

Title: General Relativity for Cosmology - Lecture 5

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Collection: General Relativity for Cosmology (Kempf)

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- Recall:
- The set $\Lambda(M)$ of differential forms on M is an associative algebra, called the Grassmann algebra over M .
 - The multiplication in $\Lambda(M)$ is the wedge product: $\wedge: \Lambda_s(M) \times \Lambda_r(M) \rightarrow \Lambda_{s+r}(M)$
 - The exterior derivative $d: \Lambda(M) \rightarrow \Lambda(M)$

But: How to obtain a directional derivative on $\Lambda(M)$?

Recall: Tangent vectors ξ are directional derivatives on $\Lambda_0(M)$!

Plan now:

A. Define an anti-derivation i_ξ of degree $k = -1$: the inner derivation.

(i_ξ will generalize feeding a tangent vector ξ to a 1-form to feeding it to a p -form.)

B. Combine d , i_ξ to obtain a derivation of degree $k = 0$: the Lie derivative

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(And the Lie derivative is going to be the directional derivative for differential forms and tensors.)

A. The "Inner Derivation":

- Assume ξ is a tangent vector field.
- Our aim: to define an anti-derivation, i_ξ , of degree $k = -1$, i.e., a linear map

$$i_\xi : \Lambda_s(\mathcal{M}) \rightarrow \Lambda_{s-1}(\mathcal{M})$$

$$i_\xi : \omega \rightarrow i_\xi(\omega)$$

which obeys the anti-Leibniz rule:

$$i_\xi(\omega \wedge \nu) = i_\xi(\omega) \wedge \nu + (-1)^r \omega \wedge i_\xi(\nu)$$

□ Definition:

$$i_\xi : \Lambda_0 \rightarrow 0$$

$$i_\xi : \Lambda_1 \rightarrow \Lambda_0$$

$$i_\xi : \omega \rightarrow \omega(\xi)$$

□ Recall: By linearity and the anti-Leibniz rule this already defines $i_\xi : \Lambda(M) \rightarrow \Lambda(M)$.

□ Proposition: $\forall \gamma \in \Lambda_s(M)$ then $i_\xi(\gamma) \in \Lambda_{s-1}(M)$

Example: * Consider $\gamma \in \Lambda_2(M) := \omega \wedge v$

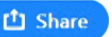
* What is $i_\xi(\gamma) \in \Lambda_1(M)$? Leibniz rule \Rightarrow

$$\begin{aligned} i_\xi(\gamma) &= i_\xi(\omega \wedge v) = i_\xi(\omega) \wedge v + (-1)^1 \omega \wedge i_\xi(v) \\ &= \omega(\xi) v - v(\xi) \omega \end{aligned}$$

* Apply $i_\xi(\gamma) \in \Lambda_1(M)$ to a tangent vector η :

$$i_\xi(\gamma)(\eta) = \omega(\xi) v(\eta) - v(\xi) \omega(\eta)$$

* Compare with claim of proposition:



Example: * Consider $\gamma := \overset{\Lambda_2(\mathcal{M})}{\omega} \wedge \overset{\Lambda_1(\mathcal{M})}{v}$

* What is $i_\xi(\gamma) \in \Lambda_1(\mathcal{M})$? Leibniz rule \Rightarrow

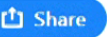
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* Compare with claim of proposition:

$$i_\xi(\gamma)(\eta) = i_\xi(\omega \wedge v)(\eta) = i_\xi(\omega \otimes v - v \otimes \omega)(\eta)$$



Properties of i_ξ :

$$\square \quad i_{\xi_1} \circ i_{\xi_2} = -i_{\xi_2} \circ i_{\xi_1}$$

\square Thus, in particular:

$$i_\xi \circ i_\xi = 0$$

(Exercise: prove this)

\square Recall: We also have $d \circ d = 0$

(Simply the evaluation of a dual vector applied to a vector in the vector space)

Recall: For $\xi \in T_p^*(M)$, $\gamma \in T_p(M)$, we have $i_\xi(\gamma) = \gamma(\xi) = \xi(\gamma)$

Definition: The inner derivation, $i_\xi(\gamma)$, of a $\gamma \in \mathfrak{X}(M)$ is also called the interior product of ξ and γ .

B. The Lie derivative, L_ξ : (algebraic definition)

Vectors $\xi: \Lambda_0(M) \rightarrow \Lambda_0(M)$ are directional derivatives.

How to generalize the notion of directional derivative to all of $\Lambda(M)$?

We have: $\square d: \Lambda_s(M) \rightarrow \Lambda_{s+1}(M)$ generalizes the notion of differential $d: \Lambda_0 \rightarrow \Lambda_1, d: f \rightarrow df$ to all of $\Lambda(M)$.

$\square i_\xi: \Lambda_s(M) \rightarrow \Lambda_{s-1}(M)$ generalizes the notion of evaluation of vectors ξ on covectors $\omega \in \Lambda_1(M)$ to all of $\Lambda(M)$.

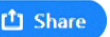
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\square $i_\xi: \Lambda_s(M) \rightarrow \Lambda_{s-1}(M)$ generalizes the notion of evaluation of vectors ξ on covectors $\omega \in \Lambda_1(M)$ to all of $\Lambda(M)$.

Spoiler: It will be: $L_\xi = d \circ i_\xi + i_\xi \circ d$



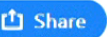
$$L_{\xi} : \Lambda_1(M) \rightarrow \Lambda_1(M)$$

i.e. it should be of degree $K=0$. In particular:

□ On functions $f \in \mathcal{F}(M) = \Lambda_0(M)$ it should be the usual directional derivative:

$$L_{\xi} : \Lambda_0(M) \rightarrow \Lambda_0(M)$$

$$L_{\xi} : f \rightarrow \xi(f) \quad \left(= \sum_{i=1}^m \xi^i(x) \frac{\partial}{\partial x^i} f(x) \right)$$



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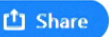
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- Recall: once we define L_{ξ} on Λ_0 and a basis of $\Lambda_1(M)$, then by linearity and the Leibniz rule, L_{ξ} will automatically be defined on all of $\Lambda(M)$.

- Consider, therefore, any $df \in \Lambda_1(M)$, e.g., the basis vectors $df = dx^i$.

↑ recall that df is the gradient



can we relate it to d and i_{ξ} ? Yes:

Cartan's equation:

Exercise: show it is a derivation

$$L_{\xi} = d \circ i_{\xi} + i_{\xi} \circ d$$

Proof:

check on $\Lambda_0(M)$: $L_{\xi} f = d \circ i_{\xi}(f) + i_{\xi}(df) = 0 + df(\xi) = \mathcal{L}(f)$

= 0 because $f \in \Lambda_0(M)$ ✓

check on basis of $\Lambda_1(M)$, e.g. $df = dx^i$: $L_{\xi} df = d \circ i_{\xi}(df) + i_{\xi} \circ ddf = d(\mathcal{L}(f))$ ✓

= $df(\xi) = \mathcal{L}(f)$ because: $d^2 = 0$

I.e., indeed, as in (D): directional derivative of gradient = gradient of directional derivative



Definition:

For any linear maps $A: \Lambda(\mathcal{M}) \rightarrow \Lambda(\mathcal{M})$, $B: \Lambda(\mathcal{M}) \rightarrow \Lambda(\mathcal{M})$
we define their commutator (or Lie-, or Poisson bracket):

$$[A, B] := A \cdot B - B \cdot A$$

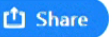
Examples of maps:

$$d: \Lambda(\mathcal{M}) \rightarrow \Lambda(\mathcal{M})$$

$$i_{\xi}: \Lambda(\mathcal{M}) \rightarrow \Lambda(\mathcal{M})$$

$$L_{\xi}: \Lambda(\mathcal{M}) \rightarrow \Lambda(\mathcal{M})$$

For the commutators of d , i_{ξ} and L_{ξ} one can prove:



Defn. 1.1.

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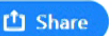
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For the commutators of d, i_{ξ} and L_{ξ} one can prove:



Since L_ξ is the directional derivative on $\Lambda(M)$:

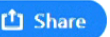
■ Can L_ξ be extended to a directional derivative for all tensor fields? **Yes!**

■ Can L_ξ be expressed as a Newton-Leibniz limit similar to

$$f'(x) = \lim_{\epsilon \rightarrow 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} \quad ? \quad \text{Yes!}$$

need an analog: a shift on a manifold, in the direction given by ξ .

To this end:



We say that a tensor field τ is invariant under the flow induced by the vector field ξ if:

$$\phi_t^*(\tau(p)) = \tau(\phi_t(p)) \quad \forall t \forall p$$

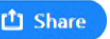
(The flow produces an image of M in M .)

image of the tensor field's value at p

tensor field's value at the image of p

Definition:

The Lie derivative is a *geom. definition*



Explicitly, in a chart:

□ $\phi: x \rightarrow \tilde{x}$ with infinitesimal flow: $\tilde{x}^i(x) = x^i + t \xi^i(x) + \mathcal{O}(t^2)$

□ Jacobian matrix: $\frac{\partial \tilde{x}^i}{\partial x^j} = \delta_j^i + t \frac{\partial \xi^i(x)}{\partial x^j} + \mathcal{O}(t^2)$
 \leftarrow we write $= \xi_{,j}^i$

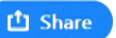
□ Inverse Jacobian: $\frac{\partial x^i}{\partial \tilde{x}^j} = \delta_j^i - t \frac{\partial \xi^i(x)}{\partial x^j} + \mathcal{O}(t^2)$

□ Image of tensor at $\tau(\tilde{x})_{i_1 \dots i_n}^{j_1 \dots j_m}$ under flow, backwards, $\tilde{x} \rightarrow x$, has the

From now, we will omit writing Σ : Twice occurring indices are always to be summed over (Einstein convention)

components:

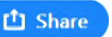
$$\begin{aligned} \phi^* \tau(\tilde{x})_{i_1 \dots i_n}^{j_1 \dots j_m} &= \tau_{j_1 \dots j_m}^{i_1 \dots i_n}(\tilde{x}) \frac{\partial x^{i_1}}{\partial \tilde{x}^{j_1}} \dots \frac{\partial x^{i_n}}{\partial \tilde{x}^{j_n}} \frac{\partial \tilde{x}^{j_1}}{\partial x^{i_1}} \dots \frac{\partial \tilde{x}^{j_m}}{\partial x^{i_m}} \\ &= \tau_{j_1 \dots j_m}^{i_1 \dots i_n}(x + t\xi) \left(\delta_{j_1}^{i_1} - t \xi_{,j_1}^{i_1} \right) \dots \left(\delta_{j_n}^{i_n} - t \xi_{,j_n}^{i_n} \right) \end{aligned}$$



$$\begin{aligned} \phi^x(\tau(x))_{j_1, \dots, j_s}^{i_1, \dots, i_s} &= \tau_{j_1, \dots, j_s}^{i_1, \dots, i_s}(\bar{x}) \frac{\partial x^{i_1}}{\partial \bar{x}^{j_1}} \dots \frac{\partial x^{i_s}}{\partial \bar{x}^{j_s}} \frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}} \dots \frac{\partial \bar{x}^{j_s}}{\partial x^{i_s}} \\ &= \tau_{j_1, \dots, j_s}^{i_1, \dots, i_s}(x + t\xi) (\delta_{j_1}^{i_1} - t\xi_{j_1}^{i_1}) \dots (\delta_{j_s}^{i_s} - t\xi_{j_s}^{i_s}) \\ &\quad \cdot (\delta_{j_1}^{i_1} + t\xi_{j_1}^{i_1}) \dots (\delta_{j_s}^{i_s} + t\xi_{j_s}^{i_s}) + o(t^2) \end{aligned}$$

$$\begin{aligned} &= \tau_{j_1, \dots, j_s}^{i_1, \dots, i_s}(x) + t \tau_{j_1, \dots, j_s}^{i_1, \dots, i_s, k}(x) \xi^k(x) \\ &\quad - t \tau_{j_1, \dots, j_s}^{i_1, \dots, i_s}(x) \xi_{j_1}^{i_1}(x) \dots - t \tau_{j_1, \dots, j_s}^{i_1, \dots, i_s}(x) \xi_{j_s}^{i_s}(x) \end{aligned}$$

$f_{,k} := \frac{\partial}{\partial x^k} f$



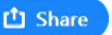
$$-t \tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}(x) \xi_{j_1, i_1}^{i_2}(x) - \dots - t \tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}(x) \xi_{j_1, i_1}^{i_r}(x)$$

$$+ t \tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}(x) \xi_{j_1, i_1}^{i_1}(x) + \dots + t \tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}(x) \xi_{j_1, i_1}^{i_s}(x)$$

$$\Rightarrow (L_\xi \tau)_{j_1, \dots, j_s}^{i_1, \dots, i_r}(x) = \lim_{t \rightarrow 0} \frac{1}{t} \left(\phi^{\tau(x)}(\tau(x))_{j_1, \dots, j_s}^{i_1, \dots, i_r} - \tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}(x(0)) \right)$$

$$= \tau_{j_1, \dots, j_s, k}^{i_1, \dots, i_r}(x) \xi^k(x) - \tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}(x) \xi_{j_1, i_1}^{i_1}(x) - \dots$$

$$+ \tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}(x) \xi_{j_1, i_1}^{i_1}(x) + \dots + \tau_{j_1, \dots, j_s}^{i_1, \dots, i_r}(x) \xi_{j_1, i_1}^{i_s}(x)$$



$$L_{[\xi, \eta]} = [L_\xi, L_\eta] \quad (= L_\xi \circ L_\eta - L_\eta \circ L_\xi)$$

Does it still obey a Leibniz rule?

Yes: $L_\xi(\tau \otimes \sigma) = L_\xi(\tau) \otimes \sigma + \tau \otimes L_\xi(\sigma)$

(tensors form an algebra w. respect to multiplication \otimes)