Title: General Relativity for Cosmology - Lecture 3

Speakers: Achim Kempf

Collection: General Relativity for Cosmology (Kempf)

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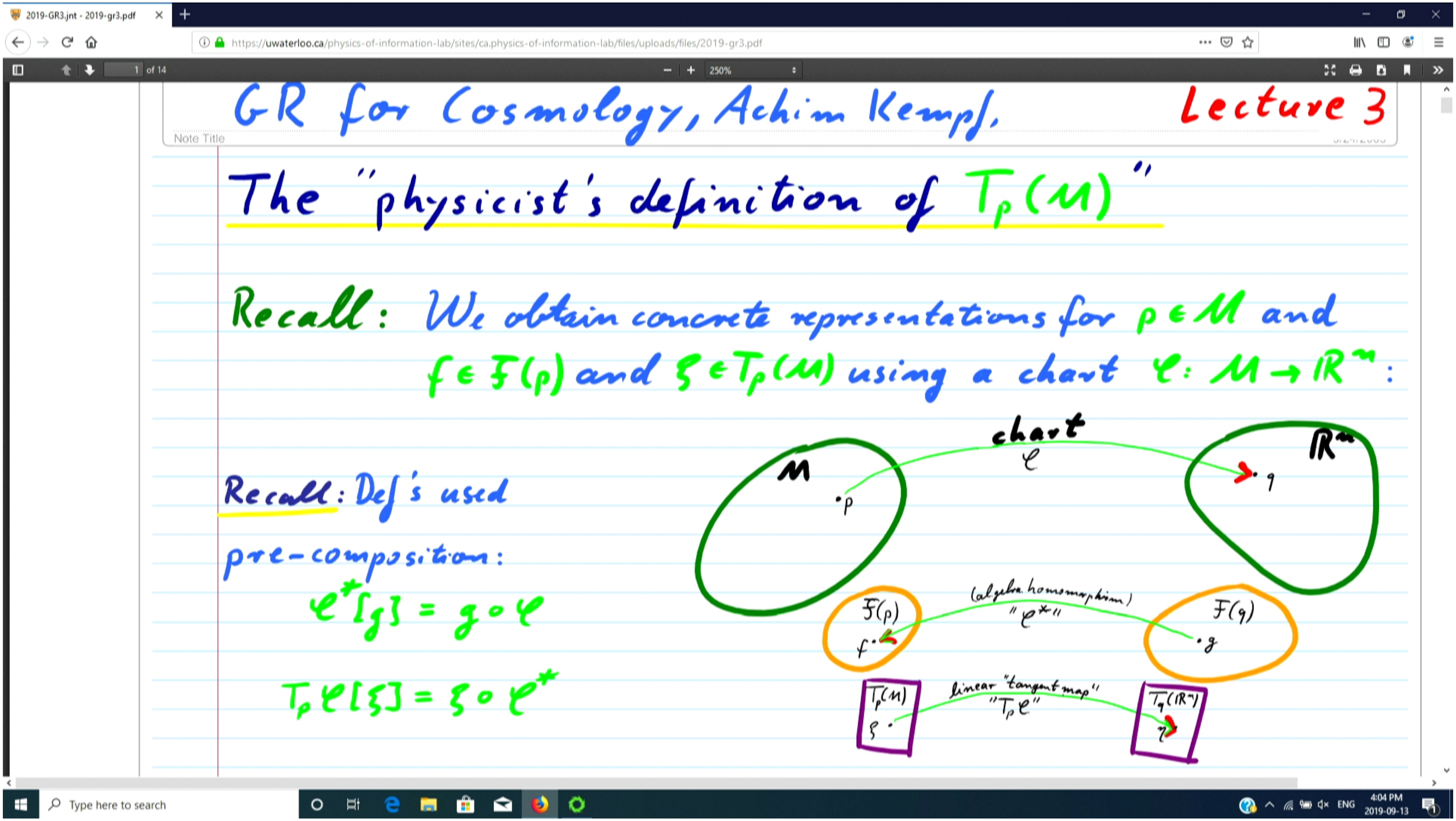


GR for Cosmology, Achim Kempf, Lecture 3

The "physicist's definition of $T_p(M)$"

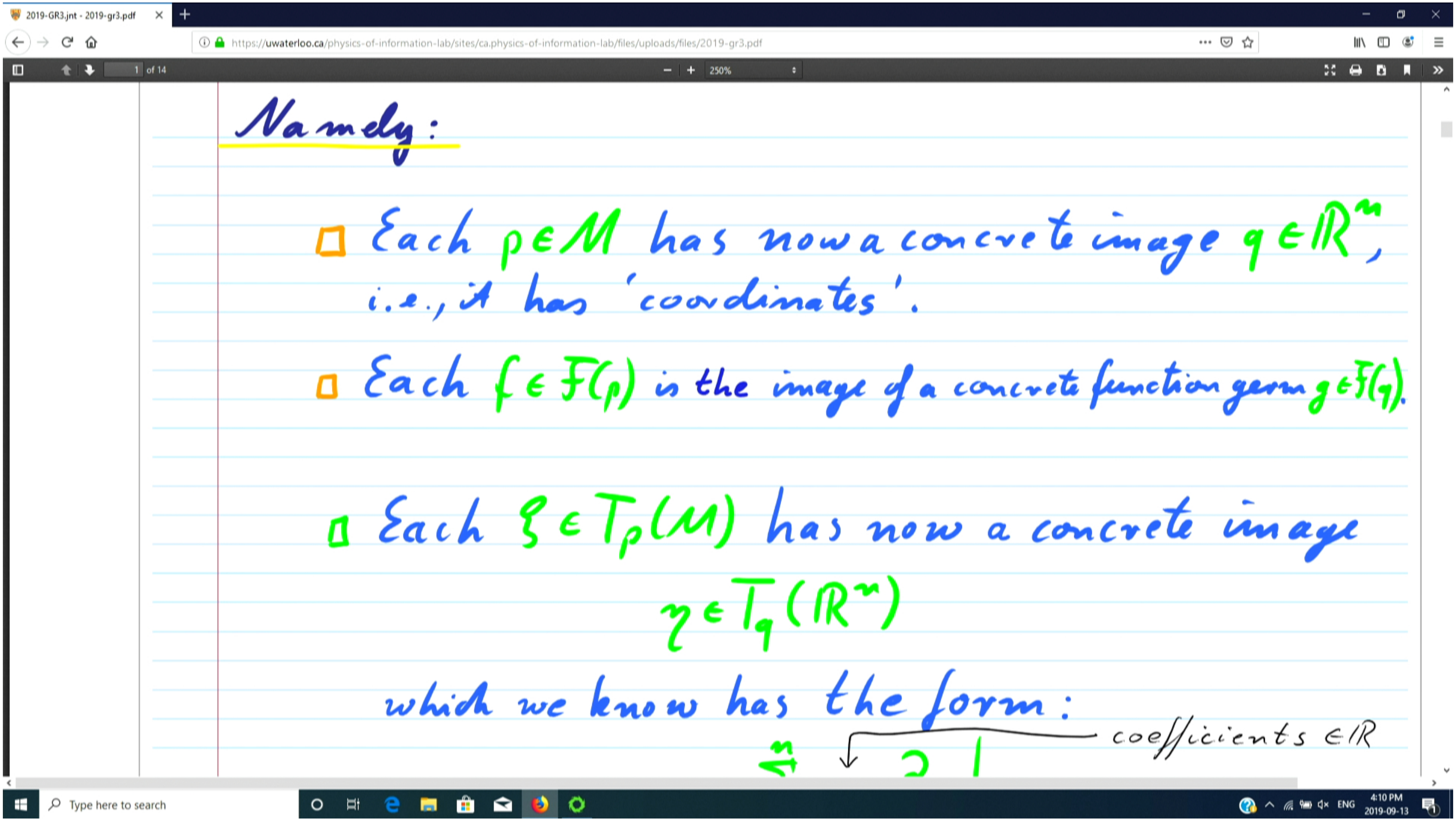
Recall: We obtain concrete representations for $p \in M$ and $f \in \mathcal{F}(p)$ and $\xi \in T_p(M)$ using a chart $\varphi : M \to \mathbb{R}^n$:

Recall: Def's used

pre-composition:

$$\varphi^*[g] = g \circ \varphi$$

$$T_p\varphi[\xi] = \xi \circ \varphi^*$$



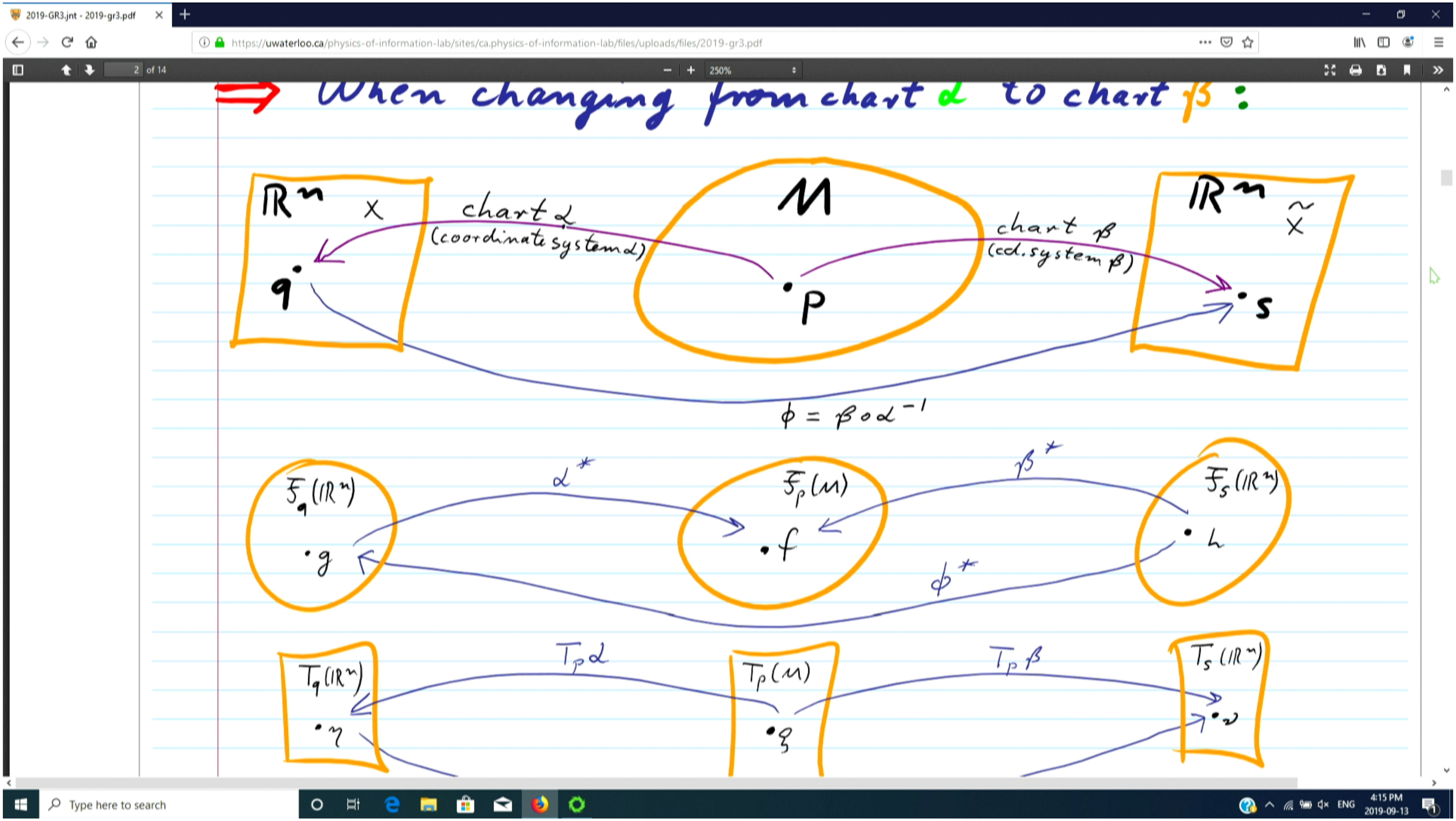
Namely:

- ☐ Each $p \in M$ has now a concrete image $q \in \mathbb{R}^n$, i.e., it has 'coordinates'.
- ☐ Each $f \in \mathcal{F}(p)$ is the image of a concrete function germ $g \in \mathcal{F}(q)$.
- ☐ Each $\xi \in T_p(M)$ has now a concrete image

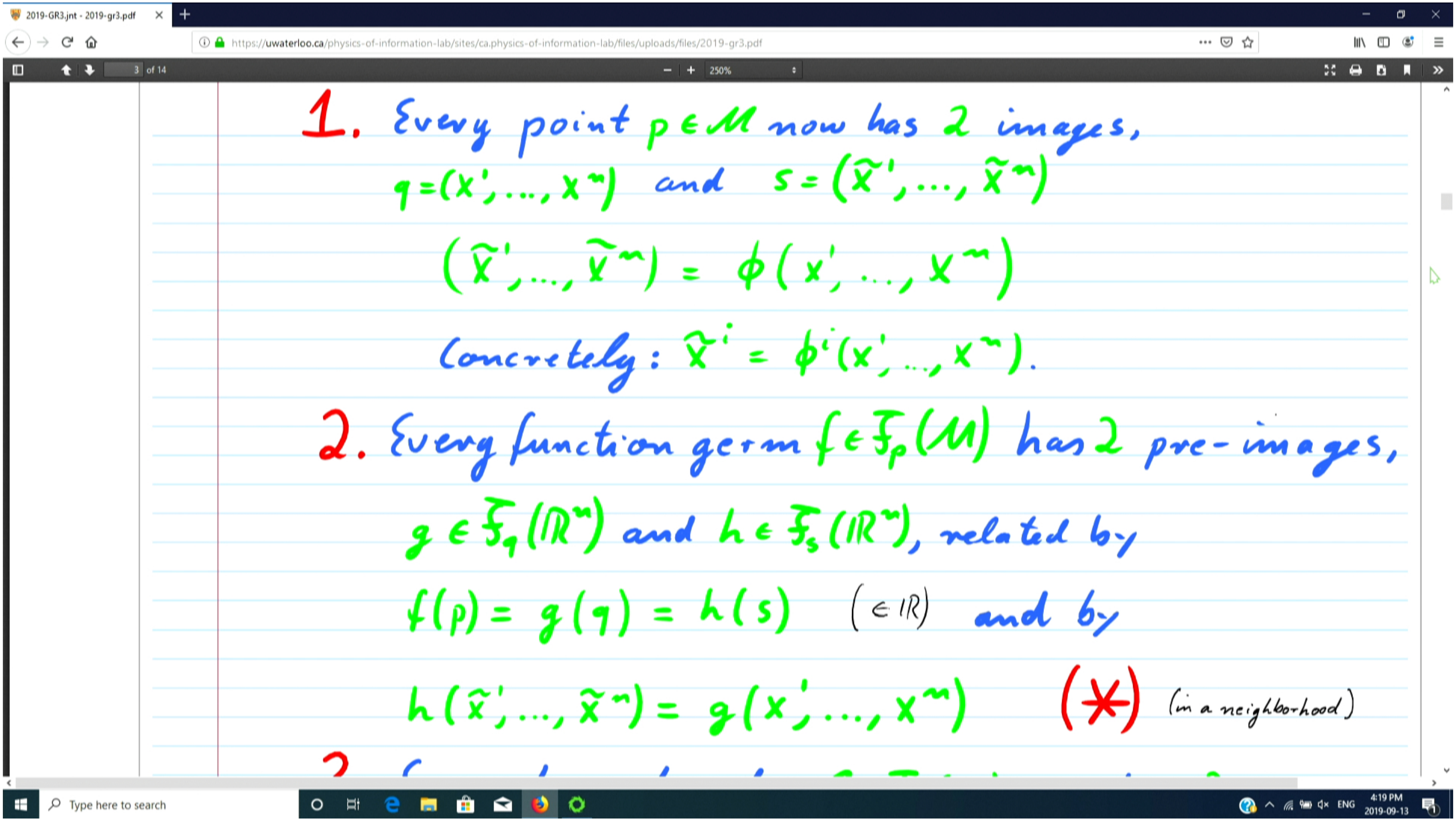
$$\eta \in T_q(\mathbb{R}^n)$$

which we know has the form:

coefficients $\in \mathbb{R}$



⟹ When changing from chart α to chart β:



1. Every point $p \in \mathcal{M}$ now has 2 images,

$q = (x^1, \dots, x^n)$ and $s = (\tilde{x}^1, \dots, \tilde{x}^n)$

$$(\tilde{x}^1, \dots, \tilde{x}^n) = \phi(x^1, \dots, x^n)$$

Concretely: $\tilde{x}^i = \phi^i(x^1, \dots, x^n)$.

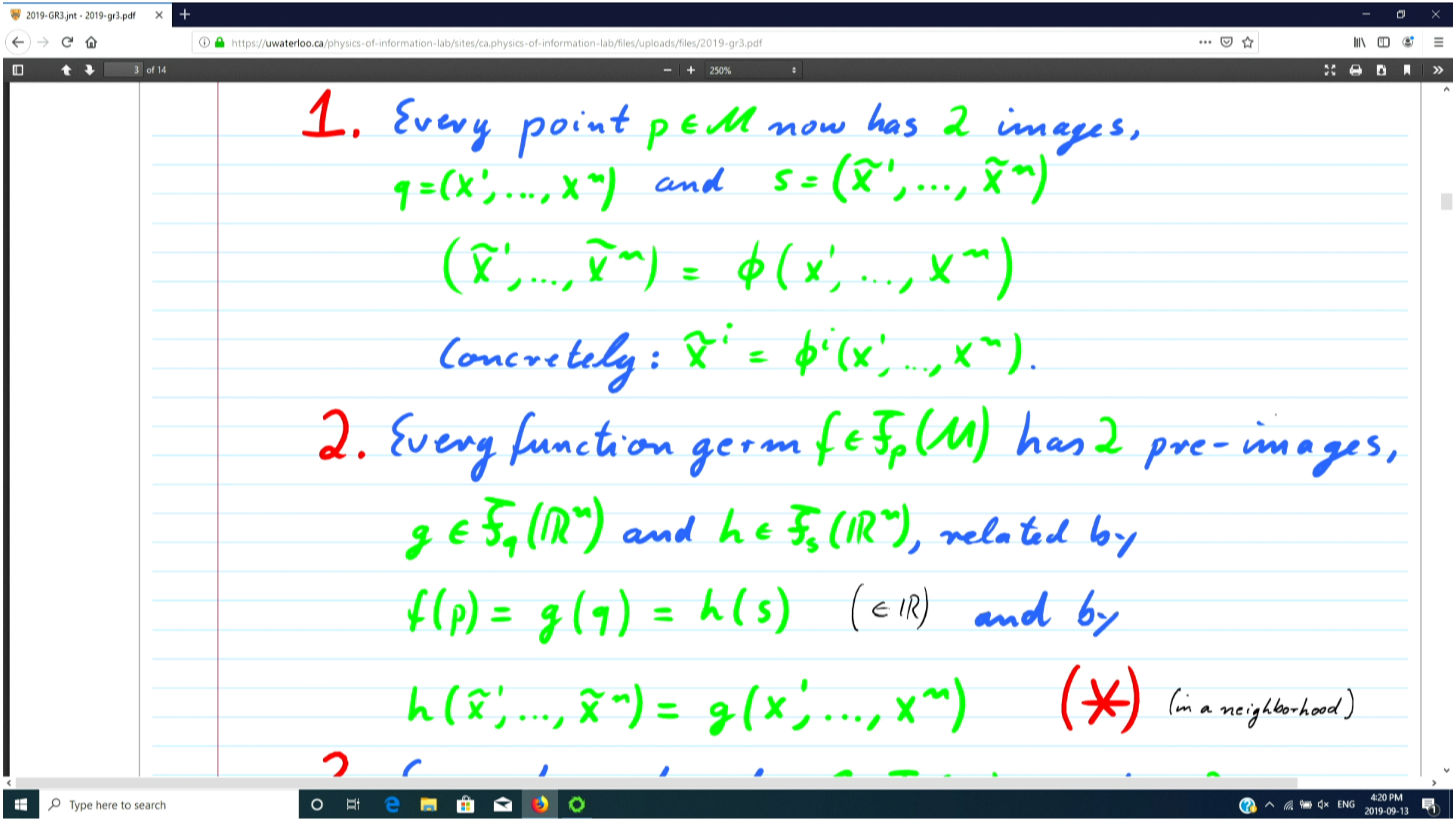
2. Every function germ $f \in \mathcal{F}_p(\mathcal{M})$ has 2 pre-images,

$g \in \mathcal{F}_q(\mathbb{R}^n)$ and $h \in \mathcal{F}_s(\mathbb{R}^n)$, related by

$f(p) = g(q) = h(s)$ $(\in \mathbb{R})$ and by

$$h(\tilde{x}^1, \dots, \tilde{x}^n) = g(x^1, \dots, x^n) \qquad (*) \quad \text{(in a neighborhood)}$$

3.



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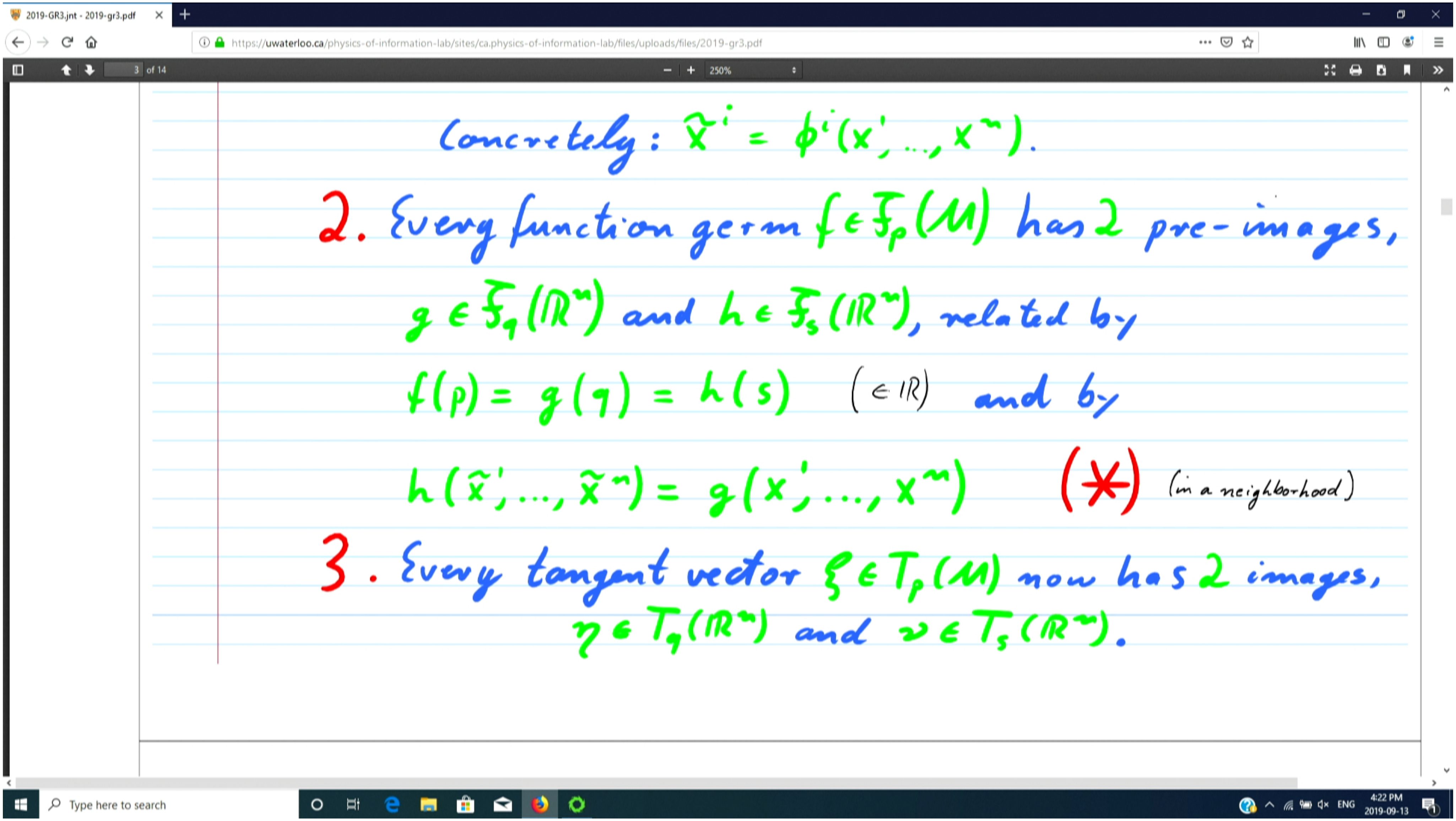


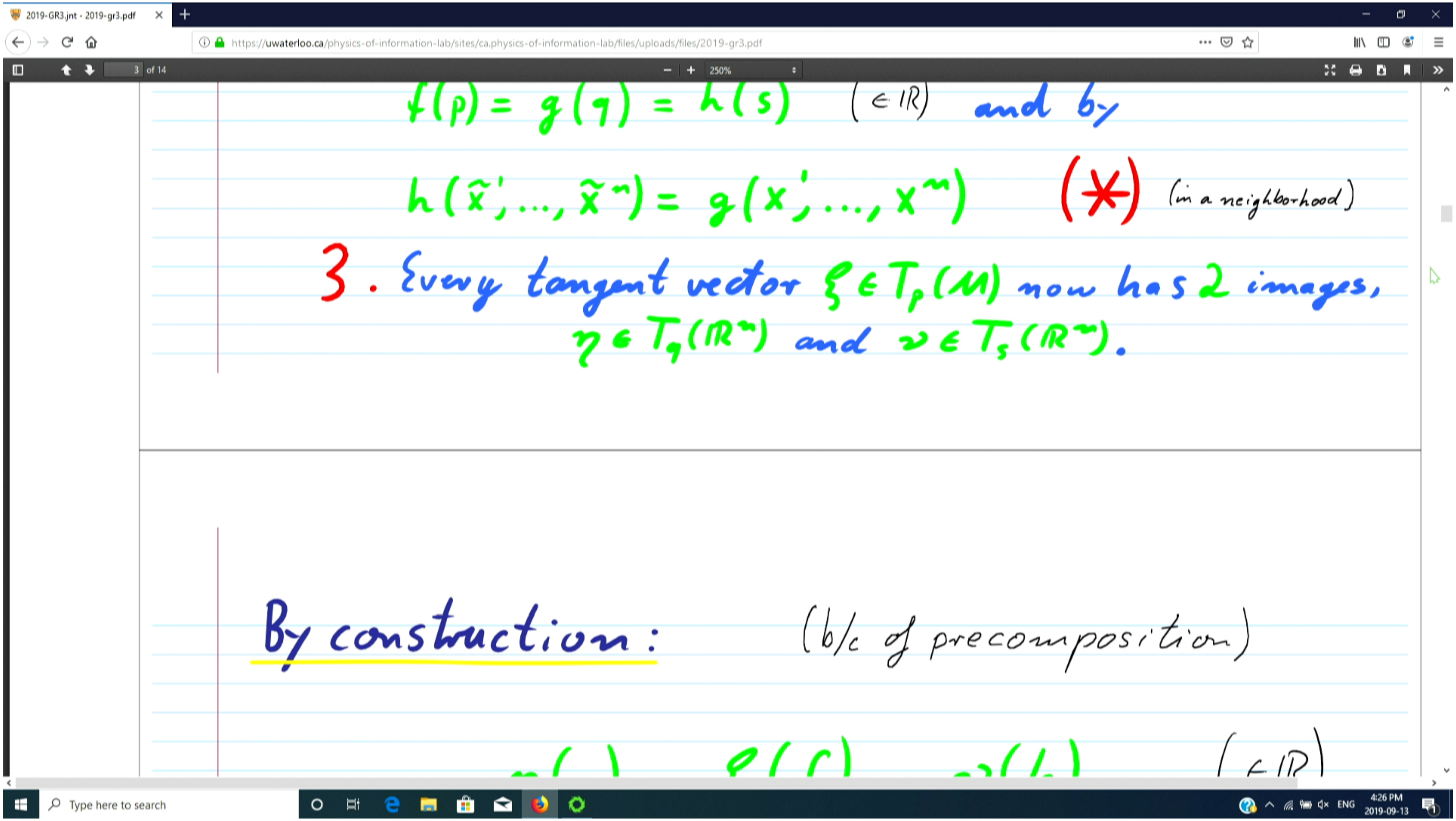
Concretely: $\tilde{x}^i = \phi^i(x^1, \ldots, x^n)$.

2. Every function germ $f \in \mathcal{F}_p(M)$ has 2 pre-images, $g \in \mathcal{F}_q(\mathbb{R}^n)$ and $h \in \mathcal{F}_s(\mathbb{R}^n)$, related by $f(p) = g(q) = h(s)$ $(\in \mathbb{R})$ and by

$$h(\tilde{x}^1, \ldots, \tilde{x}^n) = g(x^1, \ldots, x^n) \qquad (*)$$

(in a neighborhood)

3. Every tangent vector $\xi \in T_p(M)$ now has 2 images, $\eta \in T_q(\mathbb{R}^n)$ and $\nu \in T_s(\mathbb{R}^n)$.



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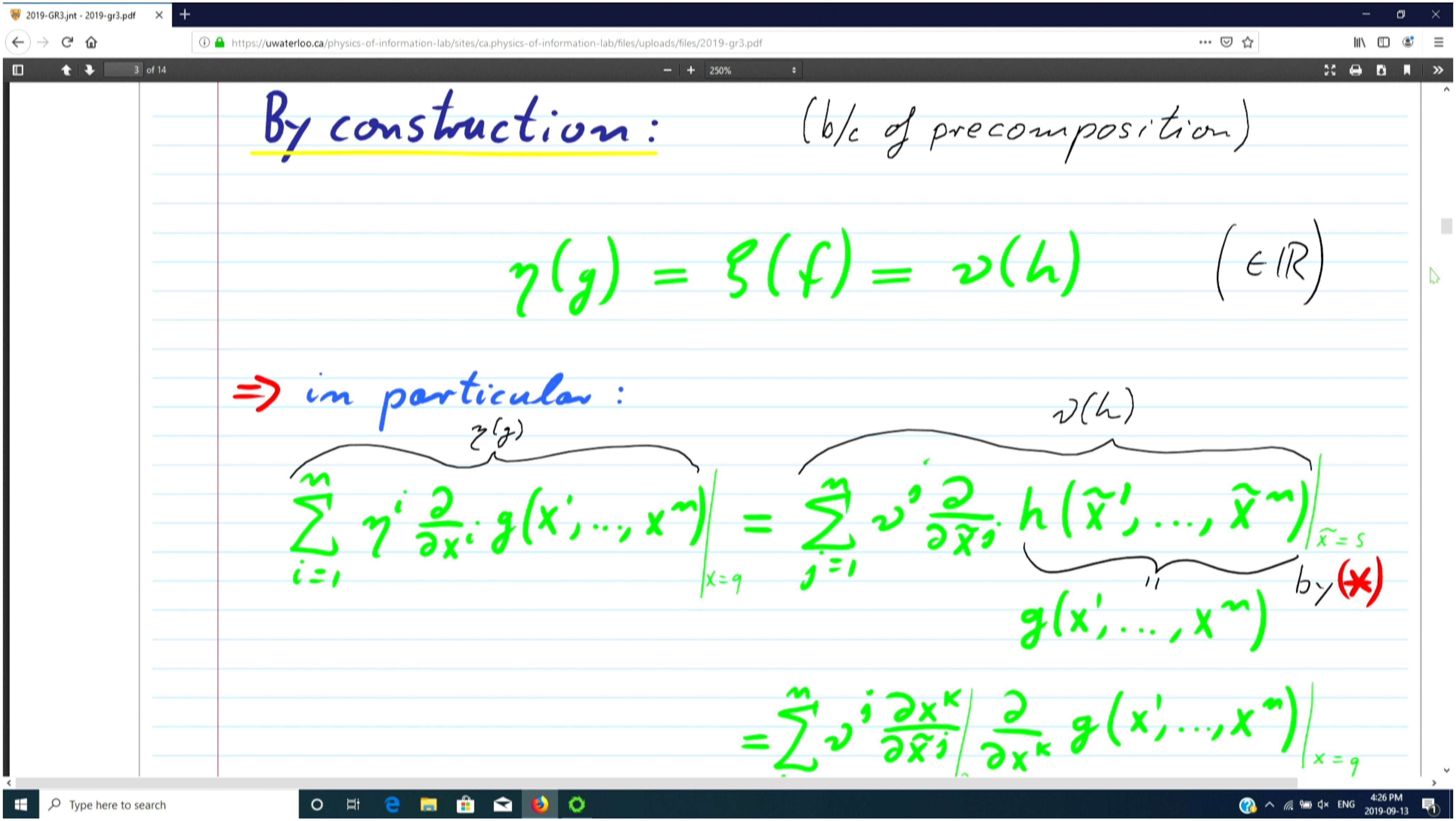
By construction: (b/c of precomposition)



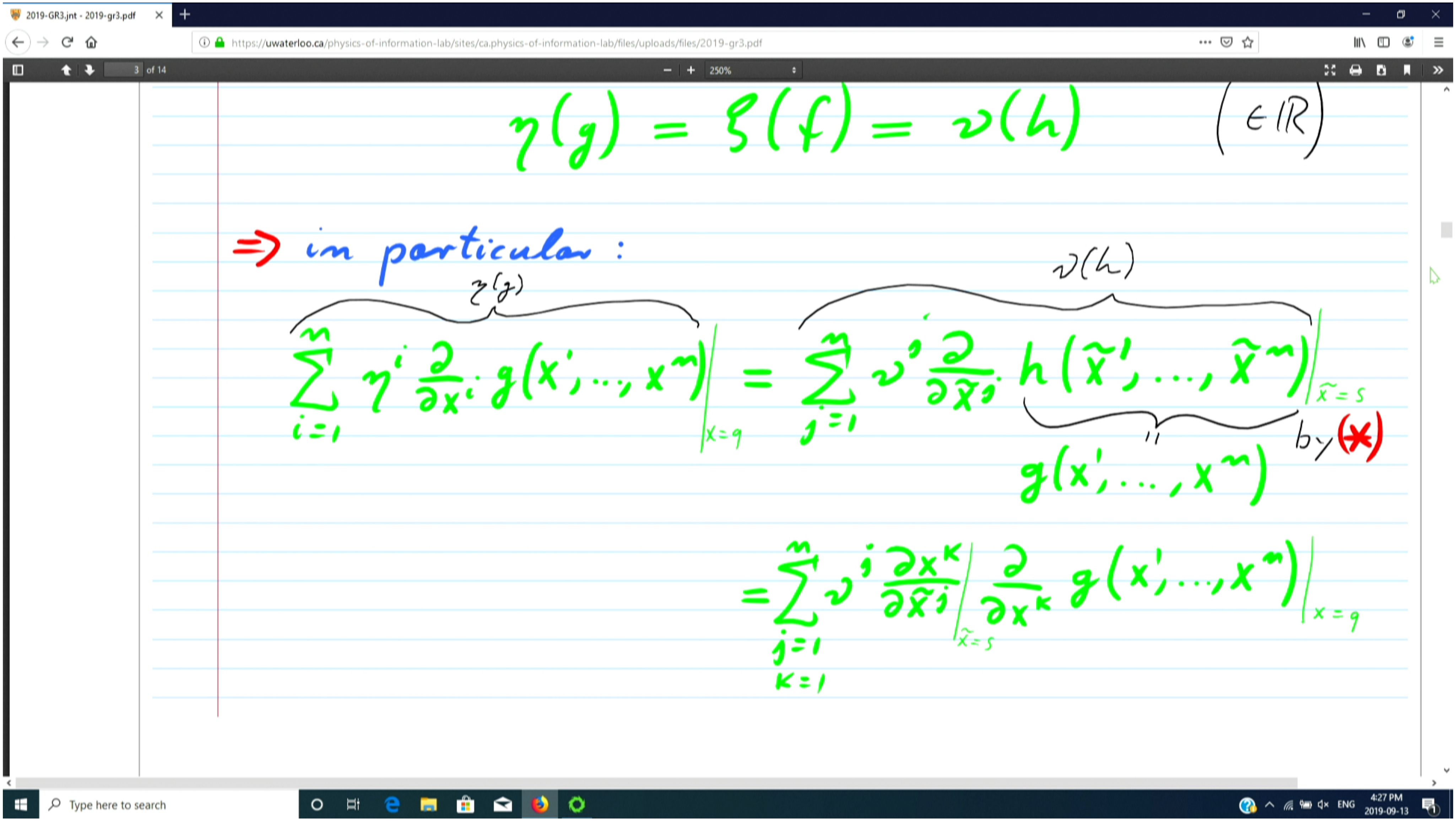
By construction: (b/c of precomposition)

$$\eta(g) = \xi(f) = \nu(h) \qquad (\in \mathbb{R})$$

⇒ in particular:

$$\overbrace{\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i} g(x^1,\dots,x^n)}^{\eta(g)}\Big|_{x=q} = \overbrace{\sum_{j=1}^{n} \nu^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1,\dots,\tilde{x}^n)}_{\overset{\text{by } (*)}{=}\; g(x^1,\dots,x^n)}}^{\nu(h)}\Big|_{\tilde{x}=s}$$

$$= \sum^{n} \nu^j \frac{\partial x^k}{\partial \tilde{x}^j}\Big| \frac{\partial}{\partial x^k} g(x^1,\dots,x^n)\Big|_{x=q}$$



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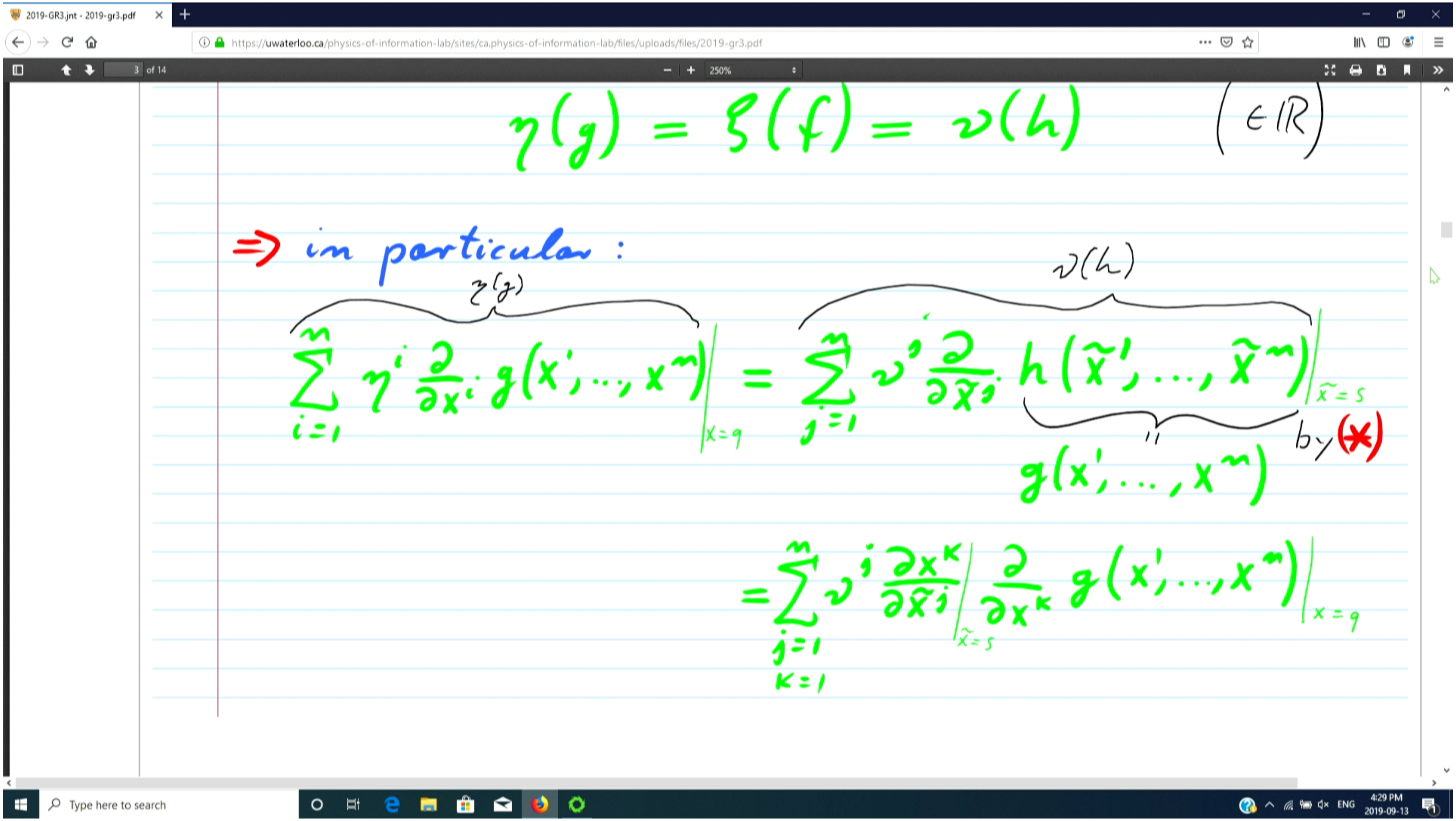
$$= \sum_{\substack{j=1\\k=1}}^{n} \nu^j \frac{\partial x^k}{\partial \tilde{x}^j}\Big|_{\tilde{x}=s} \frac{\partial}{\partial x^k} g(x^1,\dots,x^n)\Big|_{x=q}$$



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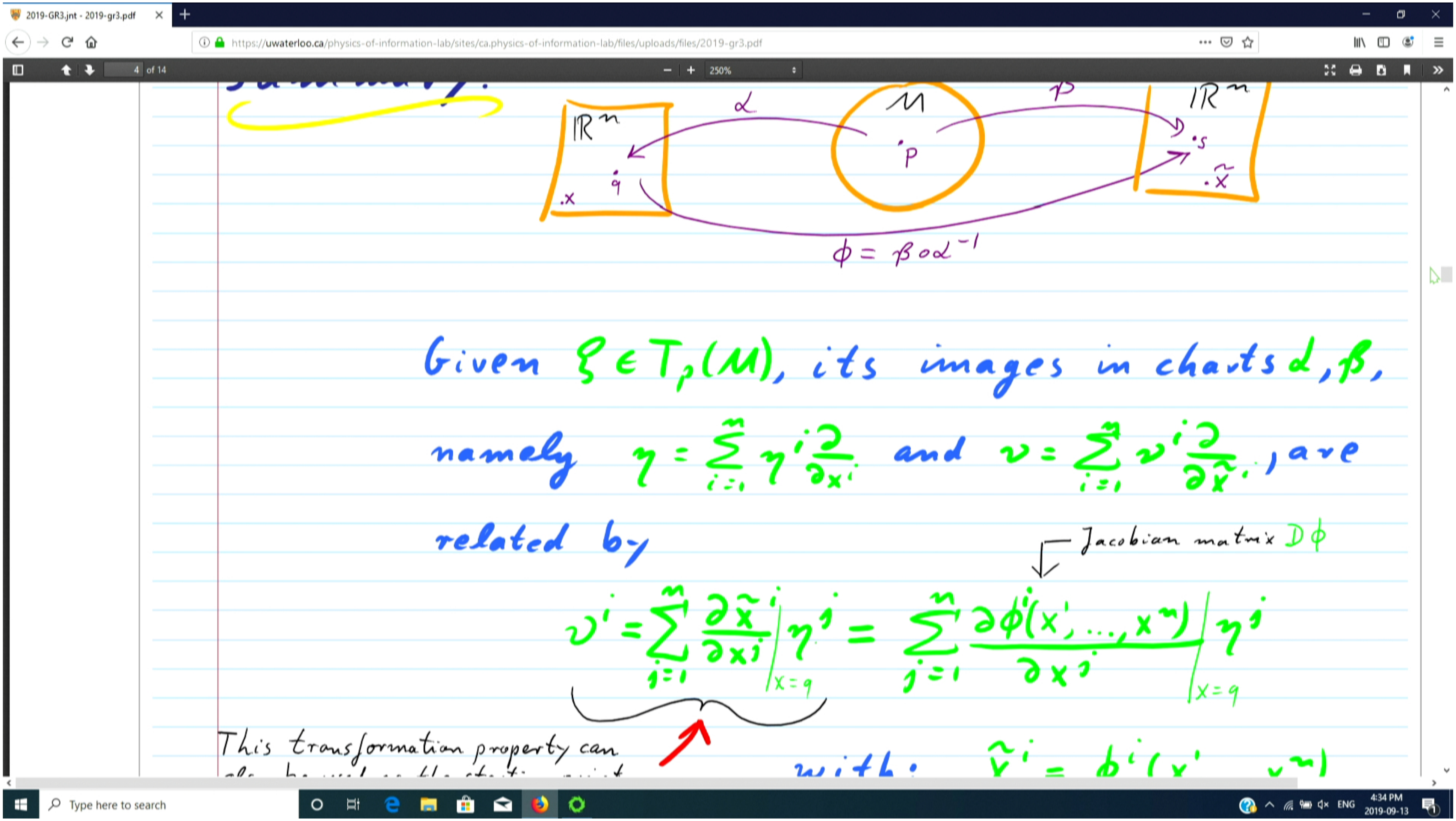


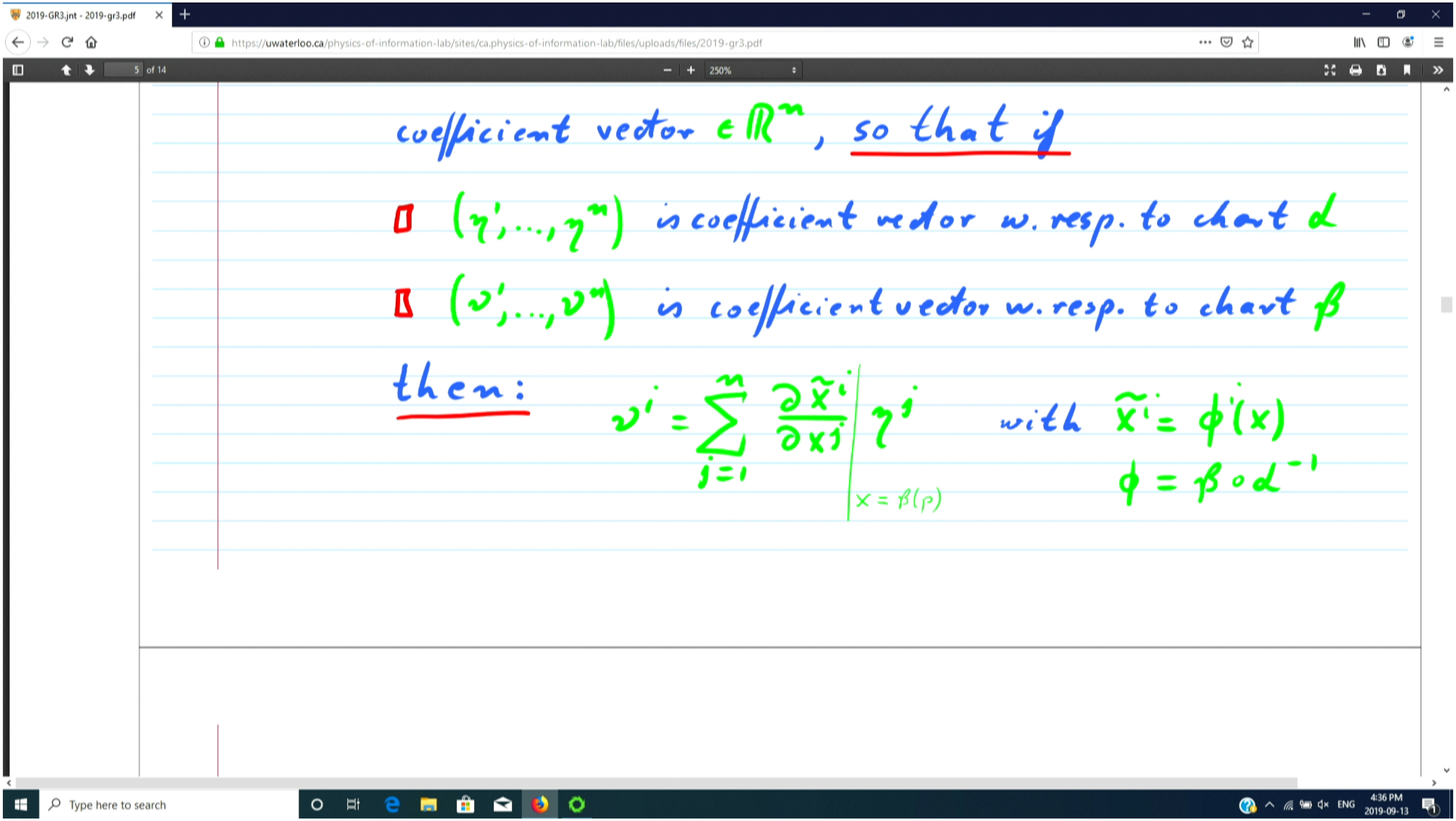
Given $\xi \in T_p(M)$, its images in charts α, β, namely $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}$ and $\nu = \sum_{i=1}^{n} \nu^i \frac{\partial}{\partial \tilde{x}^i}$, are related by

Jacobian matrix $D\phi$

$$\underbrace{\nu^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_{x=q}}\eta^j = \sum_{j=1}^{n} \frac{\partial \phi^i(x^1, \dots, x^n)}{\partial x^j}\Big|_{x=q} \eta^j$$

This transformation property can

with: $\tilde{x}^i = \phi^i(x^1 \quad x^n)$



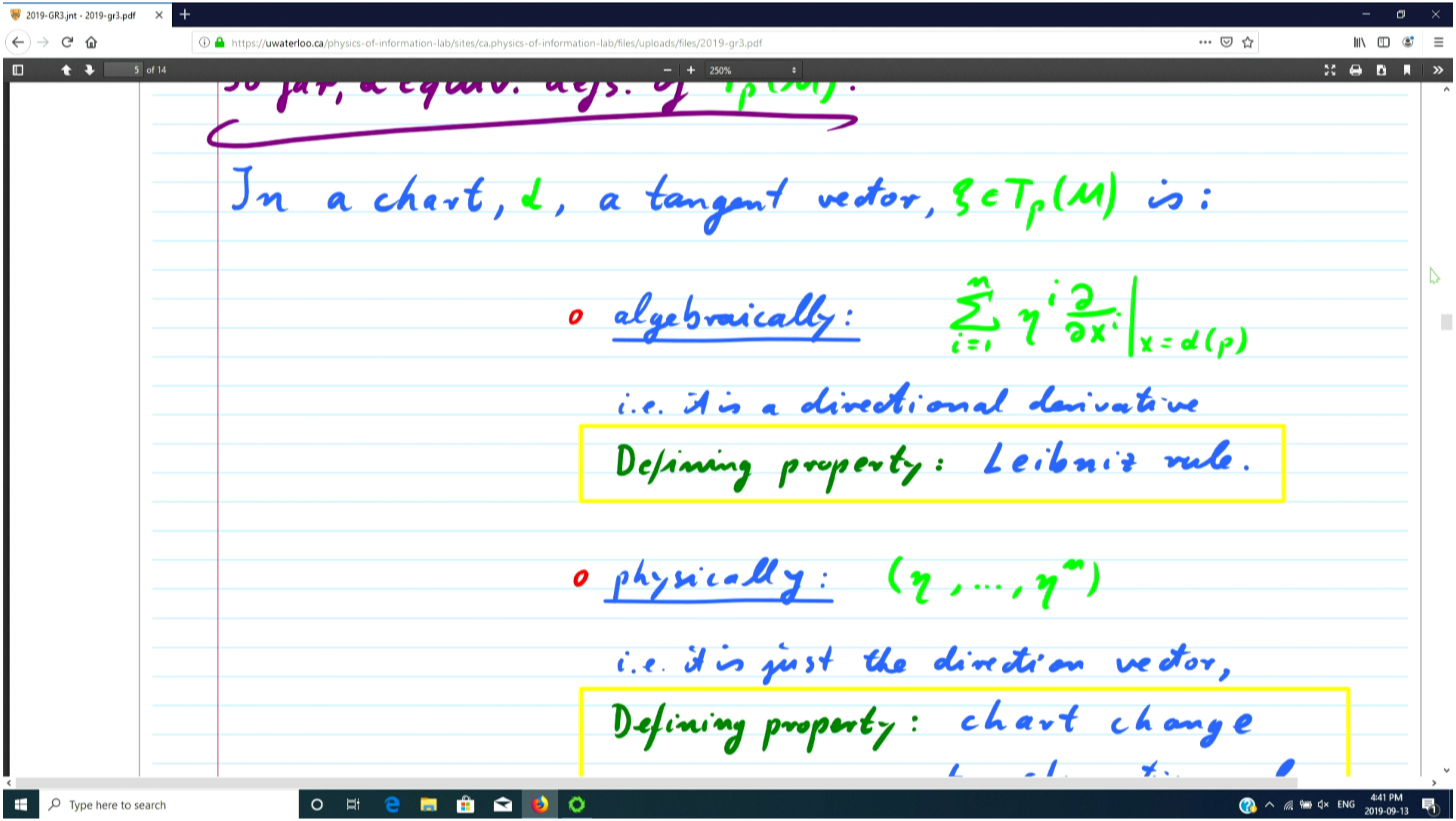
coefficient vector $\in \mathbb{R}^n$, so that if

- $(\eta^1, \dots, \eta^n)$ is coefficient vector w. resp. to chart α
- $(v^1, \dots, v^n)$ is coefficient vector w. resp. to chart β

then:

$$v^i = \sum_{j=1}^{n} \left. \frac{\partial \tilde{x}^i}{\partial x^j} \right|_{x = \beta(p)} \eta^j \qquad \text{with} \quad \tilde{x}^i = \phi^i(x)$$

$$\phi = \beta \circ \alpha^{-1}$$



So far, 2 equiv. defs. of $T_p(M)$:

In a chart, α, a tangent vector, $\xi \in T_p(M)$ is:

- algebraically: $\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=\alpha(p)}$

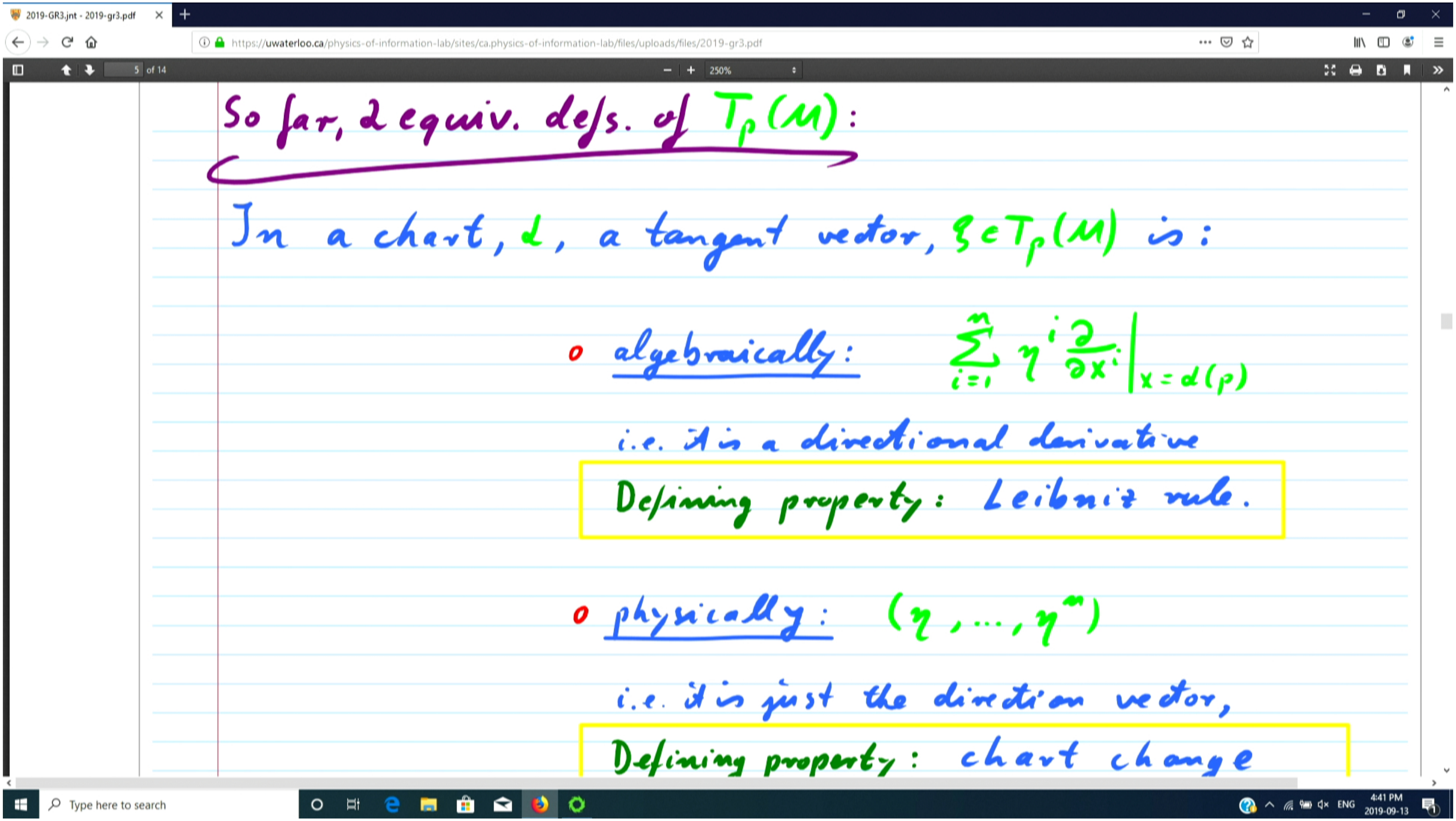
  i.e. it is a directional derivative

  Defining property: Leibniz rule.

- physically: $(\eta, \dots, \eta^n)$

  i.e. it is just the direction vector,

  Defining property: chart change



So far, 2 equiv. defs. of $T_p(M)$:

In a chart, d, a tangent vector, $\xi \in T_p(M)$ is:

- algebraically: $\sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=d(p)}$

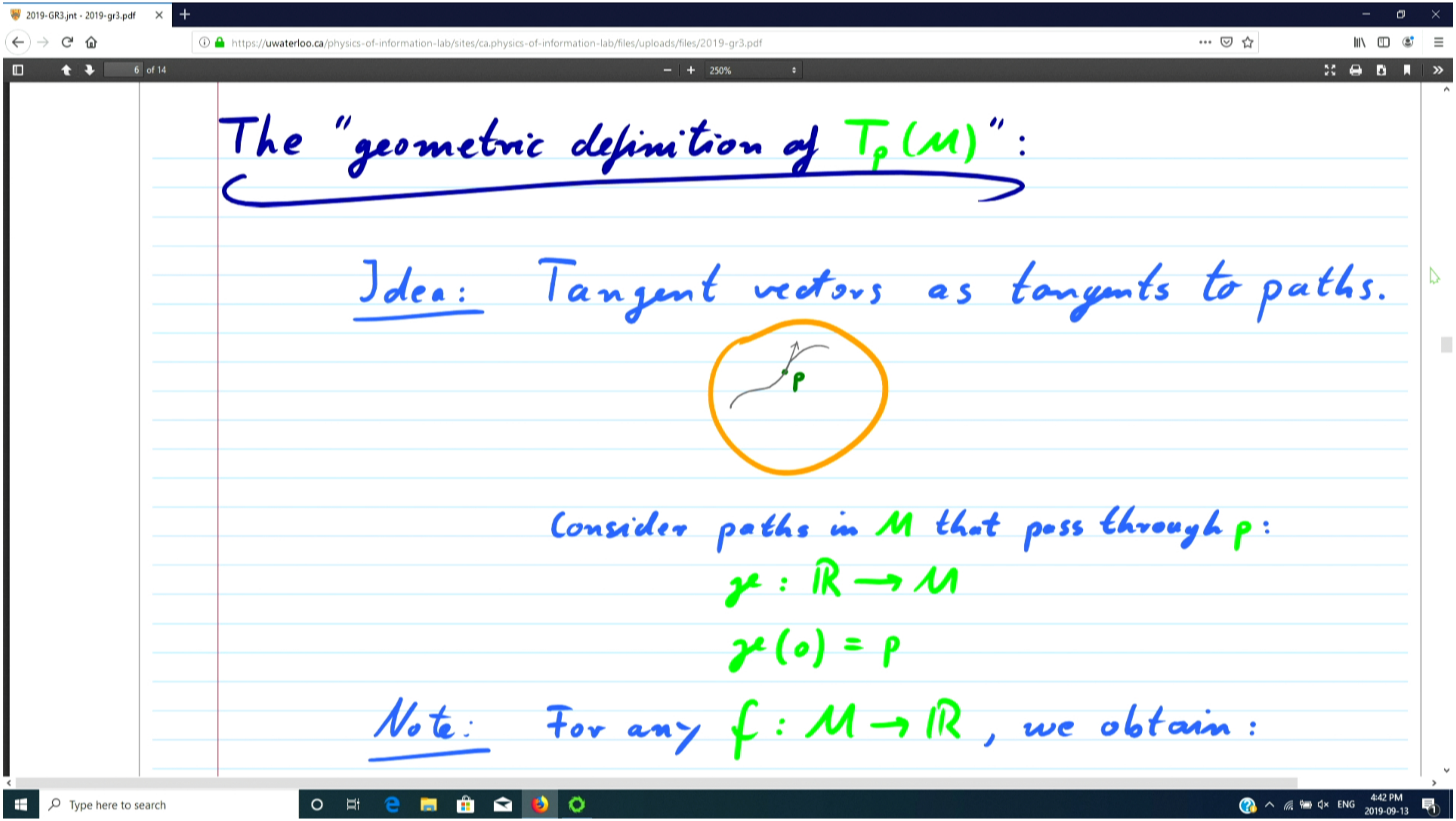
  i.e. it is a directional derivative

  Defining property: Leibniz rule.

- physically: $(\eta, \dots, \eta^n)$

  i.e. it is just the direction vector,

  Defining property: chart change



The "geometric definition of $T_p(M)$":

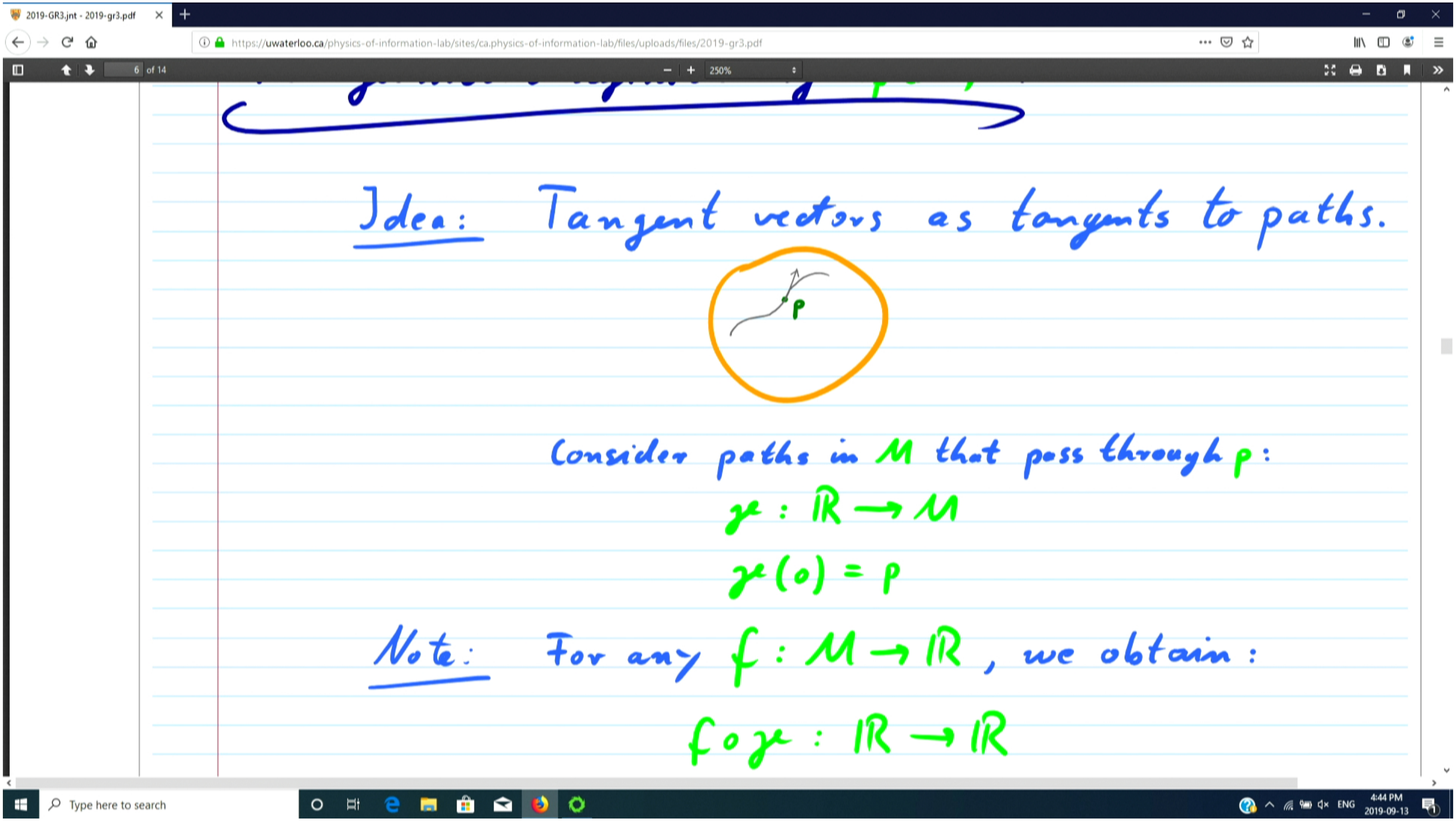
Idea: Tangent vectors as tangents to paths.

Consider paths in M that pass through p:

$$\gamma : \mathbb{R} \rightarrow M$$

$$\gamma(0) = p$$

Note: For any $f : M \rightarrow \mathbb{R}$, we obtain:



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Consider paths in M that pass through p:

$$\gamma : \mathbb{R} \to M$$

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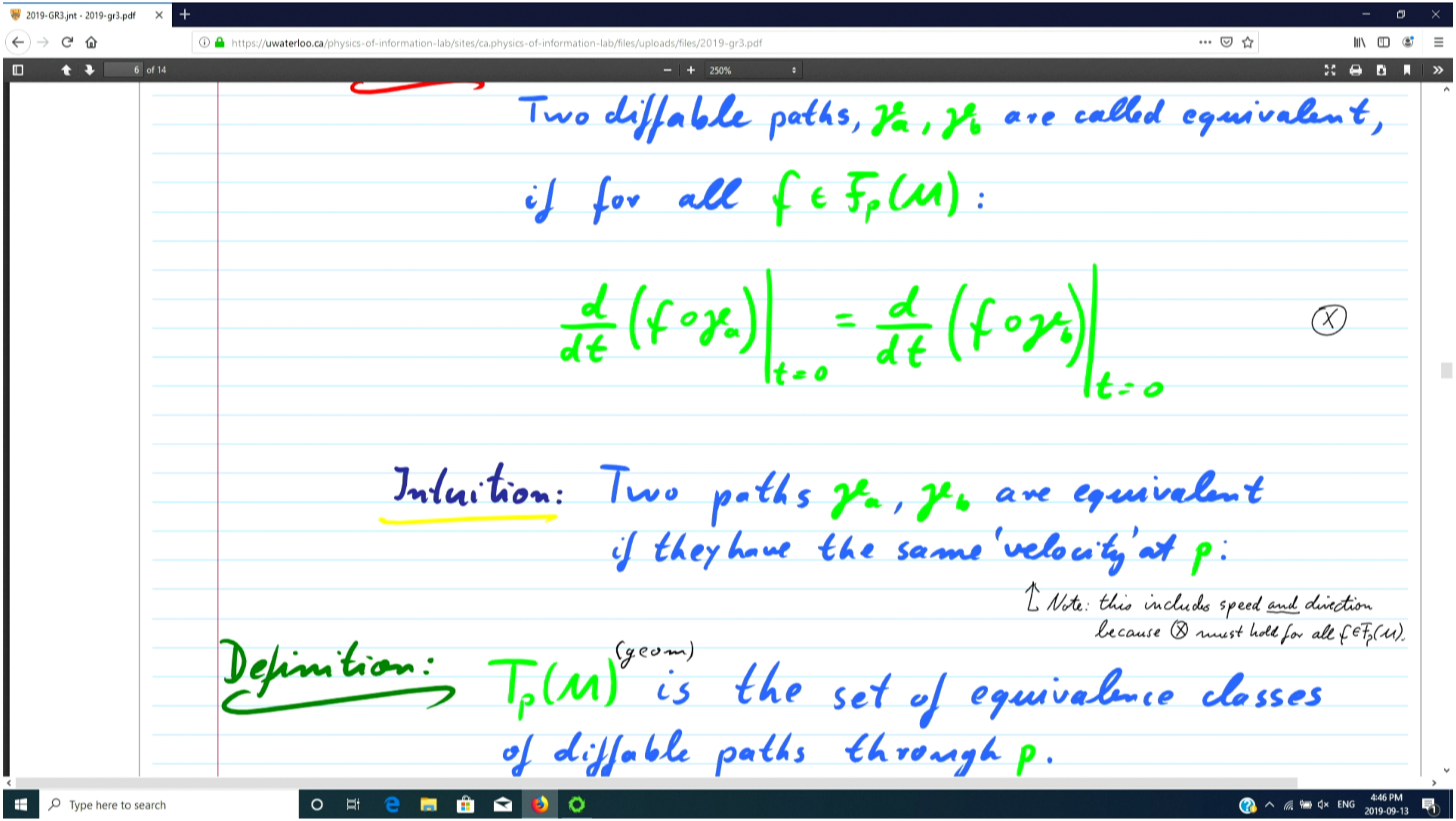
$$f \circ \gamma : \mathbb{R} \to \mathbb{R}$$



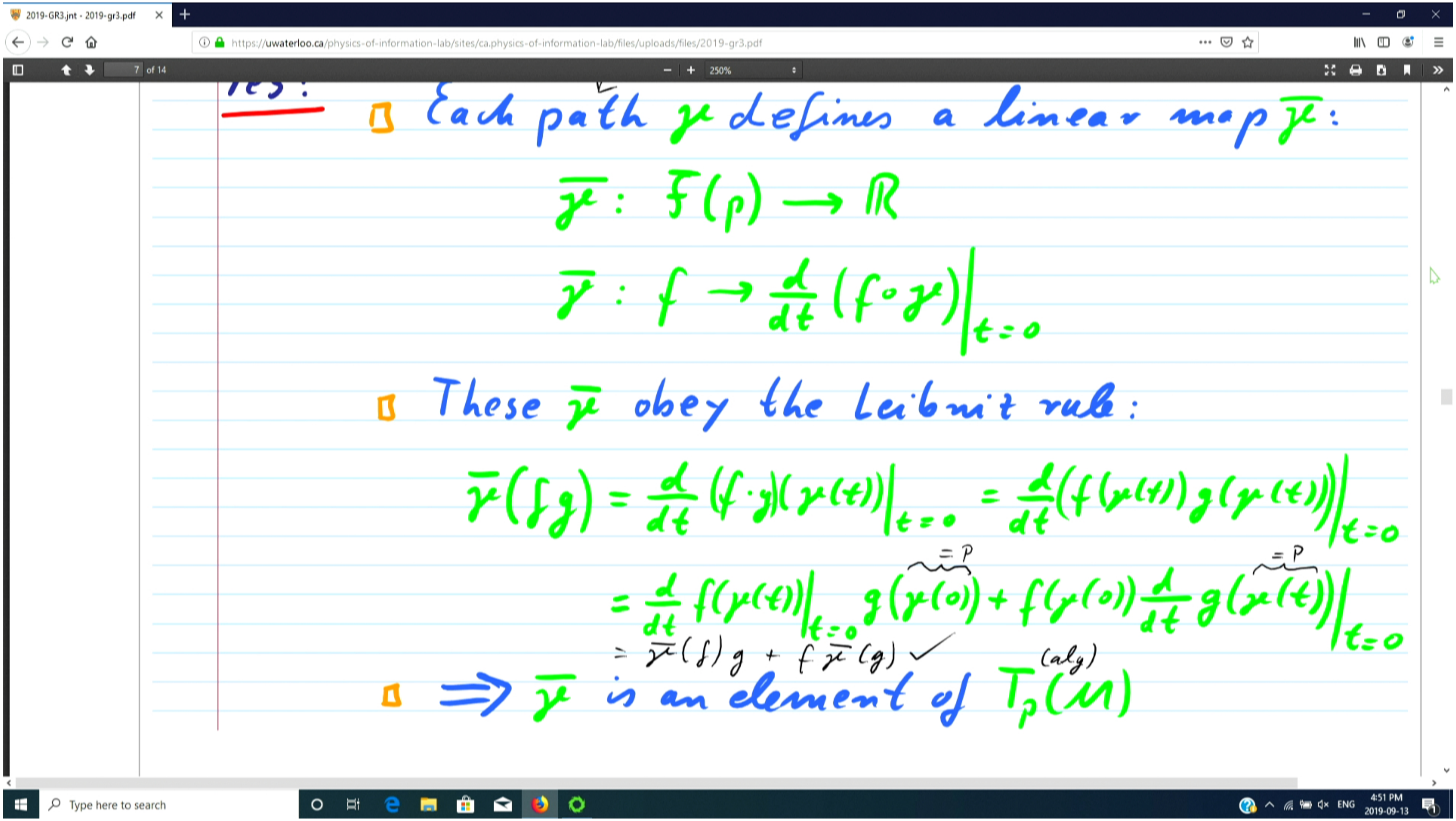
Two diffable paths, γ_a, γ_b are called equivalent, if for all $f \in \mathcal{F}_p(\mathcal{M})$:

$$\frac{d}{dt}\left(f \circ \gamma_a\right)\Big|_{t=0} = \frac{d}{dt}\left(f \circ \gamma_b\right)\Big|_{t=0} \qquad \otimes$$

Intuition: Two paths γ_a, γ_b are equivalent if they have the same 'velocity' at p:

↑ Note: this includes speed and direction because $\otimes$ must hold for all $f \in \mathcal{F}_p(\mathcal{M})$.

Definition: $T_p(\mathcal{M})^{(geom)}$ is the set of equivalence classes of diffable paths through p.



Yes:

- Each path γ defines a linear map $\bar{\gamma}$:

$$\bar{\gamma} : \mathcal{F}(p) \longrightarrow \mathbb{R}$$

$$\bar{\gamma} : f \rightarrow \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

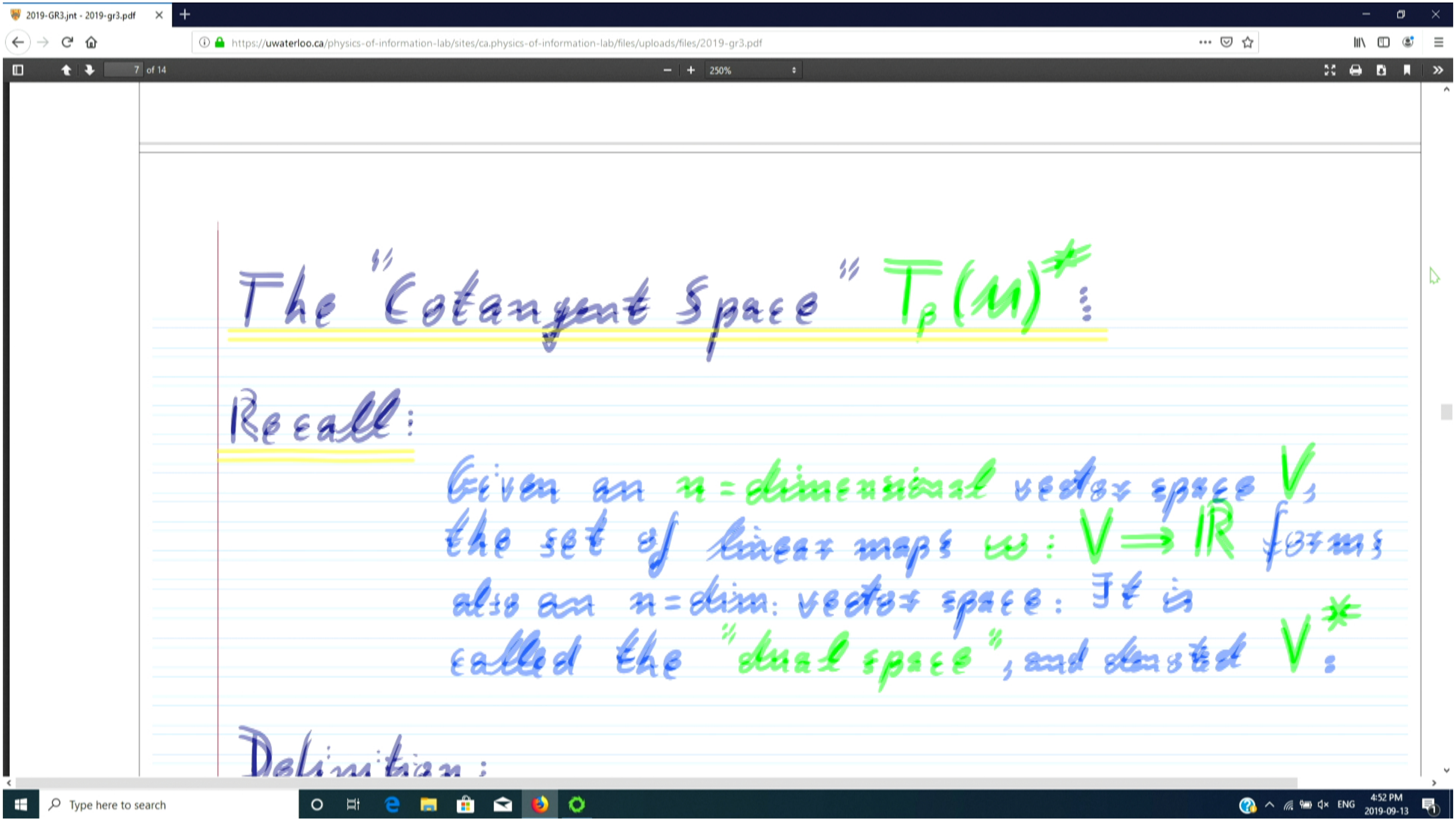
- These $\bar{\gamma}$ obey the Leibniz rule:

$$\bar{\gamma}(fg) = \frac{d}{dt}(f \cdot g)(\gamma(t))\Big|_{t=0} = \frac{d}{dt}\left(f(\gamma(t))\,g(\gamma(t))\right)\Big|_{t=0}$$

$$= \frac{d}{dt} f(\gamma(t))\Big|_{t=0} g(\overbrace{\gamma(0)}^{=p}) + f(\gamma(0))\frac{d}{dt} g(\overbrace{\gamma(t)}^{=p})\Big|_{t=0}$$

$$= \bar{\gamma}(f)\,g + f\,\bar{\gamma}(g) \checkmark$$

- $\Longrightarrow$ $\bar{\gamma}$ is an element of $T_p^{(alg)}(M)$

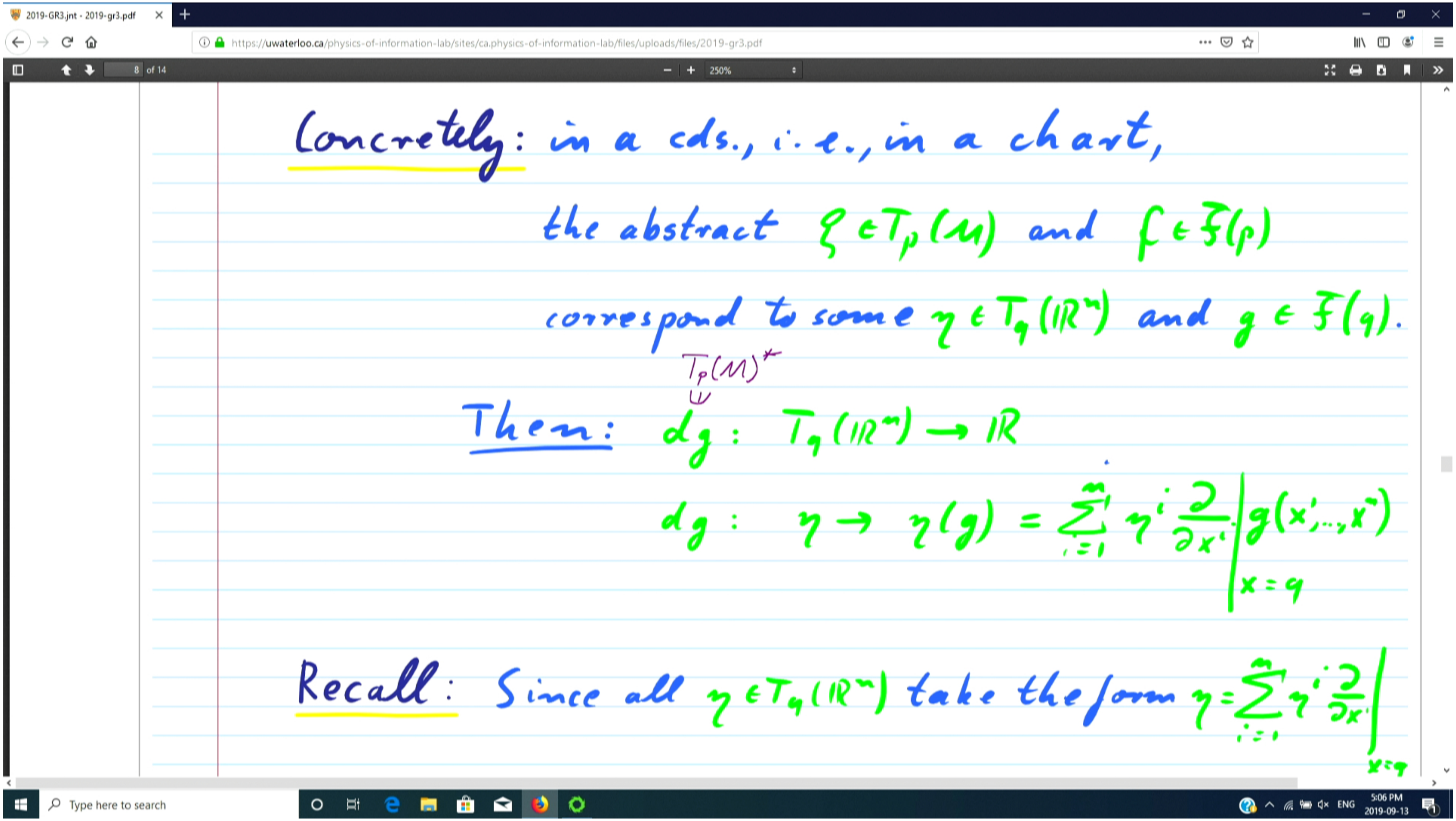


The "Cotangent Space" $T_p(M)^*$:

Recall:

Given an n-dimensional vector space V, the set of linear maps $\omega : V \Rightarrow \mathbb{R}$ forms also an n-dim. vector space: It is called the "dual space", and denoted V^*.

Definition:



Concretely: in a cds., i.e., in a chart,

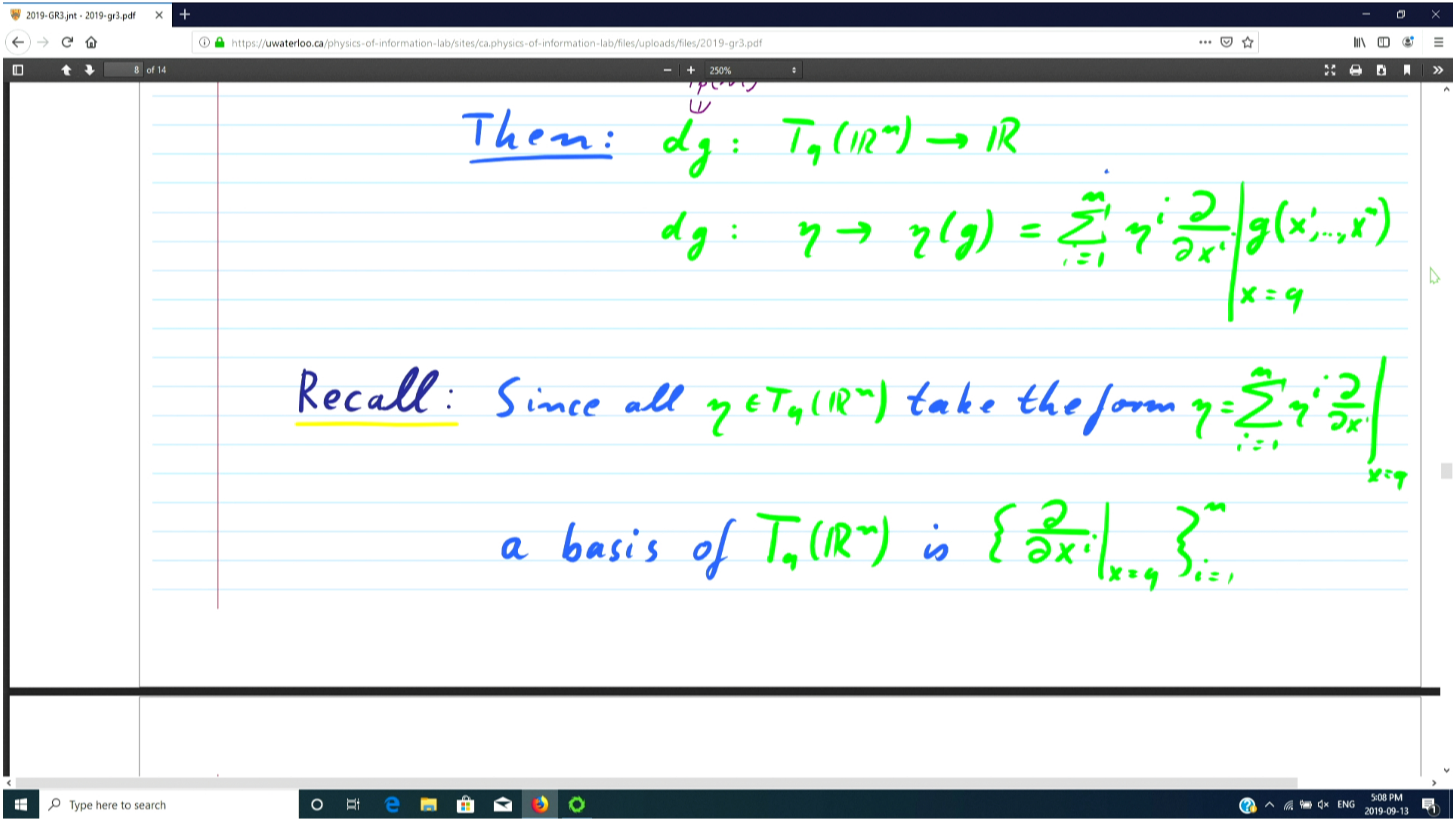
the abstract $\xi \in T_p(M)$ and $f \in \mathcal{F}(p)$

correspond to some $\eta \in T_q(\mathbb{R}^n)$ and $g \in \mathcal{F}(q)$.

Then: $\overset{\overset{T_p(M)^*}{\cup}}{dg} : T_q(\mathbb{R}^n) \to \mathbb{R}$

$$dg : \eta \to \eta(g) = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q} g(x^1, \dots, x^n)$$

Recall: Since all $\eta \in T_q(\mathbb{R}^n)$ take the form $\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q}$

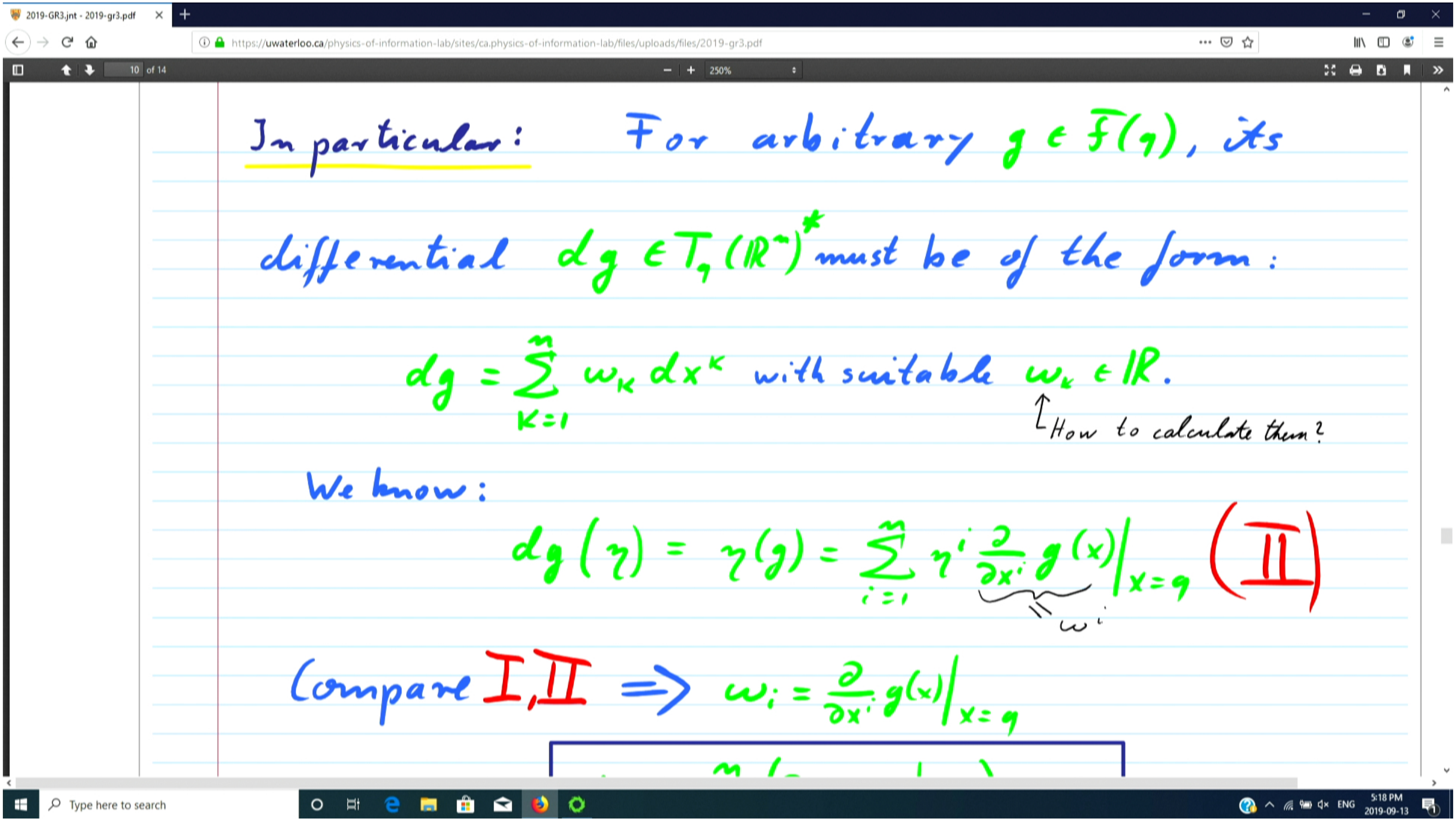


Then: $dg : T_q(\mathbb{R}^m) \to \mathbb{R}$

$$dg : \eta \to \eta(g) = \sum_{i=1}^{m} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q} g(x^1,\dots,x^m)$$

Recall: Since all $\eta \in T_q(\mathbb{R}^m)$ take the form $\eta = \sum_{i=1}^{m} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q}$

a basis of $T_q(\mathbb{R}^m)$ is $\left\{ \frac{\partial}{\partial x^i}\Big|_{x=q} \right\}_{i=1}^{m}$



In particular: For arbitrary $g \in \mathcal{F}(q)$, its differential $dg \in T_q(\mathbb{R}^n)^*$ must be of the form:

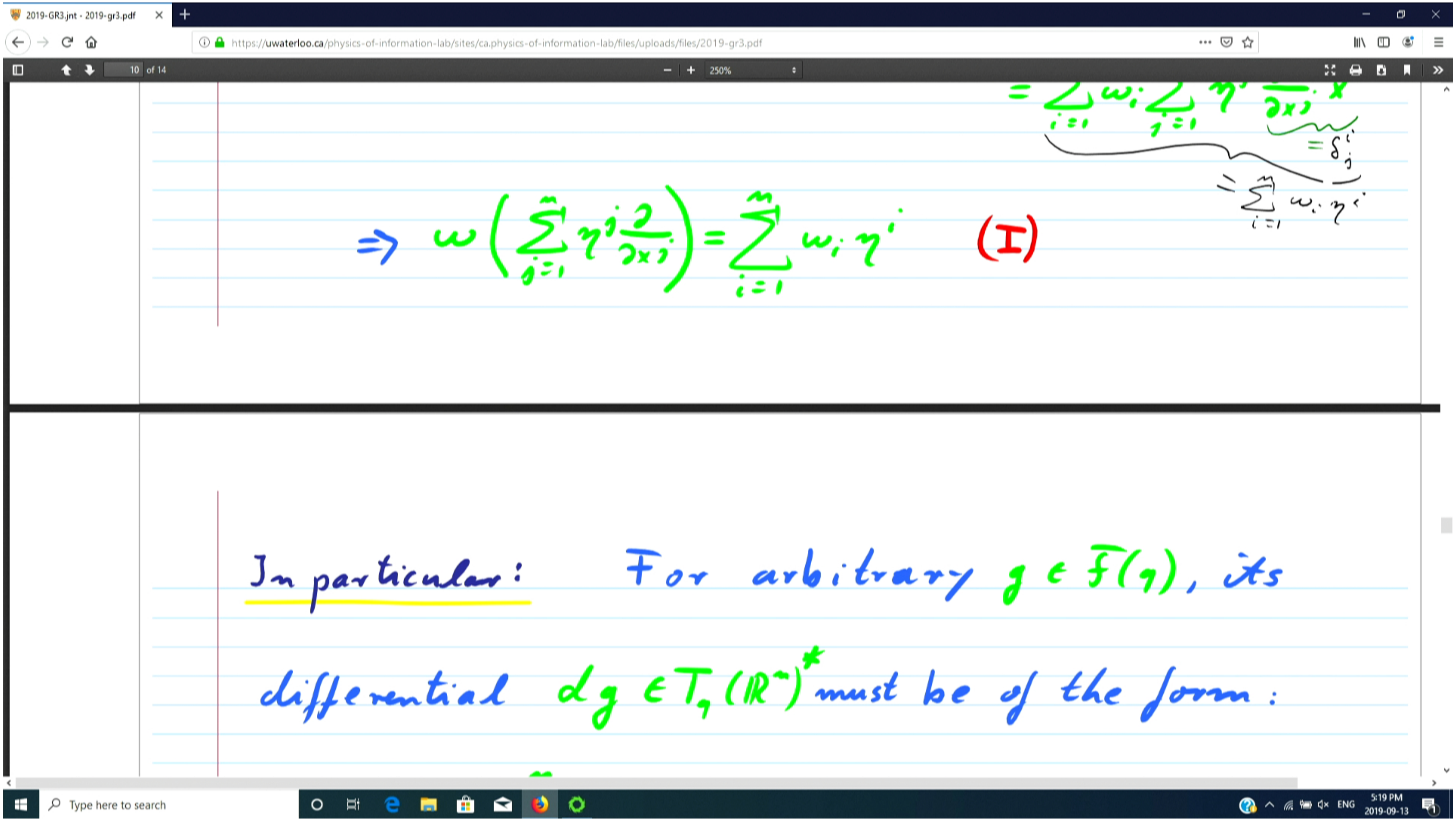
$$dg = \sum_{k=1}^{n} w_k \, dx^k$$ with suitable $w_k \in \mathbb{R}$.

↑ How to calculate them?

We know:

$$dg(\eta) = \eta(g) = \sum_{i=1}^{n} \eta^i \underbrace{\frac{\partial}{\partial x^i} g(x)}_{= \omega^i}\Big|_{x=q} \qquad (\mathrm{II})$$

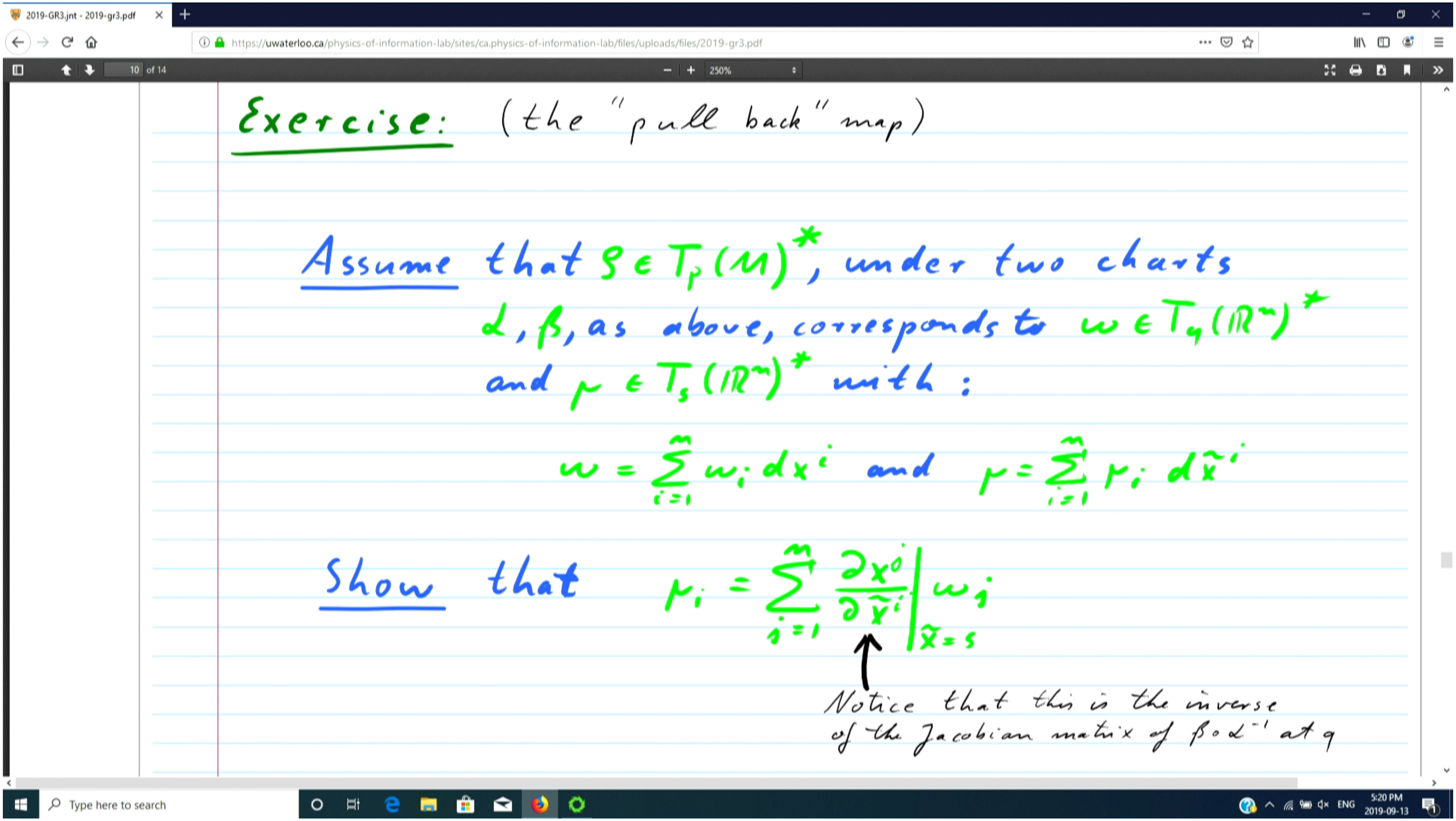
Compare I, II $\Rightarrow$ $w_i = \frac{\partial}{\partial x^i} g(x)\Big|_{x=q}$



$$= \underbrace{\sum_{i=1} \omega_i \sum_{j=1} \eta^j \underbrace{\frac{\quad}{\partial x^j} x^i}_{=\delta^i_j}}_{= \sum_{i=1}^{m} \omega_i \eta^i}$$

$$\Rightarrow \omega\left(\sum_{j=1}^{m} \eta^j \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{m} \omega_i \eta^i \qquad (I)$$

In particular: For arbitrary $g \in \mathcal{F}(q)$, its differential $dg \in T_q(\mathbb{R}^m)^*$ must be of the form:



Exercise: (the "pull back" map)

Assume that $g \in T_p(M)^*$, under two charts α, β, as above, corresponds to $w \in T_q(\mathbb{R}^m)^*$ and $\mu \in T_s(\mathbb{R}^m)^*$ with:

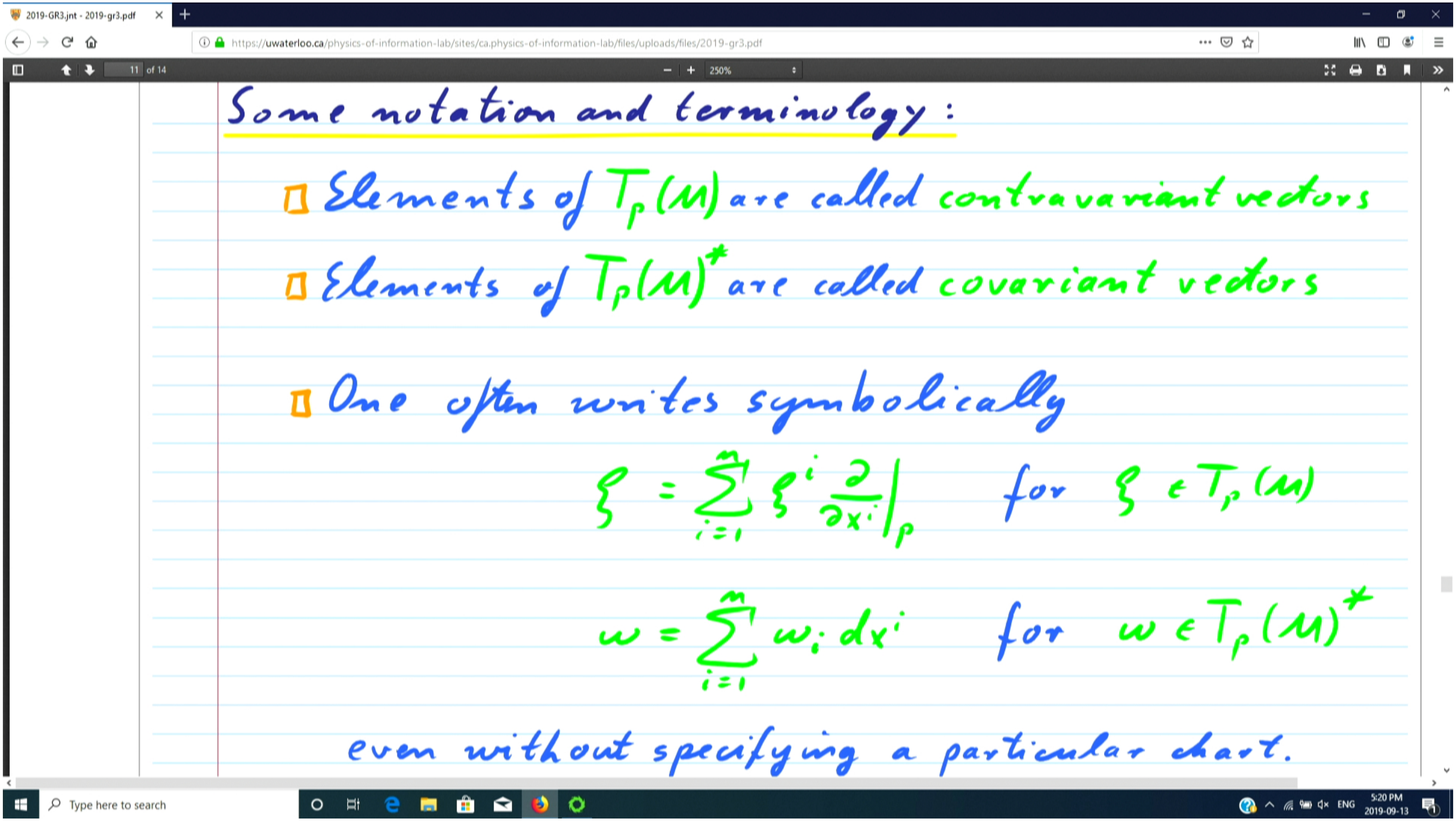
$$w = \sum_{i=1}^{m} w_i \, dx^i \quad \text{and} \quad \mu = \sum_{i=1}^{m} \mu_i \, d\tilde{x}^i$$

Show that

$$\mu_i = \sum_{j=1}^{m} \frac{\partial x^j}{\partial \tilde{x}^i}\Bigg|_{\tilde{x}=s} w_j$$

$\uparrow$

Notice that this is the inverse of the Jacobian matrix of $\beta \circ \alpha^{-1}$ at q



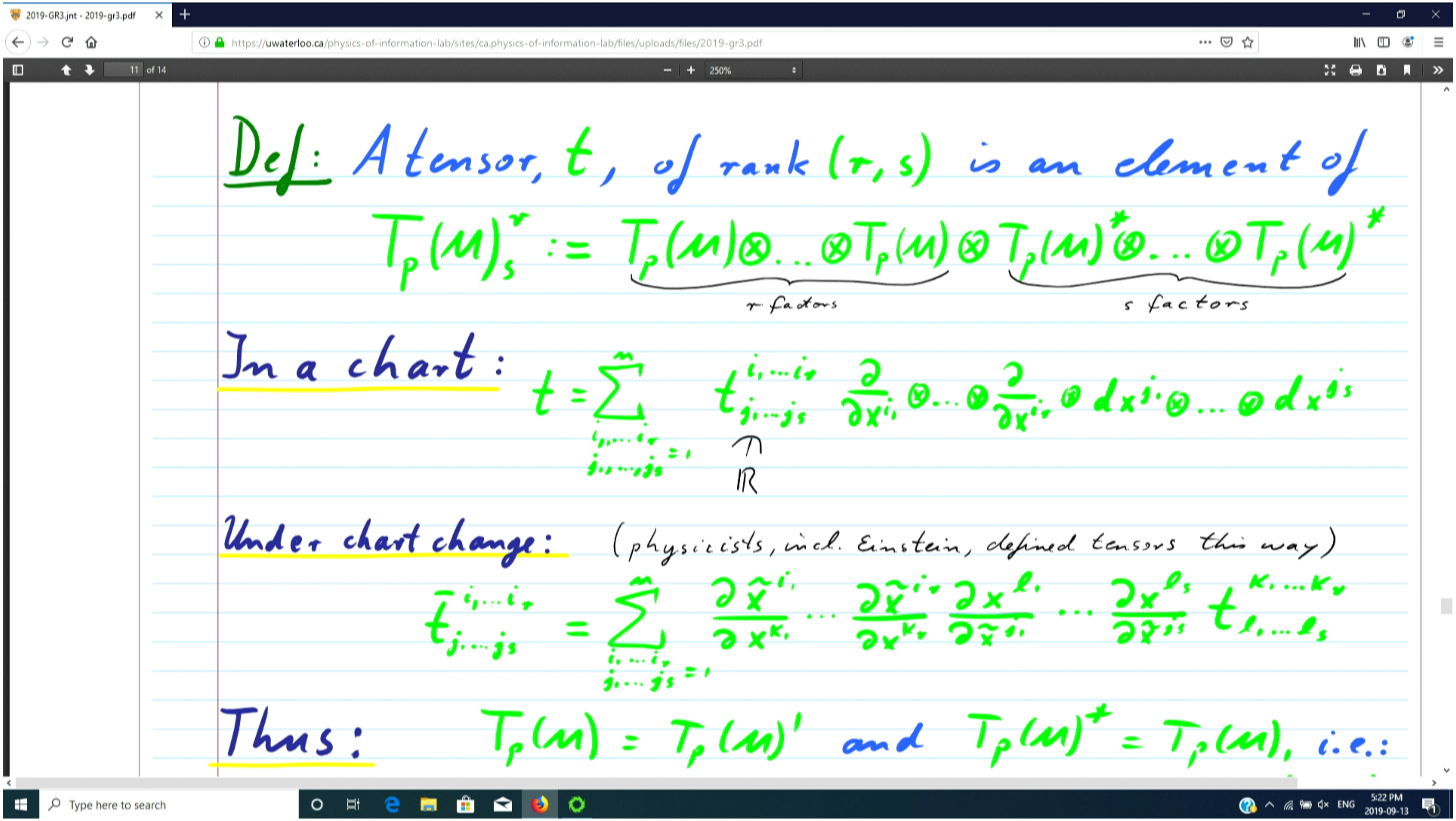
Some notation and terminology:

- Elements of $T_p(M)$ are called contravariant vectors
- Elements of $T_p(M)^*$ are called covariant vectors
- One often writes symbolically

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\Big|_p \quad \text{for} \quad \xi \in T_p(M)$$

$$\omega = \sum_{i=1}^{n} \omega_i \, dx^i \quad \text{for} \quad \omega \in T_p(M)^*$$

even without specifying a particular chart.



Def: A tensor, t, of rank (r,s) is an element of

$$T_p(M)^r_s := \underbrace{T_p(M)\otimes\ldots\otimes T_p(M)}_{r \text{ factors}}\otimes\underbrace{T_p(M)^*\otimes\ldots\otimes T_p(M)^*}_{s \text{ factors}}$$

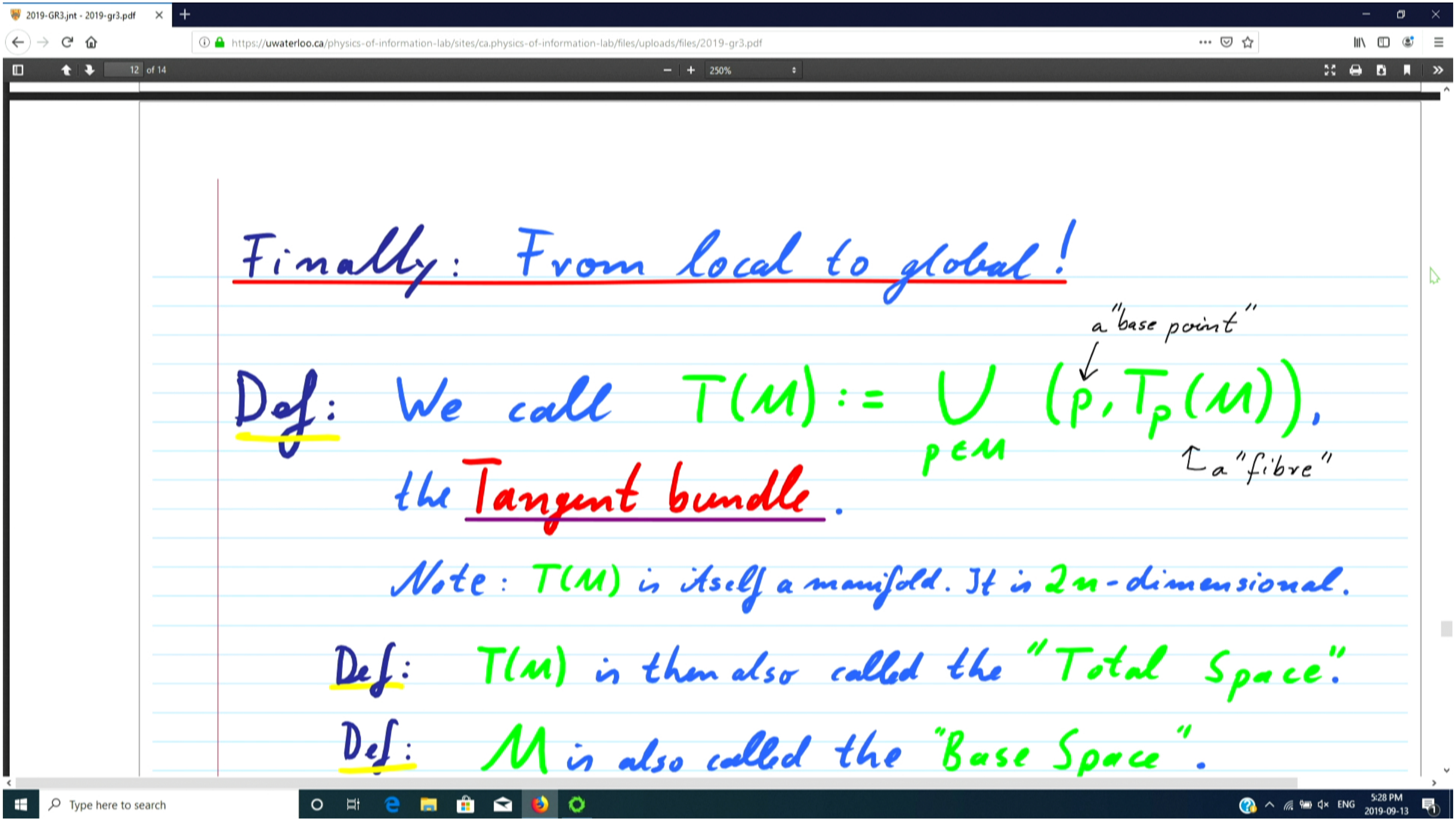
In a chart:

$$t = \sum_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_s=1}}^{m} \underset{\mathbb{R}}{\underset{\cap}{t^{i_1\ldots i_r}_{j_1\ldots j_s}}} \frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_s}$$

Under chart change: (physicists, incl. Einstein, defined tensors this way)

$$\tilde{t}^{i_1\ldots i_r}_{j_1\ldots j_s} = \sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_s=1}}^{m} \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}}\cdots\frac{\partial \tilde{x}^{i_r}}{\partial x^{k_r}}\frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}}\cdots\frac{\partial x^{l_s}}{\partial \tilde{x}^{j_s}}\, t^{k_1\ldots k_r}_{l_1\ldots l_s}$$

Thus: $T_p(M) = T_p(M)'$ and $T_p(M)^* = T_p(M)$, i.e.:



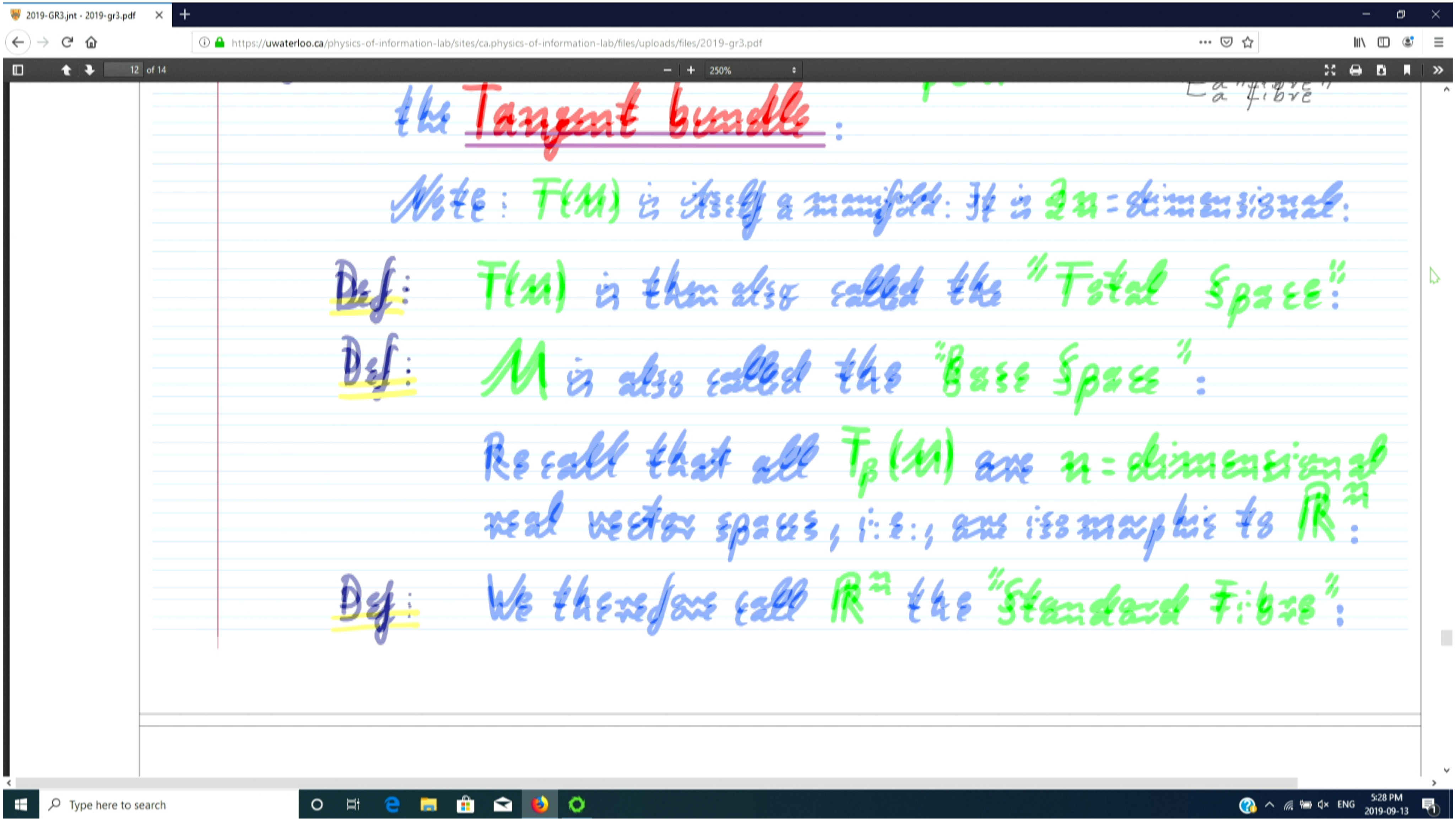
Finally: From local to global!

Def: We call $T(M) := \bigcup_{p \in M} \left(\overset{\text{a "base point"}}{\downarrow \atop p}, \underset{\uparrow \text{ a "fibre"}}{T_p(M)}\right)$, the **Tangent bundle**.

Note: $T(M)$ is itself a manifold. It is $2n$-dimensional.

Def: $T(M)$ is then also called the "Total Space".

Def: M is also called the "Base Space".



[a "fibre"

the Tangent bundle:

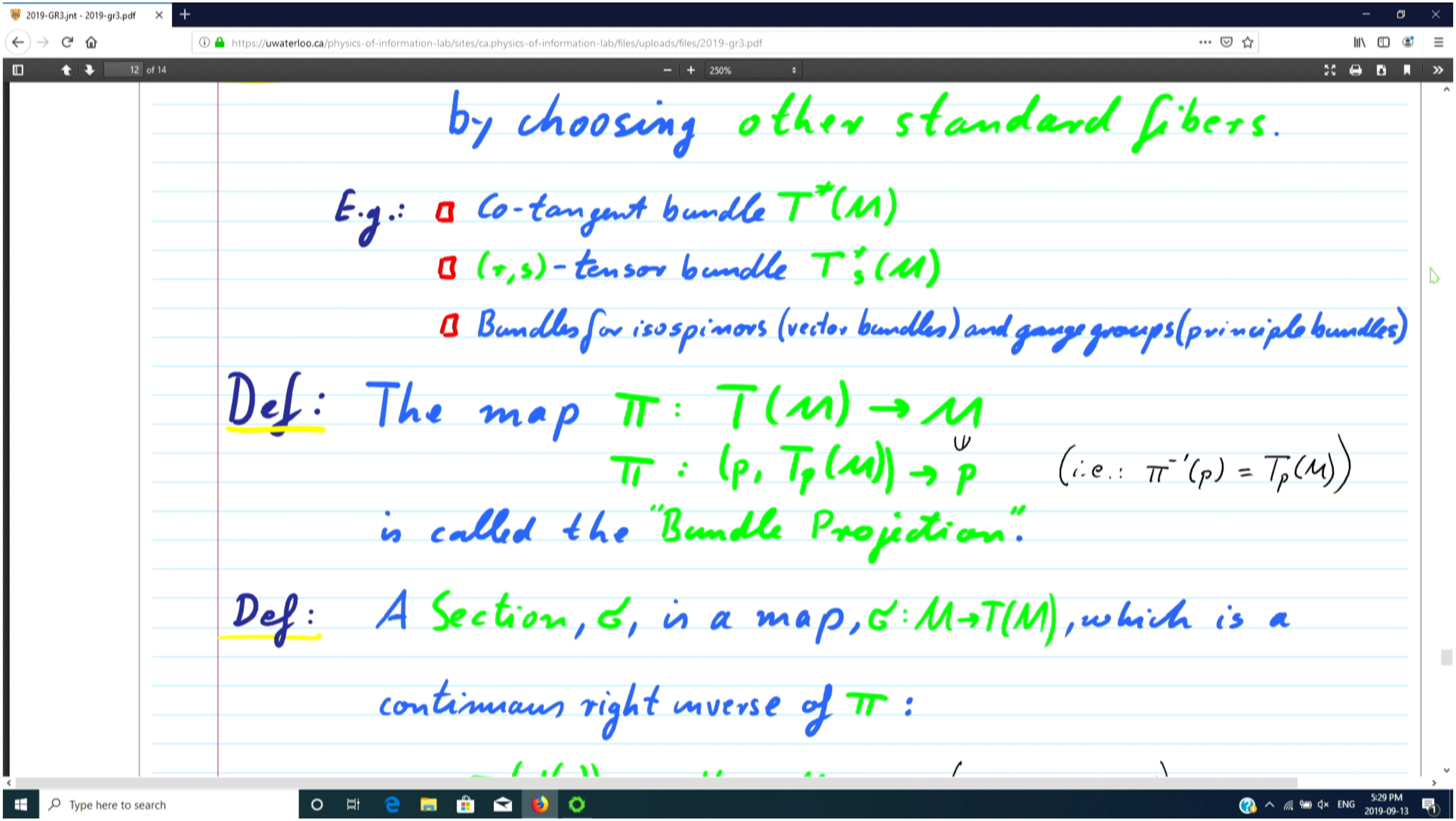
Note: T(M) is itself a manifold: It is 2n-dimensional.

Def: T(M) is then also called the "Total Space":

Def: M is also called the "Base Space":

Recall that all $T_p(M)$ are n-dimensional real vector spaces, i.e., are isomorphic to $\mathbb{R}^n$:

Def: We therefore call $\mathbb{R}^n$ the "Standard Fibre":



by choosing other standard fibers.

E.g.: ☐ Co-tangent bundle $T^*(M)$

☐ (r,s)-tensor bundle $T^r_s(M)$

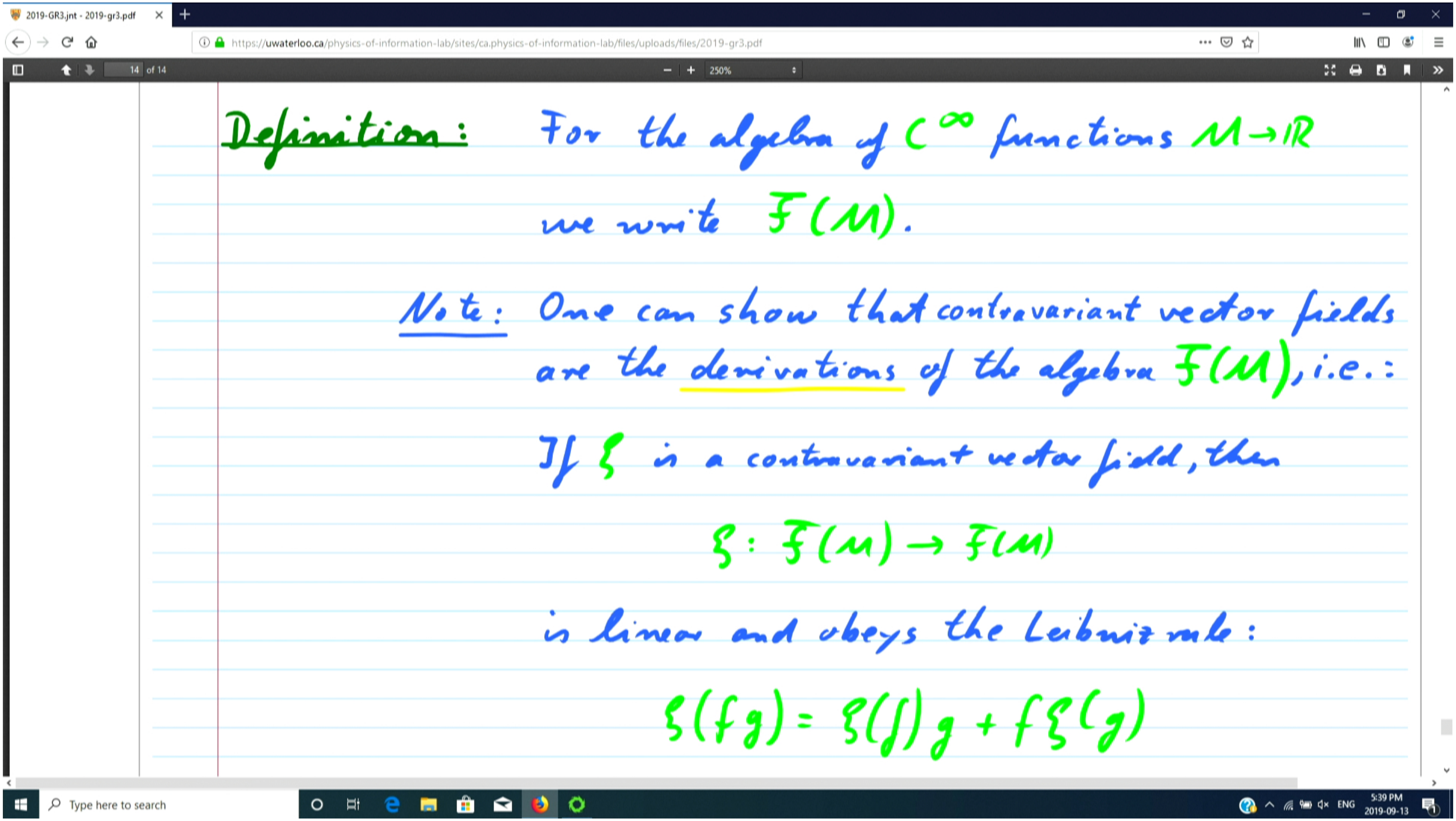
☐ Bundles for isospinors (vector bundles) and gauge groups (principle bundles)

Def: The map $\pi : T(M) \to M$

$$\pi : (p, T_p(M)) \to \overset{\in}{p} \qquad (\text{i.e.: } \pi^{-1}(p) = T_p(M))$$

is called the "Bundle Projection".

Def: A Section, σ, is a map, $\sigma : M \to T(M)$, which is a continuous right inverse of π:



Definition: For the algebra of C^∞ functions $M \to \mathbb{R}$ we write $\mathcal{F}(M)$.

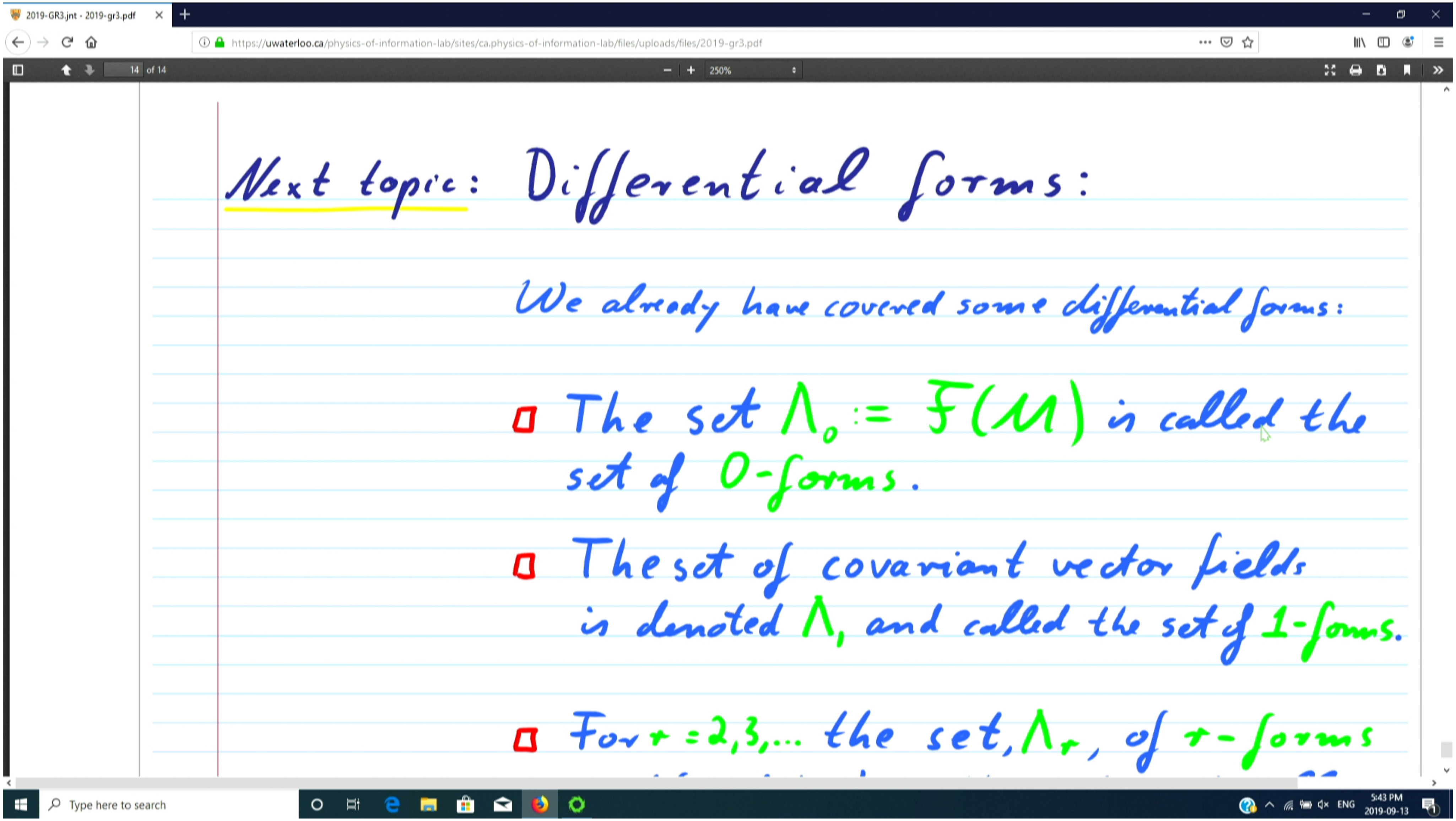
Note: One can show that contravariant vector fields are the derivations of the algebra $\mathcal{F}(M)$, i.e.:

If ξ is a contravariant vector field, then

$$\xi : \mathcal{F}(M) \to \mathcal{F}(M)$$

is linear and obeys the Leibniz rule:

$$\xi(fg) = \xi(f)g + f\xi(g)$$



Next topic: Differential forms:

We already have covered some differential forms:

- The set $\Lambda_0 := \mathcal{F}(\mathcal{M})$ is called the set of 0-forms.
- The set of covariant vector fields is denoted Λ_1 and called the set of 1-forms.
- For $r = 2, 3, \dots$ the set, Λ_r, of r-forms