Title: Locality or non-locality? Fermionic entanglement on the torus

Speakers: Ignacio Reyes

Series: Quantum Fields and Strings

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Abstract: Understanding entanglement in QFTs is a challenging topic that involves many aspects. One important probe for this is the modular (or entanglement) Hamiltonian, which is closely related to the Unruh effect. We determine this operator for the chiral fermion at finite temperature on the circle using complex analysis, and show that it exhibits surprising new features. This simple system illustrates how a modular flow can transition from complete locality to complete non-locality as a function of temperature, thus bridging the gap between previously known limits. We also derive the first exact results for the entanglement and relative entropies of this system for the different spin sectors. Based on [1905.05768,1906.02207].

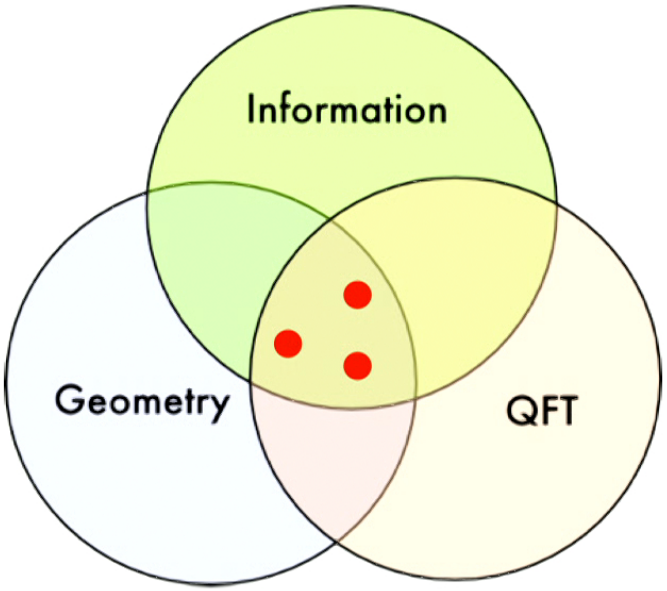
Local or non-local?
Fermionic entanglement on the torus

Ignacio Reyes

Max Plank Institut Potsdam

[1905.05768,1906.02207] with P Fries (Würzburg)

Perimeter Institute
August 2019



'70s Black hole thermodynamics

'06 Holographic entanglement entropy

'13 Complexity & black holes

... and many more

Outline

Introduction: entanglement Hamiltonians

The system/question: chiral fermion on the torus

Take-home message/solution: new Hamiltonian

The tools: resolvent + complex analysis

Entanglement

One measure: entanglement entropy

$$S_V = -\mathrm{tr}_V \left(\rho_V \log \rho_V\right)$$

Entanglement (modular) Hamiltonian

$$\rho_V := e^{-\mathcal{K}_V}$$

Relative entropy as measure of distinguishability of states

$$D(\rho|\sigma) = \mathrm{Tr}\left(\rho \log \rho\right) - \mathrm{Tr}\left(\rho \log \sigma\right) = \Delta\langle K\rangle - \Delta S$$

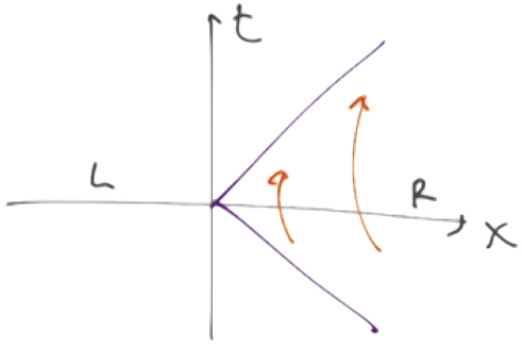
'First law of entanglement' $\quad d\langle \mathcal{K}_V\rangle = dS_V$

Quantum info, many body physics, AdS/CFT, ...

Simplest case: Unruh effect

Reduce vacuum into right half space $\rho_R = \mathrm{tr}_L |\Omega\rangle\langle\Omega|$

Vacuum prepared by euclidean path integral on plane



$$\rho_R := e^{-\mathcal{K}_R}$$

$$\mathcal{K}_R = \int_{x>0} dx\, \beta(x)\, T_{00} \quad , \quad \beta(x) = 2\pi x$$

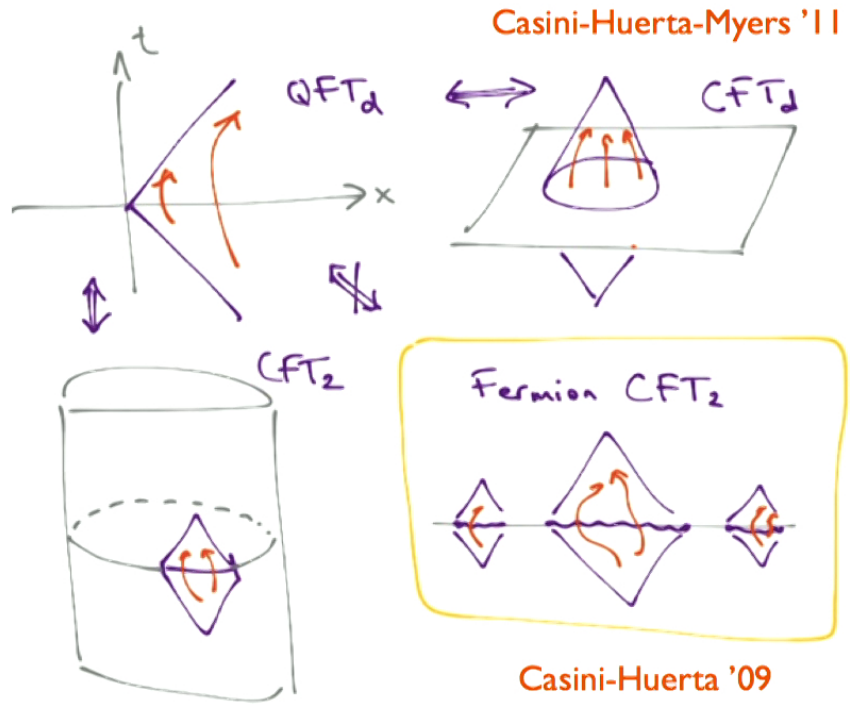
Here K is Noether charge of boost symmetry → local

What is known?

Vacuum

Rindler-like (local)

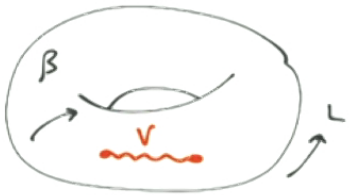
$$\mathcal{K}_{\rm loc} = \int_V dx\, \beta_V(x) T(x)$$



Non-trivial regions / topology → non-local, non-universal

The question

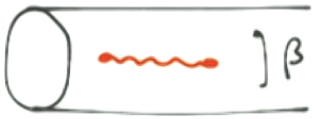
One interval on the torus?



4 spin sectors → R-NS , NS-NS (x,t)

β →0 : thermal state on $\mathbb{R}$

Local Cardy-Tonni '16



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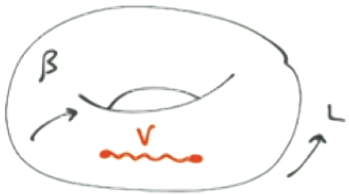

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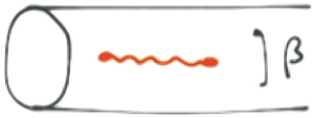
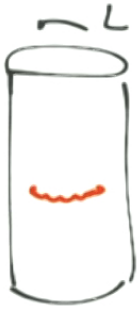


β →∞: two vacua on S^1 Klich-Vaman-Wong '15

NS : Local C-T '16

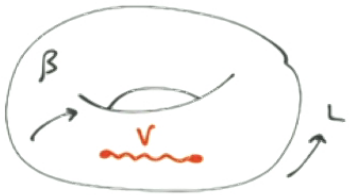


R : Local + completely non-local

$$\mathcal{K}_V \sim \int_{V^2} dx dy\, \alpha(x,y)\psi^\dagger(x)\psi(y)$$

The question

One interval on the torus?



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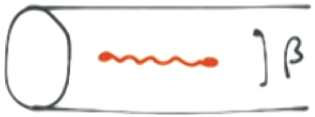

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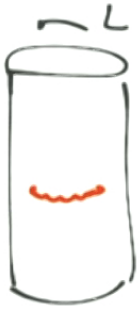

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$$\mathcal{K}_V \sim \int_{V^2} dxdy\, \alpha(x,y)\psi^\dagger(x)\psi(y)$$

How is the transition Locality → Non-locality ?

Take-Home message

The solution

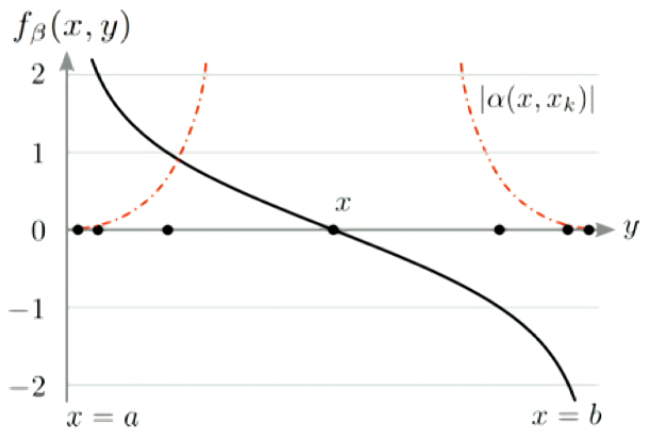
A new Hamiltonian

Fries-IR '19

At finite β : $\mathcal{K} = \mathcal{K}_{\text{loc}} + \mathcal{K}_{\text{bi-loc}}$ $\qquad \mathcal{K}_{\text{loc}} = \int_V dx\, \beta(x) T(x)$

Infinite but discrete bi-locality $\qquad \pm = {}^{R}_{NS}$

$$\mathcal{K}^{\pm}_{\text{bi-loc}} = \sum_{k\neq 0} (\pm 1)^k \int_{V^2} dx dy\, \alpha(x,y)\, \psi^\dagger(x)\psi(y)\, \delta(f_\beta(x,y) - k)$$



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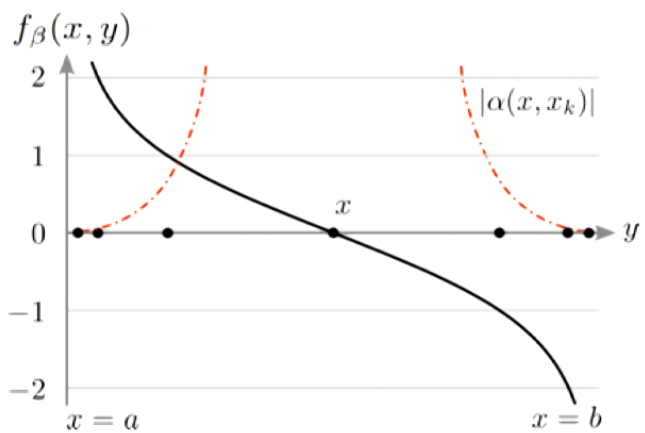


Solutions accumulate near the endpoints

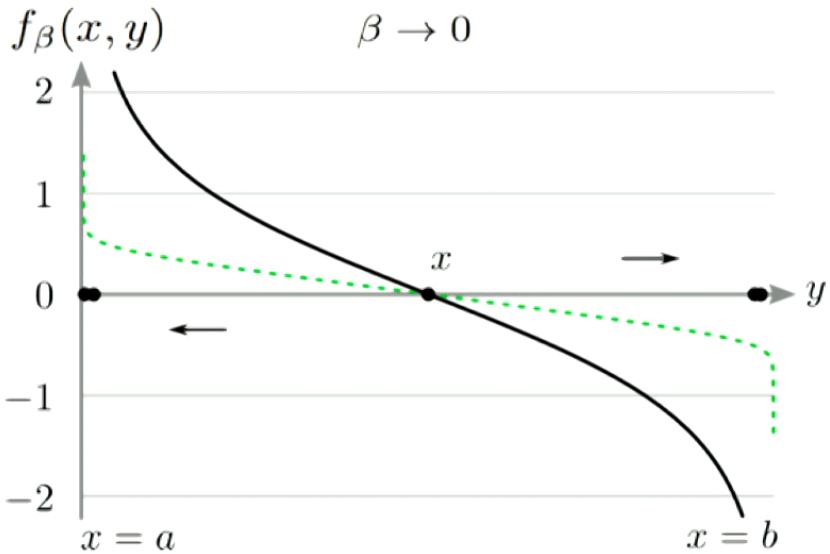
Non-local coupling: energy is 'redshifted'

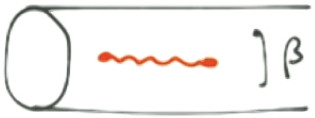
$$|\alpha(x,y)| \xrightarrow{y\to a} \frac{1}{\beta}(y-a)^{1+1/2L}$$

Bi-local at high temperature

High temperature: all solutions move towards endpoints and vanish → lose spin sectors

Recover local - universal Cardy-Tonni '16





Local

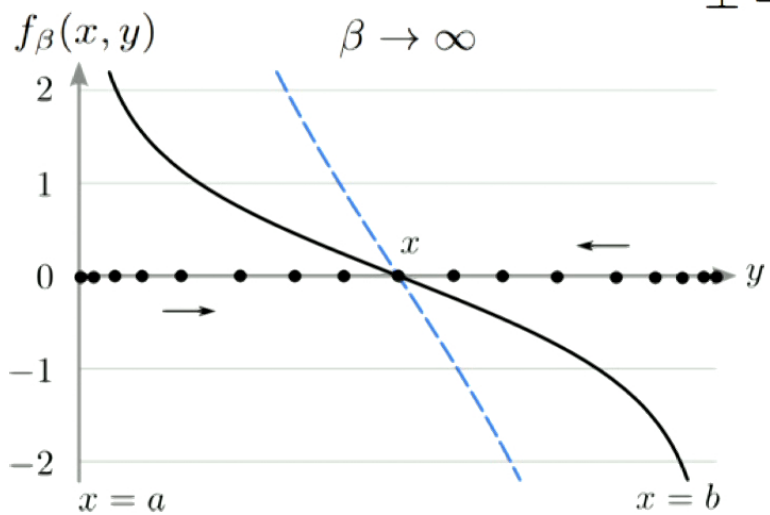
Bi-local at low temperature

Curve gets steeper inside → solutions 'condense'

Reproduces Riemann integral → non-local term!

$$\mathcal{K}^{\pm}_{\text{bi-loc}} = \sum_{k\neq 0} (\pm 1)^k \int_{V^2} dx dy\, \alpha(x,y)\, \psi^\dagger(x)\psi(y)\, \delta(f_\beta(x,y) - k)$$

$\pm = \begin{matrix} R \\ NS \end{matrix}$



NS	R
Local	Non-local
K-V-W '15	C-T '16

Free Fermions

Torus: propagator is completely fixed by the periodicities

$$G(x,y) \sim \frac{\vartheta_\nu(x-y|q)}{\vartheta_1(x-y|q)} \quad , \quad \nu = 2,3 = (\mathrm{R},\mathrm{NS}),(\mathrm{NS},\mathrm{NS})$$

and having a simple pole $\quad G \sim (2\pi i(x-y))^{-1} \quad , \quad x \to y$

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Reduced density matrix: enough to reproduce correlator

$$\mathrm{Tr}[\rho_V \psi(x)\psi^\dagger(y)] = G(x,y) \qquad x,y \in V$$

Then K is a quadratic fermionic operator Peschel'02

$$\mathcal{K}_V = \int_{V^2} dxdy\, K_V(x,y)\psi^\dagger(x)\psi(y)$$

with $\quad K_V = -\log[G_V^{-1} - 1]$ $\qquad$ how to compute log?

The Resolvent method

Holomorphic function of an operator

$$\log\left(G^{-1}-1\right) = \frac{1}{2\pi i}\oint_{\gamma} d\xi \, \log\left(\xi^{-1}-1\right) R_{\xi} \quad , \quad R_{\xi} = \frac{1}{\xi - G}$$

R is the 'resolvent' and γ must encircle entire spectrum of G

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The inverse means

$$\int_V dz\ R_\xi(x,z)\Big[\xi\delta(z-y) - G(z,y)\Big] = \delta(x-y)$$

Find resolvent → K, entropies, etc. easily follow

Plane Casini-Huerta '09, Cylinder Klich-Vaman-Wong '15

Rewrite as contour! Fries-IR '19

Change of variables $R_\xi(x,z) \to F_\xi(x,z)$

$$G(x,y) - F_\xi(x,y) - \frac{1}{\xi - 1/2}\int_V dz\; G(x,z)F_\xi(z,y) = 0$$

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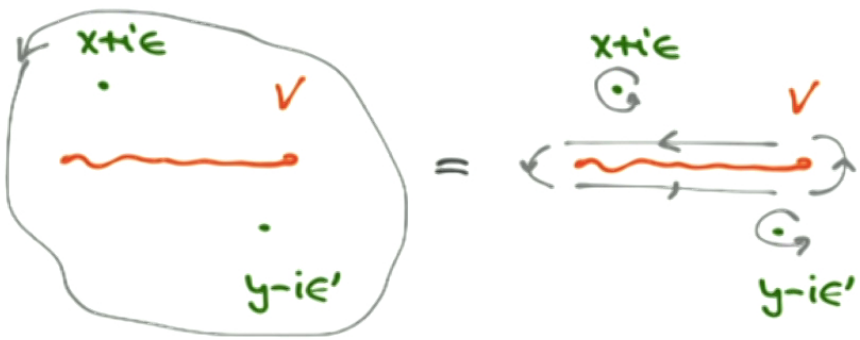
Key: secretly looks like a contour integral with zero residue !

$$\oint_\gamma dz\, G(x,z)F_\xi(z,y) = 0$$



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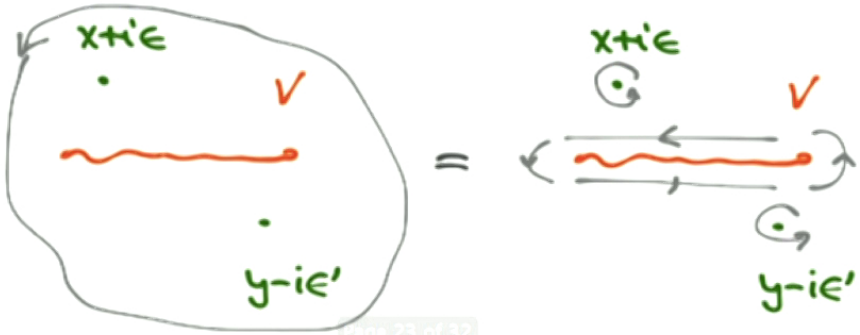
Find function with correct poles and cuts



Finding F... on the torus

Candidate: elliptic fn has zero residue in fundamental domain

$$\oint_\gamma dz\, G(x,z) F_\xi(z,y) = 0$$

$$G(x,y) - F_\xi(x,y) - \frac{1}{\xi - 1/2} \int_V dz\; G(x,z) F_\xi(z,y) = 0$$

Recipe

- Match z-periodicities of F to those of G

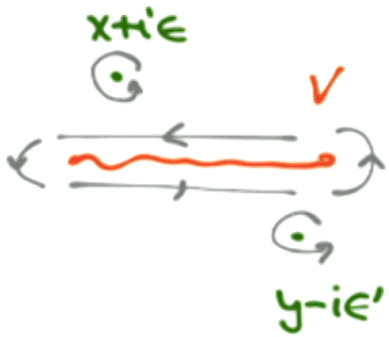
$$G(x,z)F(z,y) \quad , \quad z \sim z + n + m\tau$$

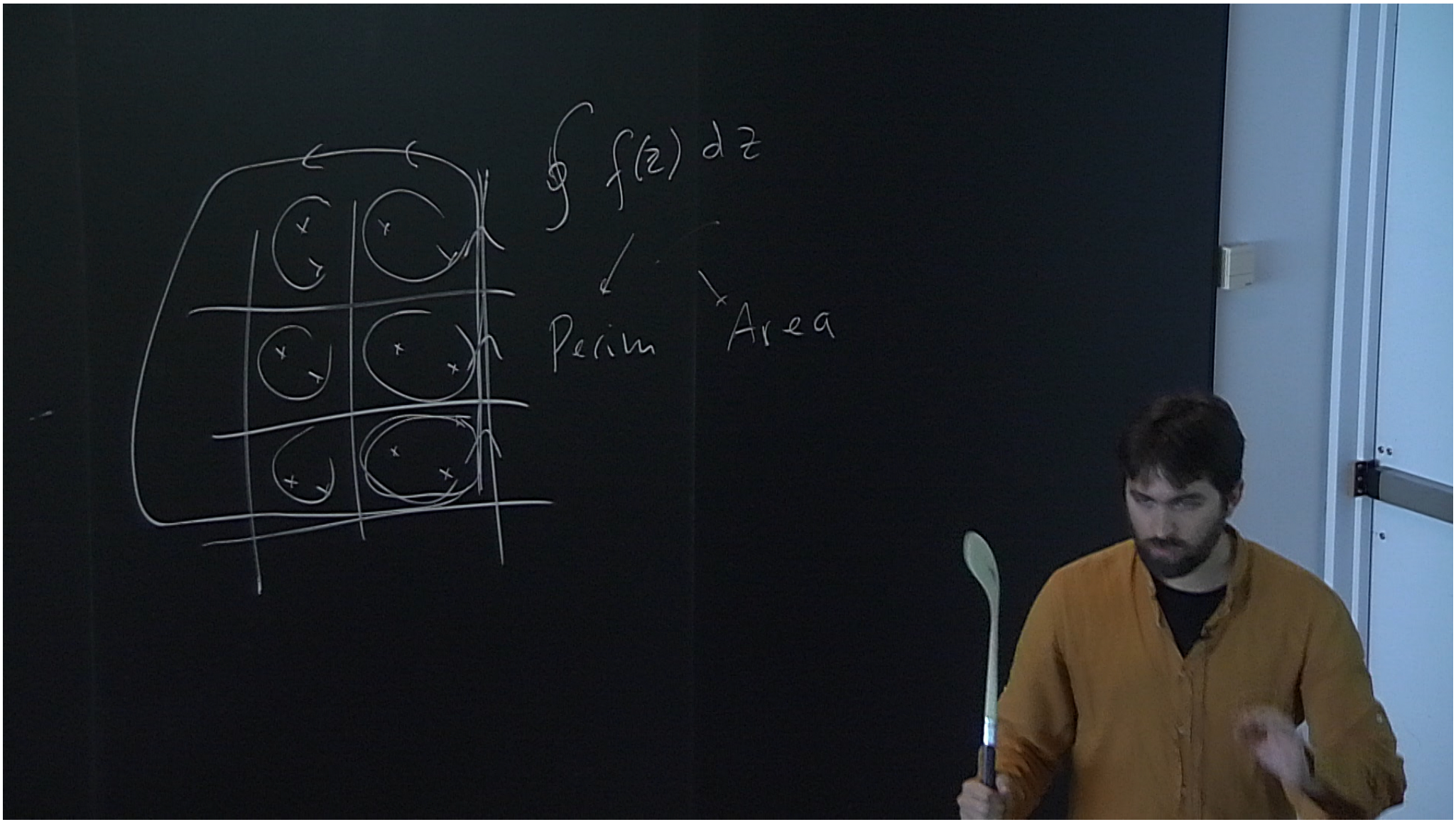
- Location of pole and branch cut

$$F(z \to y) \sim (2\pi i(z-y))^{-1}$$

- Boundary conditions at cut

$$F(z^+,y)/F(z^-,y) = \text{const} \quad , \quad \text{Res } F(\partial V) = 0$$




$\oint f(z)\,dz$
Perim
Area

The solution for F

$$F_\xi(z,y) = e^{-2\pi h} G_\nu(z-y|\tau, Lh)\left[\frac{\Omega(z|\tau)}{\Omega(y|\tau)}\right]^{ih}$$

$$h = \frac{1}{2\pi}\log\frac{\xi+1/2}{\xi-1/2}$$

$$G_\nu(z|\tau,\mu) = \frac{\eta^3(\tau)}{i\vartheta_1(z|\tau)}\frac{\vartheta_\nu(z-i\mu|\tau)}{\vartheta_\nu(-i\mu|\tau)},$$

$$\Omega(x|\tau) = -\frac{\vartheta_1(x-a|\tau)}{\vartheta_1(x-b|\tau)}$$

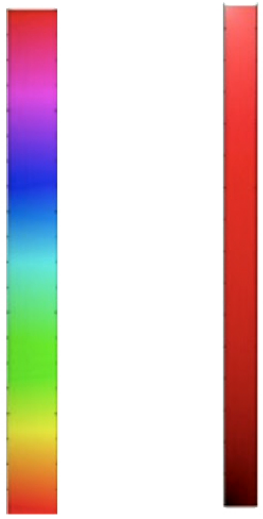
The elliptic function

$$G(x,z)F(z,y) = re^{i\theta}$$

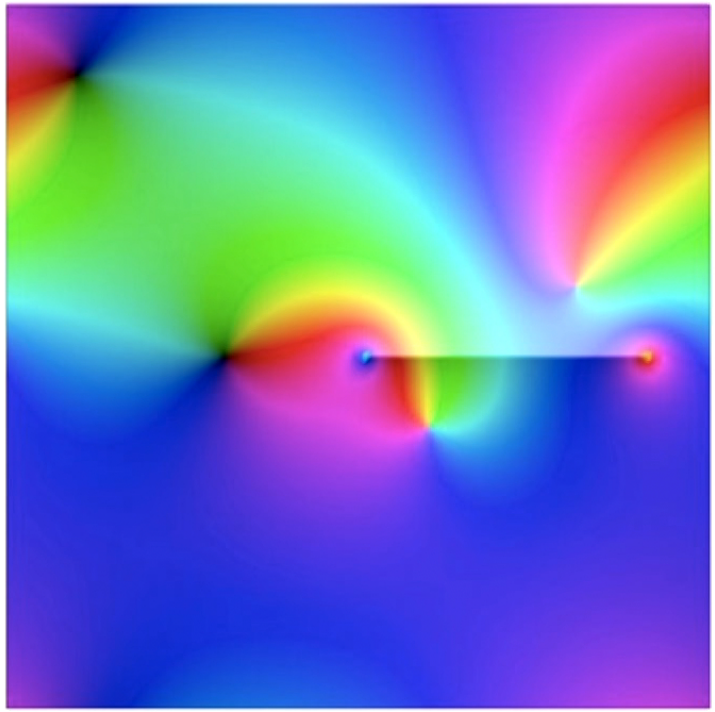
$t \sim t + i\beta$

$\theta = 2\pi$ $\quad$ $r = \infty$

$e^{i\theta} =$



$\theta = 0$ $\quad$ $r = 0$



$x \sim x + 1$

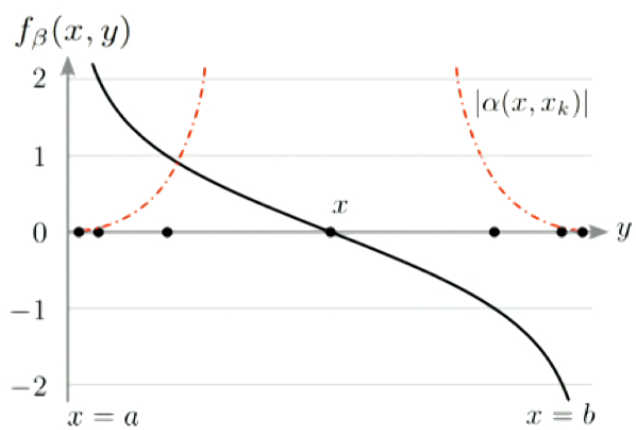
From Resolvent to Hamiltonian

Putting everything back together,

$$K_V = -\log[G|_V^{-1} - 1]$$

$$\log\left(G^{-1} - 1\right) = \frac{1}{2\pi i}\oint_\gamma d\xi \;\log\left(\xi^{-1} - 1\right)\; R_\xi$$

... and the result is



Entropies

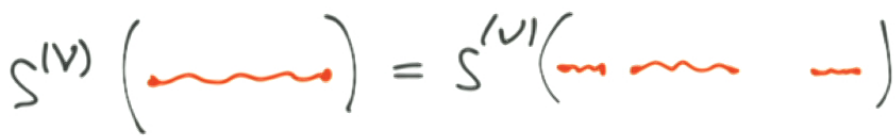
Rényi entropies known for n=2,3,... but analytic continuation is hard [Herzog-Nishioka '13, Azeyanagi-Nishioka-Takayanagi '07]

$$S = \lim_{n \to 1} \frac{1}{1-n} \log \mathrm{Tr}\left(\rho_V^n\right)$$

Compute directly from the resolvent [Fries-IR '19]

$$S = S^{(0)} + S^{(\nu)}$$

Spin-term only depends on the *total* length of subregion



Relative entropy: well defined in AQFT

$$D(\rho|\sigma) = \mathrm{Tr}\left(\rho \log \rho\right) - \mathrm{Tr}\left(\rho \log \sigma\right) = \Delta\langle K \rangle - \Delta S$$

Summary

Studied torus fermionic entanglement via resolvent

$\mathcal{K}_V$	Cardy-Tonni '16	Fries-IR 1 905.05768
S	Calabrese-Cardy '04	Fries-IR 1906.02207

New entanglement Hamiltonian

- Infinite set of bi-local couplings
- Transition from locality to non-locality

Solved via complex analysis on torus