Title: General Relativity for Cosmology - Lecture 4

Speakers: Achim Kempf

Collection: General Relativity for Cosmology (Kempf)

Date: September 17, 2019 - 4:00 PM

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GR for Cosmology, Achim Kempf — Lecture 4

Note Title

Differential forms (also called "exterior differential forms")

Why? Every p-dim. integration is an integration over a differential p-form!

Preparation:

Consider the cotangent space $T_p(M)^*$ at p:

- Each $\omega \in T_p(M)^*$ is a lin. map:

$$\omega : T_p(M) \to \mathbb{R}$$

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1 of 17 220%

Why: Every p-dim. integration is an integration over a differential p-form!

Preparation:

Consider the cotangent space $T_p(\mathcal{M})^*$ at p:

- Each $\omega \in T_p(\mathcal{M})^*$ is a lin. map:

$$\omega : T_p(\mathcal{M}) \to \mathbb{R}$$

$$\omega : \xi \to \omega(\xi)$$

- Each such ω is a covariant tensor, of rank $(0,1)$

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1 of 17
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More generally, consider the covariant tensors of rank $(0,r)$:

- Recall: $T_p(M)_r := \overbrace{T_p(M)^* \otimes \dots \otimes T_p(M)^*}^{r \text{ factors}}$
- Each $v \in T_p(M)_r$ is a multi-linear map:

$$v : T_p(M)^r \to \mathbb{R}$$

- In particular, if $\xi_1, \dots, \xi_r \in T_p(M)$ then:

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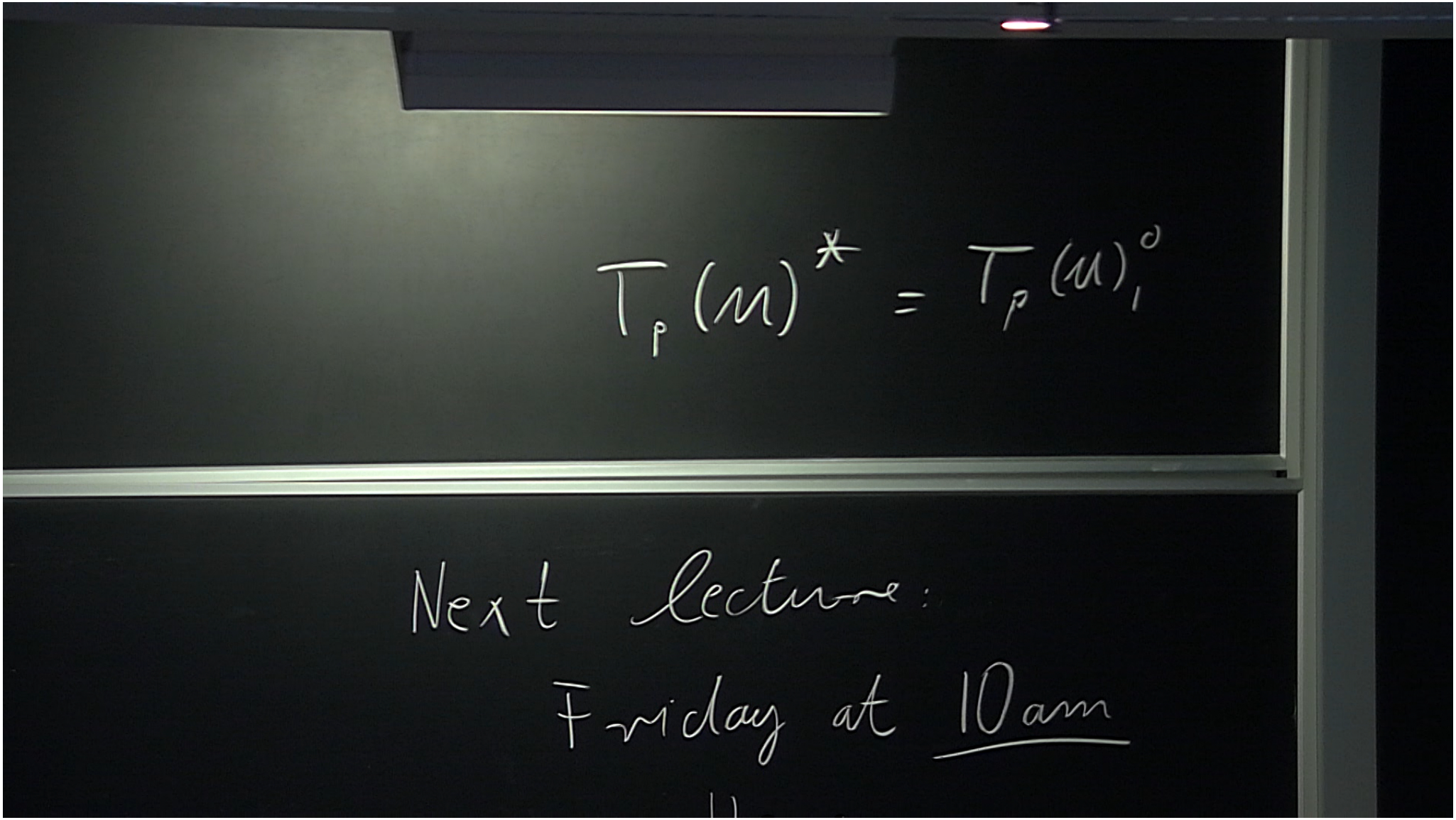
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1 of 17
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$$\omega : T_p(M)^* \to \mathbb{R}$$

- In particular, if $\xi_1, \ldots, \xi_r \in T_p(M)$ then:

$$\omega : \xi_1 \times \ldots \times \xi_r \to \omega(\xi_1, \ldots, \xi_r)$$

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$T_p(u)^* = T_p(u)^\circ$
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2 of 17
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Definition: If $r > 1$ and $v \in T_p(M)_r$, then we define the "anti-symmetric part of v" as the image $\tilde{v} = A(v)$ of v under the linear antisymmetrization map A:

$$\tilde{v}(\xi_1, \ldots, \xi_r) = A(v)(\xi_1, \ldots, \xi_r)$$

$$:= \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\, v(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(r)})$$

the sign (± 1) of the permutation σ

S_r = group of all $r!$ permutations of $(1, 2, \ldots, r)$

Why consider these?

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antisymmetrization map A:

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the sign (± 1) of the permutation σ

Σ group of all $r!$ permutations of $(1, 2, \dots, r)$

Why consider these?

They will be key for integration! Only antisym. cov. tensors transform under chart change so as to match the Jacobian determinant arising in integrals when changing charts.

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Concretely:

- Consider $v := df \otimes dg$, for $f, g \in \mathcal{F}_p(\mathcal{U})$
- Then $v(\xi_1, \xi_2) = df(\xi_1)\, dg(\xi_2)$ $\left(\text{which is} = \xi_1(f)\,\xi_2(g)\right)$
- Apply A:

$$\tilde{v}(\xi_1, \xi_2) = A v(\xi_1, \xi_2) = \frac{1}{2}\left(df(\xi_1)\, dg(\xi_2) - df(\xi_2)\, dg(\xi_1)\right)$$

- $\Rightarrow$ We can also write:

$$A(df \otimes dg) = \frac{1}{2}(df \otimes dg - dg \otimes df)$$

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Concretely:

- Consider $v := df \otimes dg$, for $f, g \in \mathcal{F}_p(M)$
- Then $v(\xi_1, \xi_2) = df(\xi_1)\, dg(\xi_2)$ $\left(\text{which is} = \xi_1(f)\,\xi_2(g)\right)$
- Apply A:

$$\tilde{v}(\xi_1, \xi_2) = A v(\xi_1, \xi_2) = \frac{1}{2}\left(df(\xi_1)\, dg(\xi_2) - df(\xi_2)\, dg(\xi_1)\right)$$

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$$A(df \otimes dg) = \frac{1}{2}(df \otimes dg - dg \otimes df)$$

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Proposition: A is a projector, i.e., it obeys

$$A \circ A = A$$

Check in above example:

$$A \circ A(df \otimes dg) = A\left(\tfrac{1}{2} df \otimes dg - \tfrac{1}{2} dg \otimes df\right)$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2} df \otimes dg - \tfrac{1}{2} dg \otimes df\right) - \tfrac{1}{2}\left(\tfrac{1}{2} dg \otimes df - \tfrac{1}{2} df \otimes dg\right)$$

$$= \tfrac{1}{2}(df \otimes dg - dg \otimes df)$$

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$$= \tfrac{1}{2}(df \otimes dg - dg \otimes df)$$

$$= A(df \otimes dg)$$

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Definition:

For $r > 1$ we define the space of differential r-forms (or 'exterior' r-forms) $\Lambda_r(p)$ at $p \in M$ as the subspace of totally anti-symmetric tensors of rank $(0, r)$:

$$\Lambda_r(p) := A\, T_p(M)_r$$

(A: a projector on a vector space; $T_p(M)_r$: a vector space; $\Lambda_r(p)$: a vector space)

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ENG 4:24 PM 2019-09-17

as the subspace of totally anti-symmetric tensors

of rank $(0,r)$:

a projector on a vector space

a vector space

a vector space

$$\Lambda_r(p) := A\, T_p(M)_r$$

$\rightsquigarrow$ So if $\omega \in \Lambda_r(p)$ then $\tilde{\omega} = A(\omega) = \omega$

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Definition: For $r = 0$ we define the set of differential 0-forms at $p \in M$ as:

$$\Lambda_0(p) := \mathbb{R}$$

(for 0 forms on the entire manifold we will have $\Lambda_0 := \mathcal{F}(M)$)

For $r = 1$ we define the set of differential 1-forms (or "Pfaffian forms") at $p \in M$ through:

$$\Lambda_1(p) := T_p^*(M),$$

Set theoretic:

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4 of 17
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$\Lambda_0(p) := \mathbb{R}$ (for 0 forms on the entire manifold we will have $\Lambda_0 := \mathcal{F}(M)$)

For $r = 1$ we define the set of differential 1-forms (or "Pfaffian forms") at $p \in M$ through:

$$\Lambda_1(p) := T_p(M)^*$$

Strategy now:

Define multiplication → obtain algebra → obtain derivations

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4 of 17
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The wedge product:

Def: If $\omega \in \Lambda_s(p)$, $\nu \in \Lambda_r(p)$, and $r, s \neq 0$ then the wedge product $\wedge$ yields a new differential form:

$$\wedge : \Lambda_r(p) \times \Lambda_s(p) \to \Lambda_{r+s}(p)$$

$$\wedge : (\omega, \nu) \to \omega \wedge \nu = \underbrace{\frac{(s+r)!}{s!\,r!}}_{\text{a normalization factor}} A(\omega \otimes \nu)$$

Def: For $c \in \Lambda_0$, $\omega \in \Lambda_s$ we have $c \wedge \omega = c\omega$

Note: $dx^i \wedge dx^i = 0 \ \forall i$

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Def: For $c \in \Lambda_0$, $\omega \in \Lambda_s$ we have $c \wedge \omega = c\omega$

Note: $dx^i \wedge dx^i = 0 \ \forall i$

Example: For dx^i, dx^j we obtain:

$$dx^i \wedge dx^j = \left(dx^i \otimes dx^j - dx^j \otimes dx^i\right)$$

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- bi-linear:

$$(\omega + \nu) \wedge \eta = \omega \wedge \eta + \nu \wedge \eta$$

$$(a\omega) \wedge \nu = \omega \wedge (a\nu) = a(\omega \wedge \nu) \quad \text{if } a \in \mathbb{R}$$

- associative:

$$(\omega \wedge \nu) \wedge \eta = \omega \wedge (\nu \wedge \eta)$$

- "graded" commutative:

$$\omega \wedge \nu = (-1)^{rs}\, \nu \wedge \omega \quad \text{if } \omega \in \Lambda_r, \nu \in \Lambda_s$$

(E.g. for $dx^i \in \Lambda_1(p)$: $dx^i \wedge dx^j = -dx^j \wedge dx^i$ since $r = s = 1$.)

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Properties of $\wedge$:

- bi-linear:

$$(\omega + \nu) \wedge \eta = \omega \wedge \eta + \nu \wedge \eta$$

$$(a\omega) \wedge \nu = \omega \wedge (a\nu) = a(\omega \wedge \nu) \quad \forall a \in \mathbb{R}$$

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5 of 17
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5 of 17
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$$(\omega \wedge \nu) \wedge \eta = \omega \wedge (\nu \wedge \eta)$$

- "graded" commutative:

$$\omega \wedge \nu = (-1)^{rs}\, \nu \wedge \omega \quad \text{if } \omega \in \Lambda_r, \nu \in \Lambda_s$$

(E.g. for $dx^i \in \Lambda_1(p)$: $dx^i \wedge dx^j = -dx^j \wedge dx^i$ since $r = s = 1$.)

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- We can use $\wedge$ to build bases of $\Lambda_r(p)$:

Assume: $\{\theta^i\}_{i=1}^n$ is a basis of $\Lambda_1 = T_p(M)^*$.

(for example $\theta^i = dx^i$)

Then: $\{\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_r}\}_{1 \le i_1 < i_2 < \dots < i_r \le n}$

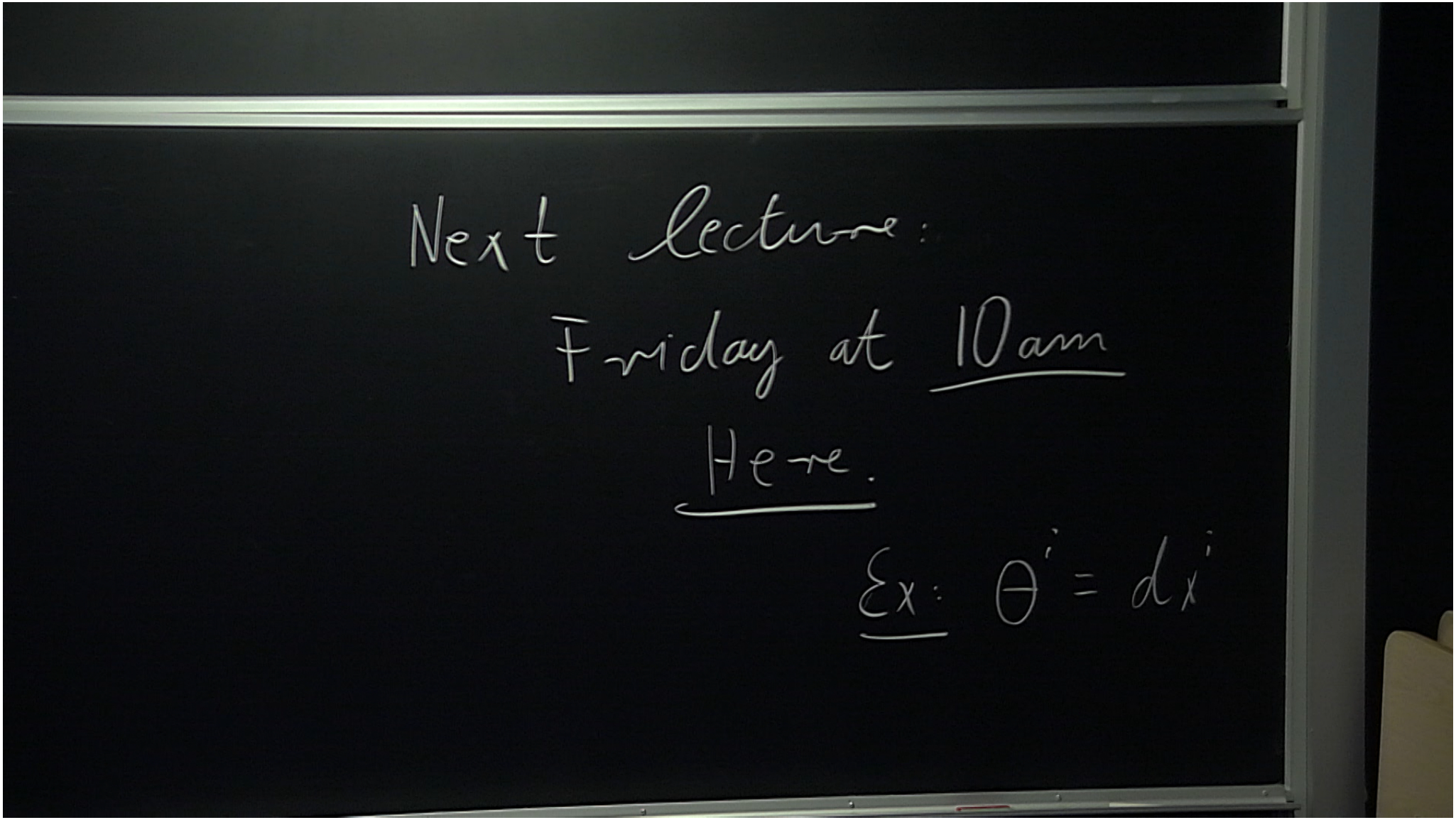
is a basis of $\Lambda_r(p)$ for $r > 1$.

Exercise: show this

- **Therefore:**

$$\dim(\Lambda_r(p)) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

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Here.
Ex: $\theta^i = dx^i$

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(for example $\theta^i = dx^i$)

Then: $\{\theta^{i_1} \wedge \theta^{i_2} \wedge \dots \wedge \theta^{i_r}\}_{1 \le i_1 < i_2 < \dots < i_r \le n}$

Exercise: show this

is a basis of $\Lambda_r(p)$ for $r > 1$.

Therefore:

$$\dim(\Lambda_r(p)) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$\Rightarrow$ no diff. forms of degree $r > n$!

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6 of 17
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Example: For $p \in M = \mathbb{R}^3$ we have bases:

$$\Lambda_1 = \text{span}(dx^1, dx^2, dx^3)$$
$$\Lambda_2 = \text{span}(dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^2 \wedge dx^3)$$
$$\Lambda_3 = \text{span}(dx^1 \wedge dx^2 \wedge dx^3)$$

Definition:

$\Lambda(p) := \bigoplus_{i=0}^{n} \Lambda_i(p)$ equipped with the multiplication $\wedge$, is an associative algebra, called the exterior algebra

$(\dim(\Lambda) = 2^n)$

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6 of 17
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Definition:

$\Lambda(p) := \bigoplus_{i=0}^{n} \Lambda_i(p)$ equipped with the multiplication $\wedge$, is an associative algebra, called the exterior algebra or the Grassmann algebra over $T_p(M)$.

$(\dim(\Lambda) = 2^n)$

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Generalization to fields:

- A differential form field is a mapping that associates to each $p \in M$ an element:

$$\omega(p) \in \Lambda(p)$$

It is usually also called simply a differential form and denoted ω.

- These fields form the

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6 of 17 — 220%

These fields form the

Grassmann algebra of differential forms, $\Lambda(M)$.

Recall:

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7 of 17 220%

Recall:

Given an algebra, it is often useful to consider derivations of the algebra, i.e., to consider linear maps that obey the Leibnit rule.

(similar to how we defined tangent vectors as derivations of $\mathcal{F}(M)$)

Here: For the algebra $\Lambda(M)$, let us consider

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7 of 17 220%

Here: For the algebra $\Lambda(M)$, let us consider the exterior and the inner derivations:

Def.:

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7 of 17
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Definition:

A linear map $\mathcal{I} : \Lambda(M) \to \Lambda(M)$ is called a derivation of degree k, if:

$\mathcal{I} : \Lambda_s(M) \to \Lambda_{s+k}(M)$ for all s

$\mathcal{I} : \alpha \wedge \beta \to \mathcal{I}(\alpha) \wedge \beta + \alpha \wedge \mathcal{I}(\beta)$

for all $\alpha, \beta \in \Lambda(M)$.

Leibniz rule

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8 of 17
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Definition:

A linear map $\Phi : \Lambda(M) \to \Lambda(M)$ is called an anti-derivation of degree k, if for all $\alpha \in \Lambda^1(M)$, $\beta \in \Lambda(M)$:

$\Phi : \Lambda_s(M) \to \Lambda_{s+k}(M)$ for all s and

$\Phi : \alpha \wedge \beta \to \Phi(\alpha) \wedge \beta + (-1)^q \alpha \wedge \Phi(\beta)$

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A linear map $\Phi : \Lambda(M) \to \Lambda(M)$ is called an anti-derivation of degree K, if for all $\alpha \in \Lambda^q(M)$, $\beta \in \Lambda(M)$:

$\Phi : \Lambda_s(M) \to \Lambda_{s+k}(M)$ for all s and

$\Phi : \alpha \wedge \beta \to \Phi(\alpha) \wedge \beta + (-1)^q \alpha \wedge \Phi(\beta)$

"Anti-Leibniz rule"

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Proposition: (as we will show constructively)

Because of the Leibniz rule and linearity, any (anti-) derivation

$$\Phi : \Lambda(M) \to \Lambda(M)$$

is already fully determined by its action only on $\Lambda_0(M)$ and on

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9 of 17
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The exterior derivative

The exterior derivative,

$$d: \Lambda(M) \to \Lambda(M)$$

is the anti-derivation of degree $K=1$ which is defined through:

a) $d: \Lambda_0(M) \to \Lambda_1(M)$

$d: f \to df$

} action of d on $\Lambda_0(M)$

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9 of 17
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The exterior derivative,

$$d: \Lambda(M) \to \Lambda(M)$$

is the anti-derivation of degree K=1 which is defined through:

a) $d: \Lambda_0(M) \to \Lambda_1(M)$
$d: f \to df$
} action of d on $\Lambda_0(M)$

b) $d: dx^i \to 0$ for all i. } action of d on a basis of $\Lambda_1(M)$!

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9 of 17
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In a chart:

We had: $d : f(x) \to df(x) = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} \cdot dx^i$

Now we have more generally:

$$\beta = \sum_{i_1 < \ldots < i_s} \underbrace{\beta_{i_1,\ldots,i_s}(x)}_{\Lambda_0(M)} \wedge \underbrace{dx^{i_1} \wedge \ldots \wedge dx^{i_s}}_{\Lambda_s(M)} \in \Lambda_s(M)$$

recall: $f \wedge \omega = f\omega$ when $f \in \Lambda_0$ and $\omega \in \Lambda$

Q: So how to carry out $d : \beta \to d\beta$?

A: By applying the anti-Leibnit rule:

$$d\beta = \sum \frac{\partial \beta_{i_1,\ldots,i_s}}{\partial x^j}(x)\, dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_s}$$

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ENG 5:04 PM 2019-09-17

Now we have more generally:

$$\beta = \sum_{i_1 < \ldots < i_s} \underbrace{\beta_{i_1,\ldots,i_s}(x)}_{\Lambda_0(M)} \wedge \underbrace{dx^{i_1} \wedge \ldots \wedge dx^{i_s}}_{\Lambda_s(M)} \in \Lambda_s(M)$$

recall: $f \wedge \omega = f\omega$ when $f \in \Lambda_0$ and $\omega \in \Lambda$

Q: So how to carry out $d : \beta \to d\beta$?

A: By applying the anti-Leibnitz rule:

$$d\beta = \sum_{i_1 < \ldots < i_s} \frac{\partial \beta_{i_1,\ldots,i_s}(x)}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_s}$$

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10 of 17
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$+$ terms of the form $d(dx^{i_1} \wedge \dots \wedge dx^{i_s})$

$= 0$ because when applying Leibniz rule to $d(dx^{i_1} \wedge \dots \wedge dx^{i_s})$ we eventually arrive at $d(dx^i) = 0 \dots$!

Proposition: $d: \Lambda(M) \to \Lambda(M)$ obeys:

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Proposition: $d: \Lambda(M) \to \Lambda(M)$ obeys:

$$\boxed{d \circ d = 0}$$

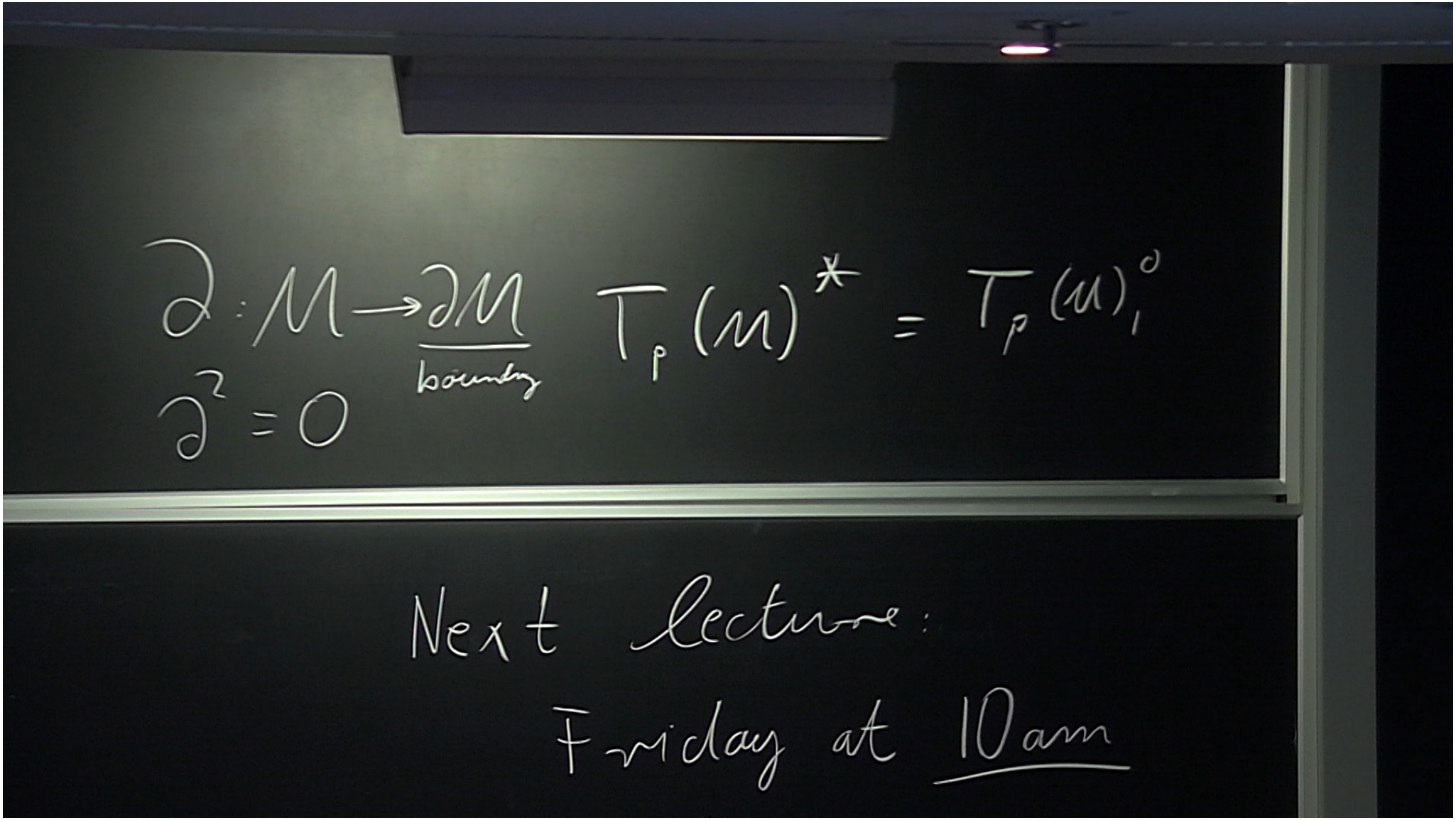
Proof:

$$d\,d(\beta) = \sum_{\substack{i,i_1 \dots i_s \\ j,k}} \underbrace{\frac{\partial^2 \beta_{i_1 \dots i_s}(x)}{\partial x^j \partial x^k}}_{\text{sym. in } j,k} \underbrace{dx^k \wedge dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}}_{dx^k \wedge dx^j = -dx^j \wedge dx^k \text{ i.e. antisym. in } j,k}$$

$$\Rightarrow \Sigma = 0$$

$$= 0$$

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$\partial : M \to \underline{\partial M}$
boundary
$\partial^2 = 0$
$T_p(M)^* = T_p(M)^\circ$
Next lecture:
Friday at 10 am

Example:

- For $M = \mathbb{R}^3$ and $f \in \mathcal{F}(M)$ we have: (e.g.: electric potential)

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i} dx^i$$

- **Notice:** $\left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3}\right)$ is the "Gradient field ∇f of f"

- **Example:** Electric field $E = d\phi$ from potential ϕ.

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Now assume $\gamma \in \Lambda_1(M)$ is an arbitrary (i.e. not necessarily gradient) covariant vector field:

$$\gamma = \sum_{i=1}^{3} \gamma_i(x)\, dx^i \quad \in \Lambda_1(\mathbb{R}^3)$$

Then:

$$\beta := d\gamma = \sum_{i,j} \frac{\partial \gamma_i(x)}{\partial x^j}\, dx^j \wedge dx^i$$

$i=j$ does not occur because $dx^i \wedge dx^i$

from $dx^j \wedge dx^i = -dx^i \wedge dx^j$

$$= \sum_{i<j} \left(\frac{\partial \gamma_i(x)}{\partial x^j} - \frac{\partial \gamma_j(x)}{\partial x^i} \right) dx^j \wedge dx^i$$

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$i=j$ does not occur because $dx^i \wedge dx^i = 0$

from $dx^j \wedge dx^i = -dx^i \wedge dx^j$

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$$= -\underbrace{\left(\frac{\partial r_1}{\partial x^2} - \frac{\partial r_2}{\partial x^1}\right)}_{\beta_3 :=} dx^1 \wedge dx^2$$

$$- \underbrace{\left(\frac{\partial r_1}{\partial x^3} - \frac{\partial r_3}{\partial x^1}\right)}_{\beta_2 :=} dx^1 \wedge dx^3$$

$$- \underbrace{\left(\frac{\partial r_2}{\partial x^3} - \frac{\partial r_3}{\partial x^2}\right)}_{\beta_1 :=} dx^2 \wedge dx^3$$

Notice: $(\beta_1(x), \beta_2(x), \beta_3(x))$ are the components of the curl

$$\beta = \nabla \times r$$

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$$= -\left(\frac{\partial \gamma_1}{\partial x^2} - \frac{\partial \gamma_2}{\partial x^1}\right) dx^1 \wedge dx^2$$

$$\beta_2 := \quad -\left(\frac{\partial \gamma_1}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^1}\right) dx^1 \wedge dx^3$$

$$-\left(\underbrace{\frac{\partial \gamma_2}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^2}}_{\beta_1 :=}\right) dx^2 \wedge dx^3$$

Notice: $(\beta_1(x), \beta_2(x), \beta_3(x))$ are the components of the curl

$$\beta = \nabla \times \gamma$$

It is called a "pseudo vector field"

$$-\underbrace{\left(\frac{\partial \gamma_2}{\partial x^3} - \frac{\partial \gamma_3}{\partial x^2}\right)}_{\beta_1 :=} dx^2 \wedge dx^3$$

Notice: $(\beta_1(x), \beta_2(x), \beta_3(x))$ are the components of the curl

$$\beta = \nabla \times \gamma$$

It is called a "pseudo vector field"

It is really a 2-form field in 3 dim.

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12 of 17 — 220%

Recall:

Gradient vector fields are curl free: $\nabla \times (\nabla f) = 0$

This is a special case of

$$d \cdot d = 0$$

because if $\beta = d\gamma$ then:

$d^2 = 0$

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12 of 17
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Definition:

- A differential form ω is called **closed** if:

$$d\omega = 0$$

E.g.: We saw that $\beta := dy$ is closed. Is this example typical?

- A differential form ω is called **exact** if there exists a ν so that

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$d\omega = 0$

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A differential form ω is called **exact** if there exists a ν so that

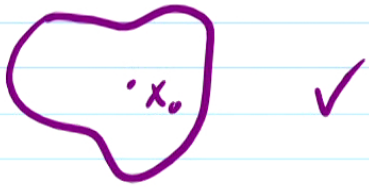
$\omega = d\nu$ (ν is like an anti-derivative!)

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How are closedness and exactness related?

This actually depends on the global topology of the manifold! (because anti-derivatives are in a sense global)

Simplest case: Assume M is contractible



✓



i.e., $\exists\, F: [0,1] \times M \to M$ (✓ continuous)

so that $F(0,x) = x \quad \forall x$

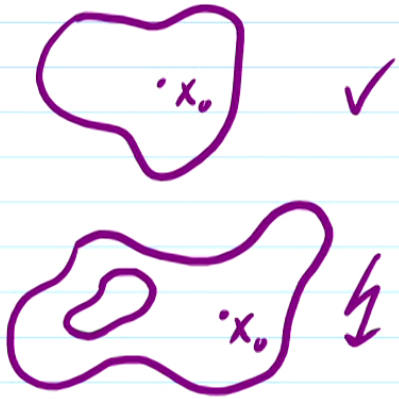
$F(1,x) = x_0 \quad \forall x$

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13 of 17
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of the manifold! (because anti-derivatives are in a sense global)

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↯

i.e., $\exists\ F: [0,1] \times M \to M$ (✓ continuous)

so that $F(0,x) = x \quad \forall x$

$F(1,x) = x_0 \quad \forall x$

↑ some fixed pt $\in M$

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Poincaré lemma:

On any contractible manifold:

γ exact $\Leftrightarrow$ γ closed

E.g.

- $\mathbb{R}^n$ is contractible

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In general: We only have

γ exact $\Rightarrow$ γ closed

which is because $d^2 = 0$.

$\Rightarrow$ We obtain a tool for classifying the "global topology" of differentiable manifolds (checking for holes, handles etc.)

How? Take a diffable manifold and calculate the vector space of closed but not exact differential forms.

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This method is a special case of a cohomology theory, called

"De Rham Cohomology"

Q: What is a cohomology theory? (roughly)

A: A cohomology theory is a map G

$$G : \{\text{differentiable manifolds}\} \to \{\text{abelian groups}\}$$

which is such that if:

$$G(M) \neq G(N) \implies M \neq N$$

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14 of 17
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A: A cohomology theory is a map G

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which is such that if:

$$G(M) \ncong G(N) \implies M \not\cong N$$

$G(M) \ncong G(N)$: i.e. not isomorphic as abelian groups

$M \not\cong N$: i.e. no diffeomorphism exists

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A look at the even bigger picture

Comment: All cohomology theories are:

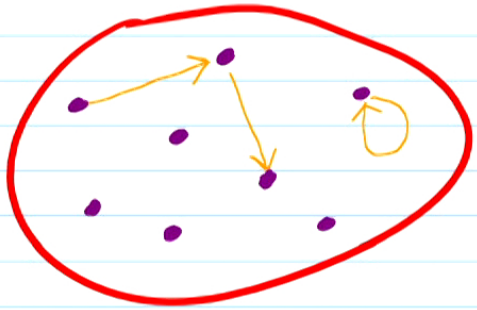
"Natural Transformations" between two "Categories!"

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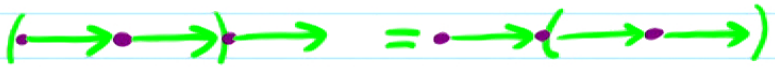
"Natural Transformations" between two "Categories".

Def: A category is a set of objects and morphisms:



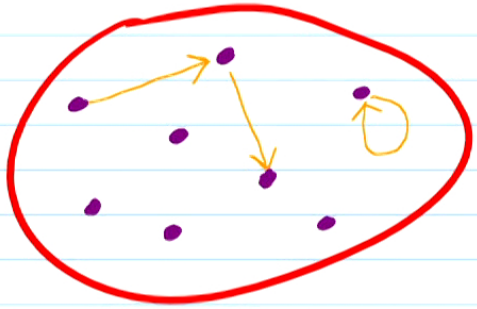
Axioms:

- $\exists\, A \to B$ and $B \to C \Rightarrow \exists\, A \to C$
- Associativity:



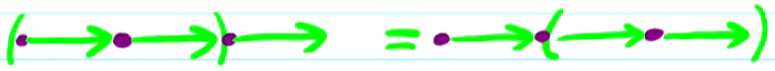
"Natural Transformations" between two "Categories"

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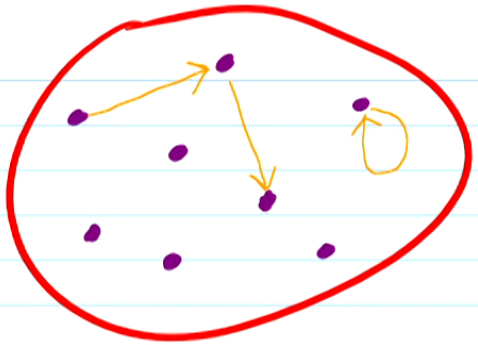
Axioms:

- $\exists\ A \to B$ and $B \to C \Rightarrow \exists\ A \to C$
- Associativity:



- $\forall \bullet\ \exists$

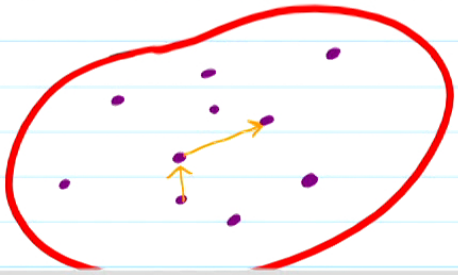
Examples:



Category of vector spaces **Vec**:

Objects: all vector spaces

Morphisms: linear transformations



Category of associative algebras **Alg**:

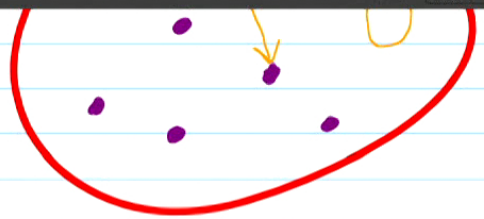
Objects: all assoc. algebras

Morphisms: algebra homomorphisms

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Objects: all vector spaces

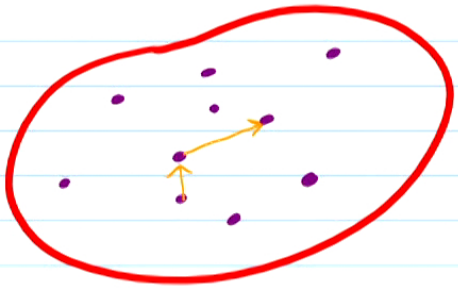
Morphisms: linear transformations



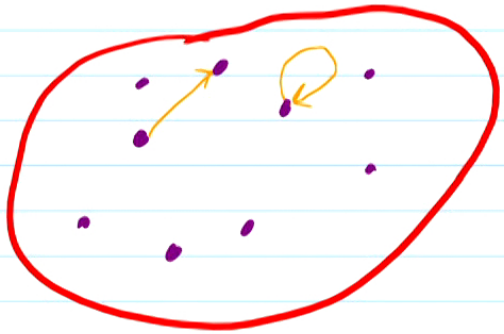
Category of associative algebras Alg:

Objects: all assoc. algebras

Morphisms: algebra homomorphisms



But also:



Category of categories Cat:

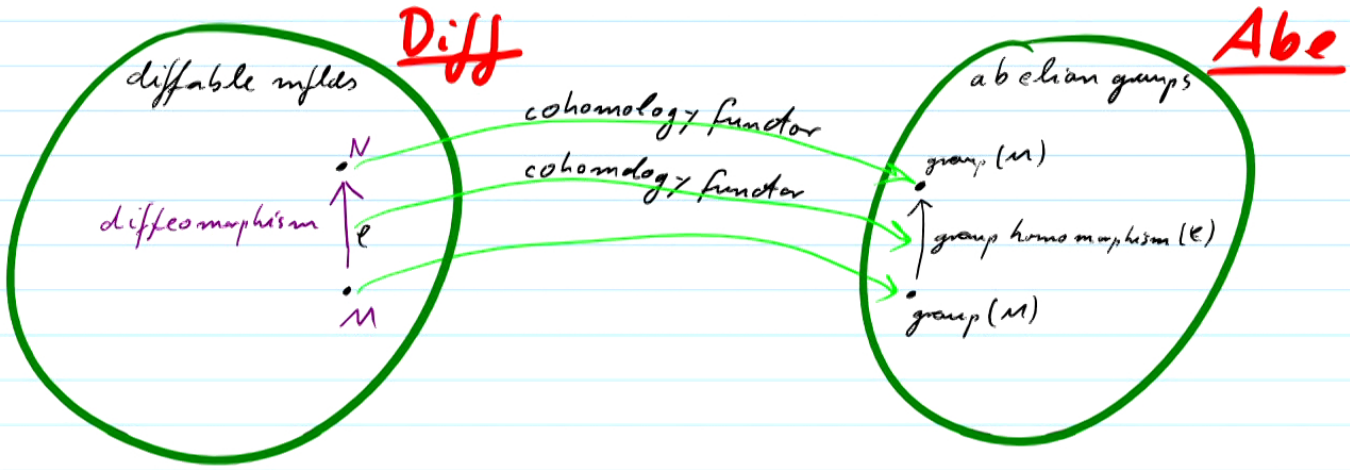
Objects: all categories

Morphisms: natural transformations
also called functors.

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A cohomology theory is a morphism between two objects in Cat, namely Diff and Abe:



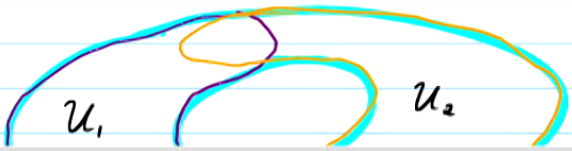
Crucial: group(M) not homomorph group(N) ⇒ M not diffeomorph to N

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Example: K-theory

- This cohomology theory uses the fact that:
- There is only one way to put a vector bundle on a contractible mfld but others can have many non-isomorphic vector bundles.
- Recall that, e.g., for a suitable vector bundle B:



$$\pi^{-1}(U_1) \simeq U_1 \times \mathbb{R}^n$$
$$\pi^{-1}(U_2) \simeq U_2 \times \mathbb{R}^n$$

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$$\pi^{-1}(U_1) \simeq U_1 \times \mathbb{R}^n$$
$$\pi^{-1}(U_2) \simeq U_2 \times \mathbb{R}^n$$
But: $B \neq M \times \mathbb{R}^n$

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17 of 17 — 220%

Q: What are the abelian groups here?

A: The vector bundles form an abelian group through Whitney's generalization of direct sum ⊕.

In this course: We'll focus on the local properties of the manifolds, such as curvature.

⇝ We mention cohomology theory issues only on the side in this course.

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17 of 17 — 220%

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Next: The inner derivation of $\Lambda(M)$.

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