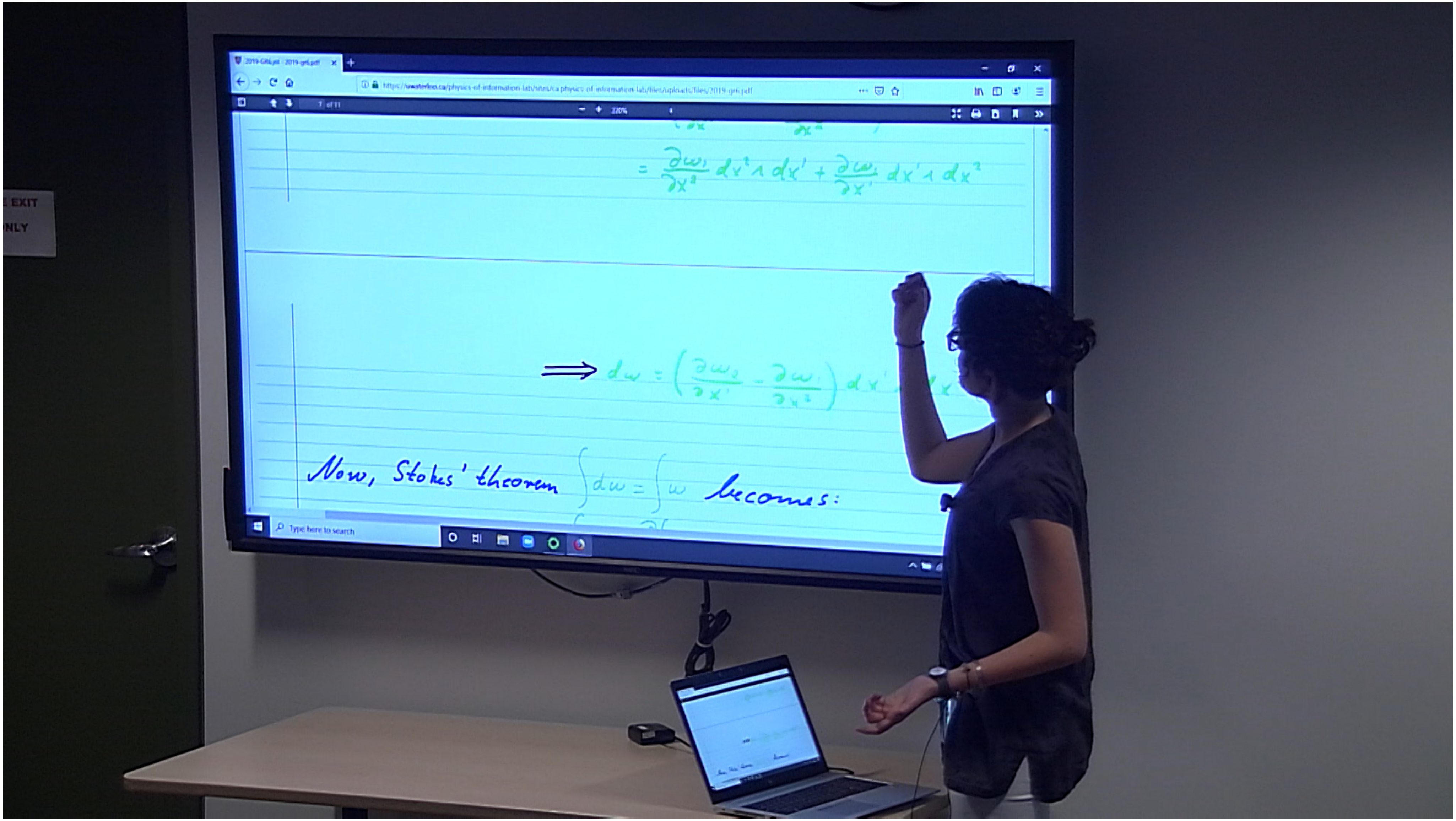

$= \frac{\partial \omega_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial \omega_2}{\partial x^1} dx^1 \wedge dx^2$
$\Longrightarrow d\omega = \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} \right) dx^1 \wedge dx^2$
Now, Stokes' theorem $\int d\omega = \int \omega$ becomes:
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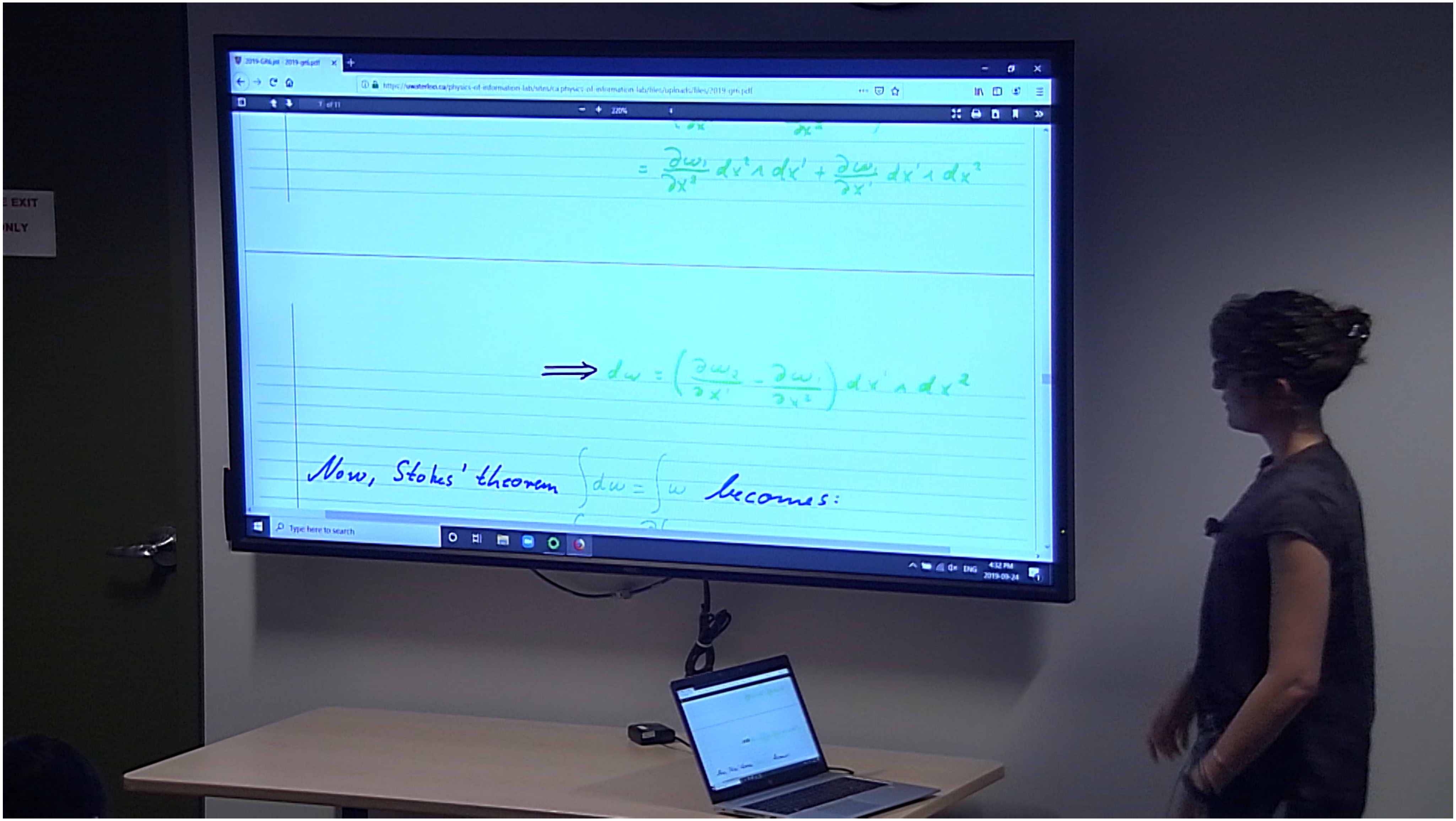
$= \frac{\partial \omega_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial \omega_2}{\partial x^1} dx^1 \wedge dx^2$
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Now, Stokes' theorem $\int d\omega = \int \omega$ becomes:

$$= \frac{\partial \omega_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial \omega_2}{\partial x^1} dx^1 \wedge dx^2$$

$$\Longrightarrow d\omega = \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} \right) dx^1 \wedge dx^2$$

Now, Stokes' theorem $\int d\omega = \int \omega$ becomes:

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$$\Longrightarrow d\omega = \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right) dx^1 \wedge dx^2$$

Now, Stokes' theorem $\int_G d\omega = \int_{\partial G} \omega$ becomes:

$$\int_G \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2}\right) dx^1 dx^2 = \int_{\partial G} \left(\omega_1 dx^1 + \omega_2 dx^2\right)$$

Recall: How to evaluate, e.g., the RHS, in practice?

- Choose a chart for ∂G, i.e., a diff'able map, invertible

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$G \qquad \partial G$

$$\int_G \left(\frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} \right) dx^1 dx^2 = \int_{\partial G} \left(\omega_1 dx^1 + \omega_2 dx^2 \right)$$

Recall: How to evaluate, e.g., the RHS, in practice?

- Choose a chart for ∂G, i.e., a diffable map, invertible map $\partial G \to \mathbb{R}$.
- Its inverse is a path: $\gamma : J \subset \mathbb{R} \to \partial G$, with $\gamma(t) = (x^1(t), x^2(t))$
- Now use $dx^i = \frac{dx^i}{dt} dt$ to obtain an integral over $J \subset \mathbb{R}$

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Special case of Green's theorem:

Assume $\omega \in \Lambda_1$ is closed, i.e., $d\omega = 0$, i.e., $\frac{\partial \omega_1}{\partial x^2} - \frac{\partial \omega_2}{\partial x^1} = 0$

Then: $\int_{\partial G} \omega = 0$

Compare: (From the residue theorem)

If a function $w: G \subset \mathbb{C} \to \mathbb{C}$ is holomorphic, i.e., it obeys the Cauchy Riemann equations, then:

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∂G

Compare: (From the residue theorem)

If a function $w: G \subset \mathbb{C} \to \mathbb{C}$ is holomorphic, i.e., it obeys the Cauchy Riemann equations, then:

$$\int_{\partial G} w(z)\, dz = 0$$

Indeed:

The Cauchy-Riemann equations mean that a diff. form is closed and co-closed. We'll define "co-closedness" later.

Special case III: (exercise)

Similarly, one can show that what is often called the Stokes theorem for $M = \mathbb{R}^3$, namely

"cross product": $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$

$$\int_G \vec{\nabla} \times \vec{w}\, dG = \int_{\partial G} \vec{w} \cdot \vec{ds}$$

$\vec{\nabla}$: $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$

$\vec{w}$: vector field

G: a 2 dim submanifold of M

∂G: 1 dim boundary of A.

"cross product": $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_1, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$

$\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$

$$\int_G \vec{\nabla} \times \vec{w} \, dG = \int_{\partial G} \vec{w} \cdot d\vec{s}$$

G: a 2dim submanifold of M

$\vec{w}$: vector field

∂G: 1dim boundary of A.

is indeed this special case:

$$w \in \Lambda_1(G) \text{ with } \vec{\nabla} \times \vec{w} = dw \in \Lambda_2(G)$$

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Before we can discuss the next example:

How to define the volume of a region $G \subset M$ of a diffable manifold M?

- In $\mathbb{R}^n$, we had: $$V = \int_G dx^1 \ldots dx^n$$

- In general, we need to choose a Volume form $$\Omega \in \Lambda_n$$ obeying $\Omega(p) \neq 0 \ \forall p \in G$. Then the ($\Omega$-dependent) volume is defined as:

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(We will later use the metric tensor to define a volume form for spacetime.)

Proposition: G orientable $\iff \exists$ volume forms Ω

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Proposition: G orientable $\Longleftrightarrow$ $\exists$ volume forms Ω (In fact ∞ many)

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Special case IV: Gauss' theorem

To obtain Gauß' theorem we need to define yet a new derivative the divergence of a vector field.

Recall: On $\mathbb{R}^n$, the divergence of a vector field, ξ, was defined as

$$\operatorname{div}\xi = \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\xi^i = \xi^i{}_{,i}$$

⇝ How to generalize to arbitrary manifolds?

Where in this course did we see $\xi^i(x)_{,i}$ before?

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Recall: $(L_\xi \tau)^{i_1 \dots i_r}_{j_1 \dots j_s}(x) = \tau^{i_1 \dots i_r}_{j_1 \dots j_s, k}(x)\,\xi^k(x) - \tau^{\bar{i} \dots i_r}_{j_1 \dots j_s}(x)\,\xi^{i_1}_{\ ,\bar{i}}(x) - \dots$

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$+ \tau^{i_1 \dots i_r}_{\bar{j}_1 \dots j_s}(x)\,\xi^{\bar{j}_1}_{;j_1}(x) + \dots\, \tau^{i_1 \dots i_r}_{j_1 \dots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{;j_s}(x)$

$\tau \in T(M)^r_s$

!

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Strategy: If we choose τ to be the volume form, which on flat $\mathbb{R}^n$ we may choose to be $\Omega = 1\, dx^1 \wedge \dots dx^n$, then the first term will drop out on $\mathbb{R}^n$ b/c $1_{,i} = 0$, and so we may be generalizing $\xi^i_{\ ,i}$ on $\mathbb{R}^n$!

Def: The Divergence of a vector field ξ with respect to a volume form, Ω, is defined to be:

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and so we may be generalizing $\xi^i_{\ ;i}$ on $\mathbb{R}^n$!

Def: The Divergence of a vector field ξ with respect to a volume form, Ω, is defined to be:

$$\mathrm{div}_\Omega \xi := L_\xi(\Omega)$$

↑ Lie derivative

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Assume $\Omega = a(x)\, dx^1 \wedge \ldots \wedge dx^n$ (volume form)

$a(x) \in \Lambda_0(M)$

and $\xi = \xi^i(x) \frac{\partial}{\partial x^i}$ (vector field)

Then:

$$\text{div}_\Omega \xi = L_\xi \Omega \overset{\text{by Leibnitz rule}}{=} \xi(a)\, dx^1 \wedge \ldots \wedge dx^n + \ldots$$

$\xi(a) = \xi^i \frac{\partial}{\partial x^i} a(x)$

$$+ a \sum_{i=1}^{n} dx^1 \wedge \ldots \wedge L_\xi(dx^i) \wedge \ldots \wedge dx^n$$

(recall: $L_\xi(dx^i) = d(\xi(x^i)) = d(\xi^j \frac{\partial}{\partial x^j} x^i) = d(\xi^j \delta_j^i) = d(\xi^i) = \frac{\partial \xi^i}{\partial x^r} dx^r$)

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□ Then:

$$\mathrm{div}_\Omega \xi = L_\xi \Omega \overset{\text{by Leibnitz rule}}{=} \underbrace{\xi(a)}_{= \xi^i \frac{\partial}{\partial x^i} a(x)} dx^1 \wedge \ldots \wedge dx^n + \ldots$$

$$+ a \sum_{i=1}^{n} dx^1 \wedge \ldots \wedge L_\xi(dx^i) \wedge \ldots \wedge dx^n$$

$$\left(\text{recall: } L_\xi(dx^i) = d(\xi(x^i)) = d\left(\xi^j \frac{\partial}{\partial x^j} x^i\right) = d(\xi^j \delta_j^i) = d(\xi^i) = \frac{\partial \xi^i}{\partial x^r} dx^r\right)$$

only dx^i term survives in wedge product

$$\Rightarrow \quad \mathrm{div}_\Omega \xi = \left(\xi^i a_{,i} + a\, \xi^i_{,i}\right) dx^1 \wedge \ldots \wedge dx^n$$

$$\Rightarrow \quad \boxed{\mathrm{div}_\Omega \xi = \frac{1}{a} (a \xi^i)_{,i}\, \Omega}$$

Notice: If $a \equiv 1$ then $\mathrm{div}_\Omega \xi = \frac{\partial \xi^i}{\partial x^i}$ as expected for the divergence in the simplest case.

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Then:

Leibnitz rule $= \xi^i \frac{\partial}{\partial x^i} a(x)$

$$\text{div}_\Omega \xi = L_\xi \Omega = \xi(a)\, dx^1 \wedge \ldots \wedge dx^n + \ldots$$

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Thus: Indeed, if $a(x) = 1\ \forall x$ then $\text{div}_\Omega \xi = \xi^i_{,i}\, dx^1 \wedge \ldots \wedge dx^n$.

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Now, we can derive Gauss' theorem from Stokes':

$\square \quad \text{div}_\Omega \xi := L_\xi \Omega \in \Lambda_n(\mathcal{M})$

$\square \quad \text{div}_\Omega \xi = (d \circ i_\xi + i_\xi \circ d)\, \Omega$

Recall: $d\Omega = 0$ because anti-symmetry doesn't allow $(n+1)$ forms.

$$\Longrightarrow \boxed{\text{div}_\Omega \xi = d \circ i_\xi(\Omega)}$$

We can now apply Stokes' theorem $\int_G dv = \int_{\partial G} v$:

$$\int_G d\, i_\xi(\Omega) = \int_{\partial G} i_\xi(\Omega)$$

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i.e.:

$$\boxed{\int_G \underbrace{\text{div}_\Omega \xi}_{n\text{-form}} = \int_{\partial G} \underbrace{i_\xi(\Omega)}_{(n-1)\text{ form}} \qquad \text{"Gauß' theorem"}}$$

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