Title: Coherence in logical quantum channels

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Abstract: In quantum error correcting codes, there is a distinction between coherent and incoherent noise. Coherent noise can cause the average infidelity to accumulate quadratically when a fixed channel is applied many times in succession, rather than linearly as in the case of incoherent noise. I will present a proof that unitary single qubit noise in the 2D toric code with minimum weight decoding is mapped to less coherent logical noise, and as the code size grows, the coherence of the logical noise channel is suppressed. In the process, I will describe how to characterize the coherence of noise using either the growth of infidelity or the relation between the diamond distance from identity and the average infidelity. I will explain how coherence in the noise on physical qubits is transformed by error correction in stabilizer codes. Then, I will sketch the proof that coherence is suppressed for the 2D toric code. The result holds even when the single qubit unitary rotation are allowed to have arbitrary directions and angles, so long as the angles are below a threshold, and even when the rotations are correlated. Joint work with John Preskill.

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## Growth of Infidelity



- The average infidelity: $\quad r(N)=1-\int_{\text {pure } \rho} \operatorname{Tr}(\rho N(\rho)) d \rho$
- After $m$ applications of a given noise channel, the average infidelity is given by

$$
\begin{array}{rll}
\text { Depolarizing: } & r\left(D^{m}\right) & =m r+\text { higher order } \\
\text { Unitary: } & r\left(\text { Unit }^{m}\right) & =m^{2} r+\text { higher order }
\end{array}
$$

## Diamond Distance from Identity

- The diamond distance from identity is defined as a max over pure states in a doubled space:

$$
|N-i d|_{\diamond}=\max _{\rho}|((N-i d) \otimes i d)(\rho)|_{1}
$$

- The diamond distance from identity is related to the average infidelity differently for coherent and incoherent channels

Depolarizing: $\quad\left\|D_{\lambda}-i d\right\|_{\diamond} \propto r$
Unitary: $\quad\left\|U n i t_{\theta}^{X}-i d\right\|_{\diamond} \propto \sqrt{r}$

## The Toric Code

- Qubits on edges of 2D square lattice with periodic boundary conditions
- Stabilizer generators are stars and plaquettes
- Logical operators are strings



## Why the Toric Code?

- Topological codes have nice properties: efficient and good decoding, high threshold, ready-made infinite families of codes that are easy to analyze
- 2 D is the simplest case and is most suitable to physical realization
- We expect that all stabilizer codes will respond in a similar way to coherent noise, but the proof is difficult so we chose the simplest example that we expect to be representative.


## Coherence in Channel Representations

- Pauli transfer matrix/Liouville representation

$$
\begin{gathered}
N(\rho)=N\left(\sum_{j} \rho_{j} \sigma^{j}\right)=\sum_{i, j} N_{i, j} \rho_{j} \sigma^{i} \\
\left\{\sigma^{i}\right\} \text { is a basis of } n \text { qubit Pauli operators }
\end{gathered}
$$

- $\chi$ matrix/process matrix representation:

$$
\begin{gathered}
N(\rho)=\sum_{i, j} \chi_{i, j} \sigma^{i} \rho \sigma^{j} \\
\left(\sigma^{i} \rho \sigma^{j}\right):=\chi_{i, j}
\end{gathered}
$$

- Incoherent components are diagonal in both representations


## Error Correction

- We will analyze one round of error correction
- We average over syndrome measurements to produce the error correction channel
- We assume perfect syndrome extraction. The errors are all bundled up into the noise channel $N$
- The logical noise channel is given by

$$
\tilde{N}=\text { Decode } \circ N \circ \text { Encode }
$$

## Logical Noise Channels

$$
\tilde{N}=\text { Decode } \circ N \circ \text { Encode }
$$

- Each component of the logical noise channel is a sum of terms from the physical noise channel
- In the $\chi$ matrix representation we can write:
$\begin{aligned} & \begin{array}{l}\text { Logical } \chi \text { matrix } \\ \text { component }\end{array}\end{aligned}\left(\tilde{L}_{a} \tilde{\rho} \tilde{L}_{b}\right)=\sum_{s, i, j}\left(E_{s} L_{a} S_{i} \rho S_{j} L_{b} E_{s}\right) \quad \begin{aligned} & \text { Physical } \chi \text { matrix } \\ & \text { components }\end{aligned}$
$\begin{array}{clrl}\tilde{L}_{a} & \text { Logical } a \text { operator on encoded qubits } & E_{s} & \text { Standard error for syndrome } s \\ \tilde{\rho} & \text { State of encoded qubits } & L_{a} & \text { Logical } a \text { on physical qubits } \\ \rho & \text { State of physical qubits } & S_{i} & \text { Stabilizer operator } i\end{array}$


## Repetition Code Calculation 1

- Consider an $n$ qubit bit flip code where $n$ is odd

- Let our noise model consist of single qubit rotations about the X axis

$$
U=\cos \theta I+i \sin \theta X \quad N(\rho)=U^{\otimes n} \rho U^{\dagger \otimes n}
$$

## Repetition Code Calculation 2

- Compute the coherent logical channel component $\tilde{\chi}_{1,0}$

$$
\begin{gathered}
\tilde{\chi}_{1,0}=\sum_{s}\left(E_{s} L_{x} \rho E_{s}\right) \\
s \text { is a syndrome } \\
L_{x} \text { is the logical X operator }
\end{gathered}
$$

- Each term in the sum corresponds to a partitioning of the logical operator into two:

$$
\begin{aligned}
& \text { ( ® ® ○ ® ○) } \rho(\bigcirc \bigcirc \otimes \bigcirc \text { ® })
\end{aligned}
$$

## Repetition Code Calculation 3

- Each syndrome and correction is a set of fewer than half of the $n$ qubits. Together with the phases that come from the factors of $i \sin \theta$ in the unitary, we have

$$
\begin{aligned}
\left(\tilde{L}_{a} \tilde{\rho}\right) & =\sum_{j=0}^{(n-1) / 2}\binom{n}{j}(-1)^{j}(i \sin \theta \cos \theta)^{n} \\
& =\binom{n-1}{\frac{n-1}{2}} i(-1)^{n+1}(\sin \theta \cos \theta)^{n}
\end{aligned}
$$

- Notice that the sum is alternating
- Cancellations are crucial to the suppression of coherence


## Repetition Code Calculation 4

- Now let us compute the incoherent logical channel component

$$
\tilde{\chi}_{1,1}=\sum_{s}\left(E_{s} L_{x} \rho L_{x} E_{s}\right)
$$

- The same logical operator appears on both sides of $\rho$.

$$
\begin{aligned}
\tilde{\chi}_{1,1} & =\sum_{j=0}^{(n-1) / 2}\binom{n}{j}(\sin \theta)^{2 n-2 j}(\cos \theta)^{2 j} \\
& =\binom{n}{\frac{n-1}{2}}(\sin \theta)^{n+1}(\cos \theta)^{n-1}+\ldots
\end{aligned}
$$

## Repetition Code Calculation 5

- We have computed exactly the coherent and incoherent components of the logical noise channel. Now compare them:

$$
\left(\tilde{L}_{1} \tilde{\rho}\right)=\frac{i(-1)^{n+1} \cos \theta}{2 \sin \theta}\left(\tilde{L}_{1} \tilde{\rho} \tilde{L}_{1}\right)
$$

- As a function of the code size $n$, the two components are related by a constant. This allows us to prove the following statement about the growth of infidelity:

$$
r\left(\tilde{N}^{m}\right) \leq m r(\tilde{N})+O\left(r(\tilde{N})^{2}\right) m^{2}
$$

## Back the Toric Code

Theorem: Consider a toric code of size $L$ with minimum weight decoding and periodic boundary condition and a noise model consisting of single qubit rotations by an angle $\theta \leq 0.19$ radians about the X or Z axis on every qubit. Then, the following bounds hold for this noise model as well as any single qubit unitary noise model with arbitrary rotation angles that has the same logical noise strength.

$$
D_{\diamond}(N-i d)^{2} \leq c r^{2} \text { for a constant is given by } c \propto\left(\frac{1}{(\sin \theta)^{2}}\right)
$$

We can also consider the growth of infidelity as we apply the logical noise channel many times. Let $r_{m}$ be the infidelity after $m$ applications, then

$$
r_{m} \leq m r\left(1+\frac{d_{L}}{2\left(d_{L}+1\right) \sin \theta} m r\right)
$$

Similar statements continue to hold when we have correlated unitary noise.

## Proof sketch

- The basic plan will be to apply something like the repetition code calculation for each logical string.
- For each string, we will perform a restricted sum over stabilizers
- The disconnected parts of the syndrome will be factored out
- We will compare coherent and incoherent contributions for each string



## The First Difficulty

- Say we just consider logical noise components with only one nontrivial logical operator:

$$
\tilde{\chi}_{a, 0}=\left(\tilde{L}_{a} \tilde{\rho}\right)
$$

- The sum over restricted syndromes for the connected logical string is not as simple as in the repetition code:

Lower weight uncorrectable error
Higher weight correctable error


## Path Counting

- Path counting has been used many times to prove thresholds for stochastic noise in the toric code.
- We can bound the number of logical strings of a given length in the square lattice.
- However, we have to keep track of phases. We are adding many terms with different phases and so a bound does not help
- We also have an asymptotic form for the number when the length is well above minimum

$$
c_{w} \sim \mu^{w} w^{\alpha-2} \quad \mu \approx 2.64 \quad \alpha=1 / 2
$$

## Truncation

- The main tool in the proof is to use the path counting expression to truncate the length of logical strings we consider.
- If the angle of rotation $\theta$ is below o.19, then we can neglect the strings longer than $L+2 k$ for some constant $k$
- The error is exponentially small in $k$
- The remaining short strings are easier to analyze. In particular, they resolve the first difficulty.


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## A Second Difficulty

- The incoherent components of the logical noise channel now involve physical coherent terms that become incoherent under error correction:

$$
\left(\tilde{L}_{a} \tilde{\rho} \tilde{L}_{a}\right)=\sum_{s, i, j}\left(E_{s} L_{a} S_{i} \rho S_{j} L_{a} E_{s}\right)
$$

$\begin{array}{clclrl}\tilde{L}_{a} & \text { Logical } a \text { operator on encoded qubits } & E_{s} & \text { Standard error for syndrome } s & L_{a} & \text { Logical } a \text { on physical qubits } \\ \tilde{\rho} & \text { State of encoded qubits } & \rho & \text { State of physical qubits } & S_{i} & \text { Stabilizer operator } i\end{array}$

- Many of the terms on the right are off-diagonal.
- We must sum over all stabilizer operators on both left and right


## Other Difficulties

- Shape of logical strings
- When we compare coherent and incoherent terms for many different strings, we don't want to double count
- Factoring disconnected piece, new exceptional terms
- Physical Y errors
- Y logical errors
- Coherent $\tilde{\chi}_{a, b}$ components with both a and b nontrivial
- Bounding unitarity in terms of the $\chi$ matrix



## Arbitrary Angles

- So far we have considered a noise model in which every qubit is rotated by the same unitary.
- We can show that this maximizes the coherence of the logical noise channel
- For inhomogeneous rotation axes and angles, among the class of noise models with the same logical infidelity, the logical coherence is bounded by our calculation with all angles equal
- Proved using a schematic description of logical noise components as functions of the individual qubit angles.


## Correlations

- We can allow for correlations between qubits, so that we no longer have one qubit unitary rotations but entangled multiqubit rotations
- We use a Hamiltonian to model the correlations that was simple but not entirely physical

$$
H=\sum_{k} h_{1} X_{k}+\sum_{i, j} h_{2} X_{i} X_{j}
$$

- The same two body term couples every pair of qubits along a logical string.
- Coherence is still suppressed in this case


## Results

Theorem: Consider a toric code of size $L$ with minimum weight decoding and periodic boundary condition and a noise model consisting of single qubit rotations by an angle $\theta \leq 0.19$ radians about the X or Z axis on every qubit. Then, the following bounds hold for this noise model as well as any single qubit unitary noise model with arbitrary rotation angles that has the same logical noise strength.

$$
D_{\diamond}(N-i d)^{2} \leq c r^{2} \text { for a constant is given by } c \propto\left(\frac{1}{(\sin \theta)^{2}}\right)
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We can also consider the growth of infidelity as we apply the logical noise channel many times. Let $r_{m}$ be the infidelity after $m$ applications, then

$$
r_{m} \leq m r\left(1+\frac{d_{L}}{2\left(d_{L}+1\right) \sin \theta} m r\right)
$$

Similar statements continue to hold when we have correlated unitary noise.

## Future Work

- For now, our proof applies only to the toric code with minimum weight decoding. We expect that a similar theorem holds for any stabilizer code and reasonable decoding scheme.
- Numerics are probably needed to test how tight our bound is on the logical coherence for a particular code size.
- Can we find a more physical model for correlations that is tractable?


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