Title: General Relativity for Cosmology - Lecture 5

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Collection: General Relativity for Cosmology (Kempf)

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Recall:

- The set $\Lambda(M)$ of differential forms on M is an associative algebra, called the Grassmann algebra over M.

- The multiplication in $\Lambda(M)$ is the wedge product: $\wedge : \Lambda_s(M) \times \Lambda_r(M) \rightarrow \Lambda_{s+r}(M)$

- The exterior derivative $d: \Lambda(M) \rightarrow \Lambda(M)$

But: How to obtain a directional derivative on $\Lambda(M)$?

Recall: Tangent vectors ξ are directional derivatives on $\Lambda_0(M)$!

Plan now:

A. Define an anti-derivation i_ξ of degree $K=-1$: the inner derivation.

(i_ξ will generalize feeding a tangent vector ξ to a 1-form to feeding it to a p-form.)

B. Combine d, i_ξ to obtain a derivation of degree $K=0$: the Lie derivative

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(And the Lie derivative is going to be the directional derivative for differential forms and tensors)

A. The "Inner Derivation":

- Assume ξ is a tangent vector field.
- Our aim: to define an anti-derivation, i_ξ, of degree $K=-1$, i.e., a linear map

$$i_\xi : \Lambda_s(M) \to \Lambda_{s-1}(M)$$
$$i_\xi : \quad \omega \to i_\xi(\omega)$$

which obeys the anti-Leibniz rule:

$$i_\xi(\omega \wedge \nu) = i_\xi(\omega) \wedge \nu + (-1)^r \omega \wedge i_\xi(\nu)$$

Definition:

$$i_\xi : \Lambda_0 \to 0$$

$$i_\xi : \Lambda_1 \to \Lambda_0$$

$$\begin{matrix} \cup & & \cup \\ i_\xi : \omega & \to & \omega(\xi) \end{matrix}$$

Recall: By linearity and the anti-Leibniz rule this already defines $i_\xi : \Lambda(M) \to \Lambda(M)$.

Proposition: If $\chi \in \Lambda_s(M)$ then $i_\xi(\chi) \in \Lambda_{s-1}(M)$

Example: * Consider $\overset{\Lambda_2(M)\,\ni}{\chi} := \overset{\Lambda_1(M)\,\ni}{\omega} \wedge \overset{\Lambda_1(M)\,\ni}{\nu}$

* What is $i_\xi(\chi) \in \Lambda_1(M)$? Leibniz rule $\Rightarrow$

$$i_\xi(\chi) = i_\xi(\omega \wedge \nu) = i_\xi(\omega) \wedge \nu + (-1)^1 \omega \wedge i_\xi(\nu)$$

$$= \omega(\xi)\,\nu - \nu(\xi)\,\omega$$

* Apply $i_\xi(\chi) \in \Lambda_1(M)$ to a tangent vector η :

$$i_\xi(\chi)(\eta) = \omega(\xi)\,\nu(\eta) - \nu(\xi)\,\omega(\eta)$$

* Compare with claim of proposition:

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* Compare with claim of proposition:

$$i_\xi(\chi)(\eta) = i_\xi(\omega \wedge \nu)(\eta) = i_\xi(\omega \otimes \nu - \nu \otimes \omega)(\eta)$$

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Properties of i_ξ :

- $i_{\xi_1} \circ i_{\xi_2} = - i_{\xi_2} \circ i_{\xi_1}$

(Exercise: prove this)

- Thus, in particular:

$$i_\xi \circ i_\xi = 0$$

(Simply the evaluation of a dual vector applied to a vector in the vector space)

- Recall: We also have $d \circ d = 0$

Recall: For $\xi \in T_p(M)$, $\gamma \in T_p^*(M)$, we have $i_\xi(\gamma) = \gamma(\xi) = \xi(\gamma)$

Definition: The inner derivation, $i_\xi(\gamma)$, of a $\gamma \in \Lambda(M)$ is also called the interior product of ξ and γ.

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B. The Lie derivative, "L_ξ": (algebraic definition)

Vectors $\xi : \Lambda_0(M) \to \Lambda_0(M)$ are directional derivatives.

How to generalize the notion of directional derivative to all of $\Lambda(M)$?

We have:

- $d : \Lambda_s(M) \to \Lambda_{s+1}(M)$ generalizes the notion of differential $d : \Lambda_0 \to \Lambda_1$, $d : f \to df$ to all of $\Lambda(M)$.
- $i_\xi : \Lambda_s(M) \to \Lambda_{s-1}(M)$ generalizes the notion of evaluation of vectors ξ on covectors $\omega \in \Lambda_1(M)$ to all of $\Lambda(M)$.

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$\boxed{\text{Spoiler: It will be: } L_\xi = d \circ i_\xi + i_\xi \circ d}$

$$L_\xi : \Lambda_r(M) \to \Lambda_r(M)$$

i.e. it should be of degree $K=0$. In particular:

- On functions $f \in \mathcal{F}(M) = \Lambda_0(M)$ it should be the usual directional derivative:

$$L_\xi : \Lambda_0(M) \to \Lambda_0(M)$$

$$L_\xi : \quad f \to \xi(f) \qquad \left(= \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i} f(x) \right)$$

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- **Recall:** once we define L_ξ on Λ_0 and a basis of $\Lambda_1(M)$, then by linearity and the Leibniz rule, L_ξ will automatically be defined on all of $\Lambda(M)$.

- Consider, therefore, any $df \in \Lambda_1(M)$, e.g., the basis vectors $df = dx^i$.

recall that df is the gradient

Can we relate it to d and i_ξ? Yes:

Cartan's equation:

$$\boxed{L_\xi = \overbrace{d\circ i_\xi + i_\xi\circ d}^{\text{Exercise: show it is a derivation}}}$$

Proof:

check on $\Lambda_0(M)$:

$$L_\xi \underset{\Lambda_0(M)}{\underset{\in}{f}} = d\circ \underbrace{i_\xi(f)}_{=0 \text{ because } f\in\Lambda_0(M)} + i_\xi(df) = 0 + df(\xi) = \xi(f) \quad ✓$$

check on basis of $\Lambda_1(M)$, e.g. $df = dx^i$:

$$L_\xi df = d\circ \overbrace{i_\xi(df)}^{=df(\xi)=\xi(f)} + i_\xi\circ \overbrace{ddf}^{\overset{\text{because: } d^2=0}{0}} = d(\xi(f)) \quad ✓$$

I.e., indeed, as in (D): directional derivative of gradient = gradient of directional derivative

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Definition:

For any linear maps $A: \Lambda(M) \to \Lambda(M)$, $B: \Lambda(M) \to \Lambda(M)$

we define their commutator (or Lie-, or Poisson bracket):

$$[A,B] := A \cdot B - B \cdot A$$

Examples of maps:

$$d: \Lambda(M) \to \Lambda(M)$$

$$i_\xi: \Lambda(M) \to \Lambda(M)$$

$$L_\xi: \Lambda(M) \to \Lambda(M)$$

For the commutators of d, i_ξ and L_ξ one can prove:

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For the commutators of d, i_ξ and L_ξ one can prove:

Since L_ξ is the directional derivative on $\Lambda(M)$:

- Can L_ξ be extended to a directional derivative for all tensor fields? Yes!

- Can L_ξ be expressed as a Newton-Leibniz limit similar to

need an analog: a shift on a manifold, in the direction given by ξ.

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(\underbrace{x+\varepsilon}) - f(x)}{\varepsilon} \quad ? \quad \text{Yes!}$$

To this end:

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We say that a tensor field τ is invariant under the flow induced by the vector field ξ if:

$$\underbrace{\phi_t^*(\tau(p))}_{\text{image of the tensor field's value at } p} = \underbrace{\tau(\phi_t(p))}_{\text{tensor field's value at the image of } p} \qquad \forall t \, \forall p$$

(The flow produces an image of M in M:)

- **Definition:**

geom. definition

The Lie derivative of ...

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Explicitly, in a chart:

- $\phi: x \to \tilde{x}$ with infinitesimal flow: $\tilde{x}^i(x) = x^i + t\xi^i(x) + O(t^2)$
- Jacobian matrix: $\frac{\partial \tilde{x}^i}{\partial x^j} = \delta^i_j + t\,\frac{\partial \xi^i(x)}{\partial x^j} + O(t^2)$

  we write $= \xi^i_{,j}$
- Inverse Jacobian: $\frac{\partial x^i}{\partial \tilde{x}^j} = \delta^i_j - t\,\frac{\partial \xi^i(x)}{\partial x^j} + O(t^2)$
- Image of tensor at $\tau(\tilde{x})^{i_1 \dots i_r}_{j_1 \dots j_s}$ under flow, backwards, $\tilde{x} \to x$, has the

From now, we will omit writing Σ: Twice occurring indices are always to be summed over (Einstein convention)

components:

$$\phi^*(\tau(x))^{i_1 \dots i_r}_{j_1 \dots j_s} = \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(\tilde{x})\,\frac{\partial x^{i_1}}{\partial \tilde{x}^{\bar{i}_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{\bar{i}_r}}\,\frac{\partial \tilde{x}^{\bar{j}_1}}{\partial x^{j_1}} \cdots \frac{\partial \tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$

$$= \tau^{\bar{i}_1 \dots \bar{i}_r}_{\bar{j}_1 \dots \bar{j}_s}(x + t\xi)\left(\delta^{i_1}_{\bar{i}_1} - t\,\xi^{i_1}_{,\bar{i}_1}\right) \cdots \left(\delta^{i_r}_{\bar{i}_r} - t\,\xi^{i_r}_{,\bar{i}_r}\right)$$

$$\phi^{*}(\tau(x))^{i_1\dots i_r}_{j_1\dots j_s} = \tau^{\bar i_1\dots\bar i_r}_{\bar j_1\dots \bar j_s}(\bar x)\, \frac{\partial x^{i_1}}{\partial \bar x^{\bar i_1}}\cdots\frac{\partial x^{i_r}}{\partial \bar x^{\bar i_r}}\,\frac{\partial \bar x^{\bar j_1}}{\partial x^{j_1}}\cdots\frac{\partial \bar x^{\bar j_s}}{\partial x^{j_s}}$$

$$= \tau^{\bar i_1\dots\bar i_r}_{\bar j_1\dots\bar j_s}(x+t\xi)\left(\delta^{i_1}_{\bar i_1} - t\xi^{i_1}_{,\bar i_1}\right)\cdots\left(\delta^{i_r}_{\bar i_r}-t\xi^{i_r}_{,\bar i_r}\right)$$

$$\cdot\left(\delta^{\bar j_1}_{j_1}+t\xi^{\bar j_1}_{,j_1}\right)\cdots\left(\delta^{\bar j_s}_{j_s}+t\xi^{\bar j_s}_{,j_s}\right)+O(t^2)$$

$$f_{,k} := \frac{\partial}{\partial x^k} f$$

$$= \tau^{i_1\dots i_r}_{j_1\dots j_s}(x) + t\,\tau^{i_1\dots i_r}_{j_1\dots j_s,k}(x)\,\xi^k(x)$$

$$- t\,\tau^{\bar i_1\dots i_r}_{j_1\dots j_s}(x)\,\xi^{i_1}_{,\bar i_1}(x) - \dots - t\,\tau^{i_1\dots\bar i_r}_{j_1\dots j_s}(x)\,\xi^{i_r}_{,\bar i_r}(x)$$

$$- t\, \tau^{\bar i \cdots i_r}_{j_1 \cdots j_s}(x)\, \xi^{i_1}_{,\bar i}(x) - \dots - t\, \tau^{i_1 \cdots \bar i}_{j_1 \cdots j_s}(x)\, \xi^{i_r}_{,\bar i}(x)$$

$$+ t\, \tau^{i_1 \cdots i_r}_{\bar j \cdots j_s}(x)\, \xi^{\bar j}_{,j_1}(x) + \dots + t\, \tau^{i_1 \cdots i_r}_{j_1 \cdots \bar j}(x)\, \xi^{\bar j}_{,j_s}(x)$$

$$\Rightarrow (L_\xi \tau)^{i_1 \cdots i_r}_{j_1 \cdots j_s}(x) = \lim_{t \to 0} \frac{1}{t} \Big(\phi^{*-1}(\tau(x))^{i_1 \cdots i_r}_{j_1 \cdots j_s} - \tau^{i_1 \cdots i_r}_{j_1 \cdots j_s}(\overbrace{x(0)}^{x\, =}) \Big)$$

$$= \tau^{i_1 \cdots i_r}_{j_1 \cdots j_s,k}(x)\, \xi^k(x) - \tau^{\bar i \cdots i_r}_{j_1 \cdots j_s}(x)\, \xi^{i_1}_{,\bar i}(x) - \dots$$

$$+ \tau^{i_1 \cdots i_r}_{\bar j \cdots j_s}(x)\, \xi^{\bar j}_{,j_1}(x) + \dots \tau^{i_1 \cdots i_r}_{j_1 \cdots \bar j}(x)\, \xi^{\bar j}_{,j_s}(x)$$

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$$L_{[\xi,\eta]} = [L_\xi, L_\eta] \quad \left(= L_\xi \circ L_\eta - L_\eta \circ L_\xi \right)$$

- Does it still obey a Leibniz rule?

Yes: $$L_\xi(\tau \otimes \sigma) = L_\xi(\tau) \otimes \sigma + \tau \otimes L_\xi(\sigma)$$

(tensors form an algebra w. respect to multiplication $\otimes$)

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